Remarkable Squares of Homotopy Types*

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Dedicated to the memory of Carlos B. de Lyra

1. Introduction. Although the category N of nilpotent groups does not admit pushouts in general, there exist non-trivial examples of bicartesian squares in N. Two particular ways of constructing such squares in N were discussed in [2]. Here we apply the results of [2] and [3] in order to construct analogous examples in the homotopy category NH of nilpotent spaces.

Recall that a commutative square

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho"] \\ Z \arrow[r, "\sigma"'] & W \end{tikzcd} \tag{1.1}$$

in the homotopy category H is called a *weak pullback* if, after turning ρ into a fibration, the induced fibre space over Z is naturally homotopy equivalent to X. It was proved in [3; Corollary II.7.6] that if the square (1.1) is a weak pullback with Y, Z, W in NH, then X is in NH provided it is connected; in this case we say that (1.1) is a *weak pullback in NH.*

Dually, the square (1.1) is called a *weak pushout* if, after turning ψ into a cofibration, the total space of the induced cofibration over Y is naturally homotopy equivalent to W. In general, W will not lie in NH even if X, Y, Z are in NH; for exemple, the fundamental group of W will fail to be nilpotent.

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We will define a *weak pushout in NH* in the following way. Consider the square (1.1), assumed to be in NH, and form the weak pushout of $\{\phi, \psi\}$ in H; we thus have a commutative diagram in H,

(1.2)
$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\bar{\rho}"] \arrow[ddr, "\rho"] & \\
Z \arrow[r, "\bar{\sigma}"'] \arrow[drr, "\sigma"'] & \overline{W} \arrow[dr, "\theta"] & \\
& & W
\end{tikzcd}$$

where $\{\bar{\rho}, \bar{\sigma}\}$ is the weak pushout of $\{\phi, \psi\}$. We say that (1.1) is a *weak pushout in NH* if we may choose θ in (1.2) to be a homology equivalence

$$\theta_* : H_*\overline{W} \cong H_*W \qquad \text{(integer coefficients)}$$

We justify this definition by the following observations.

PROPOSITION 1.1. *Let* (1.1) *be a weak pushout in NH and let*

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho'"] \\
Z \arrow[r, "\sigma'"'] & W'
\end{tikzcd}$$

be a commutative square in NH. Then there exists $\kappa : W \longrightarrow W'$ *with* $\kappa\rho = \rho'$, $\kappa\sigma = \sigma'$.

PROOF. There certainly exists $\bar{\kappa} : \overline{W} \longrightarrow W'$ with $\bar{\kappa}\bar{\rho} = \rho'$, $\bar{\kappa}\bar{\sigma} = \sigma'$. But θ induces

(1.3)
$$\theta^* : [W, M] \cong [\overline{W}, M]$$

for any nilpotent space M. Thus there exists (a unique) $\kappa : W \longrightarrow W'$ with $\kappa\theta = \bar{\kappa}$. Obviously $\kappa\rho = \rho'$, $\kappa\sigma = \sigma'$.

PROPOSITION 1.2. *Let* (1.1) *and*

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho'"] \\
Z \arrow[r, "\sigma'"'] & W'
\end{tikzcd}$$

both be weak pushouts in NH, so that there exists a homology equivalence $\theta' : \overline{W} \longrightarrow W'$ *with* $\theta'\bar{\rho} = \rho'$, $\theta'\bar{\sigma} = \sigma'$. *Then there exists a unique* $\kappa : W \longrightarrow W'$ *with* $\kappa\theta = \theta'$. *Moreover* κ *is a homotopy equivalence and* $\kappa\rho = \rho'$, $\kappa\sigma = \sigma'$.

PROOF. From (1.3) we infer a unique $\kappa : W \longrightarrow W'$ with $\kappa\theta = \theta'$. Then κ is a homology equivalence between nilpotent spaces and hence a homotopy equivalence. Finally $\kappa\rho = \kappa\theta\bar{\rho} = \theta'\bar{\rho} = \rho'$, and similarly $\kappa\sigma = \sigma'$.

Proposition 1.2 shows that the weak pushout in NH is essentially unique; and Proposition 1.1 shows that it does have the "weak pushout property" in NH. Of course, there is no guarantee that the weak pushout in NH exists; if it does then we have a Mayer-Vietoris sequence, from which we infer (for definitions see [3])

PROPOSITION 1.3. *If* (1.1) *is a weak pushout in NH and if Y, Z are of finite type* (*quasifinite*) *then X is of finite type* (*quasifinite*) *iff W is of finite type* (*quasifinite*).

We say that the square (1.1) in NH is *weakly bicartesian in NH* if it is a weak pullback in NH and a weak pushout in NH.

In Section 2 we will consider *(P, Q)-squares*, where P, Q are families of primes such that $P \cup Q = \Pi$, the family of all primes. Then a (P, Q)*-square* is a commutative diagram (1.1) in NH such that ϕ, σ are P-equivalences and ψ, ρ are Q-equivalences. We will prove:

THEOREM A. (i) *Every* (P, Q)*-square is weakly bicartesian in NH.*
(ii) *If* $\rho : Y \longrightarrow W$ *is a Q-equivalence in NH and* $\sigma : Z \longrightarrow W$ *is a P-equivalence in NH, then* $\{\rho, \sigma\}$ *may be embedded in a* (P, Q)*-square.*
(iii) *If* $\phi : X \longrightarrow Y$ *is a P-equivalence in NH and* $\psi : X \longrightarrow Z$ *is a Q-equivalence in NH then* $\{\phi, \psi\}$ *may be embedded in a* (P, Q)*-square.*

We note that (P, Q)-squares are also studied in [4] and a result similar to Theorem *A(i)* is obtained. However, Kahn's definitions do not precisely coincide with ours, so that his (P, Q)-squares are more restricted; in particular he requires the *morphisms* of the square to be nilpotent. This restriction is especially significant in considering weak pushouts.

In Section 2 we describe some natural constructions of (P, Q)-squares.

In Section 3 we study a particular square, constructed out of a nilpotent space X, which was discussed in [3] in the case when X is of finite type. Let $e_p: X \longrightarrow X_p$ be the p-localization map, $p \in \Pi$, and let

$$\check{e}: X \longrightarrow \check{X} = \prod_p X_p$$

be the map with components e_p; we call $\check{X}$ the *local expansion* of X. Let $\check{e}_0$ be the rationalization of $\check{e}$. Thus, writing $\check{X}_0$ for $(\check{X})_0$,

$$\check{e}_0: X_0 \longrightarrow \check{X}_0,$$

and we have a commutative square

$$\begin{tikzcd} X \arrow[r, "\check{e}"] \arrow[d, "r"] & \check{X} \arrow[d, "r"] \\ X_0 \arrow[r, "\check{e}_0"] & \check{X}_0 \end{tikzcd} \qquad (1.4)$$

We prove:

THEOREM B. *The square* (1.4) *is weakly bicartesian in NH.*

We will use Theorem *B* to improve results in [3] concerning the possibility of equipping a nilpotent space X with an H-space structure, inducing prescribed H-space structures on the localizations X_p of X.

We note that, in defining a weak pushout in NH, we merely require that θ may be chosen in (1.2) to be a homology equivalence. However, in the circumstances of Theorems A or B, every θ fitting into the diagram (1.2) induces the *same* $\theta_*: H_*\bar{W} \to H_*W$. This is because, in those circumstances, the functor H_n yields a bicartesian square

$$\begin{tikzcd} H_nX \arrow[r] \arrow[d] & H_nY \arrow[d] \\ H_nZ \arrow[r] & H_n\bar{W} \end{tikzcd}$$

We will adopt throughout this paper the notation and terminology of [3], except that in [3] the notation $\check{X}$ is not introduced (and $\check{e}$ appears as $\hat{e}$).

2. (P, Q)-squares. We first prove Theorem *A(ii)* in the form of the following more precise Proposition.

PROPOSITION 2.1. *If* $\rho: Y \longrightarrow W$ *is a* Q*-equivalence in* NH *and* $\sigma: Z \longrightarrow W$ *is a* P*-equivalence in* NH, *with* $P \cup Q = \Pi$, *then, in the weak pullback (in* H*)*

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"] & Y \arrow[d, "\rho"] \\ Z \arrow[r, "\sigma"'] & W \end{tikzcd} \qquad (2.1)$$

we have:

(i) X *is in* NH, *so that* (2.1) *is a weak pullback in* NH;

(ii) *in* (2.1) ϕ *is a* P*-equivalence and* ψ *is a* Q*-equivalence.*

PROOF. (i) X is connected by Corollary II.7.12 of [3], and so is in NH by Corollary II.7.6 of [3]. Then (ii) follows immediately from Corollary II.7.10 of [3], since we know that X is connected.

We next prove Theorem *A(iii)*, in a form dual to Proposition 2.1.

PROPOSITION 2.2. *If* $\phi: X \longrightarrow Y$ *is a* P*-equivalence in* NH *and* $\psi: X \longrightarrow Z$ *is a* Q*-equivalence in* NH, *with* $P \cup Q = \Pi$, *then*

(i) $\{\phi, \psi\}$ *may be embedded in a weak pushout in* NH

(2.2)
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho"] \\ Z \arrow[r, "\sigma"'] & W \end{tikzcd}$$

(ii) *in* (2.2) ρ *is a* Q*-equivalence and* σ *is a* P*-equivalence.*

PROOF. Given two families of primes P, Q and a P-equivalence $u: K \longrightarrow L$ in NH, define

$$u^Q: L_Q \longrightarrow K_{P \cap Q}$$

by $u^Q = u_{P \cap Q}^{-1} \circ (e_P)_Q$. Here $e_P: L \longrightarrow L_P$ is P-localization and $(e_P)_Q: L_Q \longrightarrow L_{P \cap Q}$ is the Q-localization of e_P. Since e_P is a P-equivalence, so is $(e_P)_Q$ and hence so is u^Q. Moreover we have the relation

$$u^Q e_Q u = e_{P \cap Q}: K \longrightarrow K_{P \cap Q}, \tag{2.3}$$

where $e_Q: L \longrightarrow L_Q$.

Now construct the weak pushout of $\{\phi, \psi\}$ in H,

(2.4)
$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\bar{\rho}"] \\ Z \arrow[r, "\bar{\sigma}"'] & \bar{W} \end{tikzcd}$$

and the weak pullback of $\{\phi^Q, \psi^P\}$ in H,

(2.5)
$$\begin{tikzcd} \bar{W} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y_Q \arrow[d, "\phi^Q"] \\ Z_P \arrow[r, "\psi^P"'] & X_{P \cap Q} \end{tikzcd}$$

By Proposition 2.1, ϕ^Q being a P-equivalence and ψ^P a Q-equivalence, (2.5) is a weak pullback in NH; and, moreover, α is a Q-equivalence and β is a P-equivalence. Since ψ is a Q-equivalence, ψ induces $\psi^*: H^*(Z; G) \cong H^*(X; G)$, for any $\mathbb{Z}_Q$-module G. Applying the Mayer-Vietoris sequence in cohomology to (2.4), we infer that

$$\bar{\rho}^*: H^*(\bar{W}; G) \cong H^*(Y; G), \text{ for any } \mathbb{Z}_Q\text{-module } G. \tag{2.6}$$

Since Y_Q is a nilpotent Q-local space, we infer from (2.6) a (unique) map $\lambda: \bar{W} \longrightarrow Y_Q$ with $\lambda\bar{\rho} = e_Q: Y \longrightarrow Y_Q$. Similarly, we infer a (unique) map $\mu: \bar{W} \longrightarrow Z_P$ with $\mu\bar{\sigma} = e_P: Z \longrightarrow Z_P$. Now, by (2.3),

$$\begin{aligned} \phi^Q\lambda\bar{\rho}\phi &= \phi^Q e_Q \phi = e_{P \cap Q}: X \longrightarrow X_{P \cap Q}, \quad \text{and} \\ \psi^P\mu\bar{\sigma}\psi &= \psi^P e_P \psi = e_{P \cap Q}: X \longrightarrow X_{P \cap Q}. \end{aligned} \tag{2.7}$$

But $\bar{\rho}\phi = \bar{\sigma}\psi$ induces an isomorphism in cohomology,

$$(\bar{\rho}\phi)^* = (\bar{\sigma}\psi)^*: H^*(\bar{W}; G) \cong H^*(X; G), \text{ for any } \mathbb{Z}_{P \cap Q}\text{-module } G,$$

and $X_{P \cap Q}$ is a nilpotent $(P \cap Q)$-local space. Thus we infer from (2.7) that

$$\phi^Q\lambda = \psi^P\mu. \tag{2.8}$$

However, (2.5) is a weak pullback in H, so that we may infer from (2.8) a map $\theta: \bar{W} \longrightarrow W$ with

$$\alpha\theta = \lambda, \ \beta\theta = \mu. \tag{2.9}$$

Now $\bar{\rho}$, e_Q and α all induce homology isomorphisms with $\mathbb{Z}_Q$ coefficients; so therefore, by the first relation (2.9), does θ. Similarly the second relation (2.9) implies that θ induces homology isomorphisms with $\mathbb{Z}_P$ coefficients. Since $P \cup Q = \Pi$, it follows that

$$\theta: H_* \bar{W} \cong H_* W.$$

Now define $\rho = \theta\bar{\rho}: Y \longrightarrow W$, $\sigma = \theta\bar{\sigma}: Z \longrightarrow W$. It follows from the definition that (2.2) is now a weak pushout in NH. Also

$$\rho_*: H_*(Y; \mathbb{Z}_Q) \cong H_*(W; \mathbb{Z}_Q)$$

so that, Y, W, being nilpotent, ρ is a Q-equivalence. Similarly σ is a P-equivalence and the proposition is completely proved.

It remains to prove Theorem A(i). Given a (P, Q)-square

(2.10)
$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho"] \\
Z \arrow[r, "\sigma"'] & W
\end{tikzcd}$$

we form, as we know from Propositions 2.1 and 2.2 that we may, the weak pullback of $\{\rho, \sigma\}$ in NH and the weak pushout of $\{\phi, \psi\}$ in NH, and obtain the commutative diagram

(2.11)
$$\begin{tikzcd}
X \arrow[rrr, "\phi"] \arrow[dr, "\alpha"] \arrow[ddd, "\psi"'] & & & Y \arrow[dl, "\rho'"'] \arrow[ddd, "\rho"] \\
& X' \arrow[urr, "\phi'"] \arrow[ddl, "\psi'"] & W' \arrow[ddr, "\beta"] & \\
& & & \\
Z \arrow[uurr, "\sigma'"] \arrow[rrr, "\sigma"'] & & & W
\end{tikzcd}$$

where $\{\phi', \psi'\}$ is the weak pullback of $\{\rho, \sigma\}$ in NH and $\{\rho', \sigma'\}$ is the weak pushout of $\{\phi, \psi\}$ in NH. The maps α, β then exist by the weak pullback property and the weak pushout property (Proposition 1.1) respectively.

By Proposition 2.1(ii) ϕ' is a P-equivalence. Since ϕ is a P-equivalence it follows that α is a P-equivalence. Similarly α is a Q-equivalence, so that, since $P \cup Q = \Pi$, α is a homotopy equivalence. In the same way we invoke Proposition 2.2(ii) to infer that β is a homotopy equivalence, and so complete the proof of Theorem A.

REMARK. A space is said to be prenilpotent [1] if it has the property of admitting a homology equivalence to a nilpotent space. Our proof of Proposition 2.2 shows that if $\phi: X \longrightarrow Y$ is a P-equivalence in NH and $\psi: X \longrightarrow Z$ is a Q-equivalence in NH with $P \cup Q = \Pi$, and if we form the weak pushout (2.4) of $\{\phi, \psi\}$ in H, then $\overline{W}$ is prenilpotent. Of

course, our definition of the weak pushout in NH makes it obvious that, given any maps $\phi: X \longrightarrow Y$, $\psi: X \longrightarrow Z$ in NH, then $\{\phi, \psi\}$ admits a weak pushout in NH iff the weak pushout in H is prenilpotent.

We record the following additional property of (P, Q)-squares.

PROPOSITION 2.3. *Let* (2.10) *be a* (P, Q)*-square. Then* X *and* W *are of finite type (quasifinite) iff* Y *and* Z *are of finite type (quasifinite).*

PROOF. Suppose X and W are of finite type. Then $(H_n Y)_P \cong (H_n X)_P$ is a finitely generated $\mathbb{Z}_P$-module and $(H_n Y)_Q \cong (H_n W)_Q$ is a finitely generated $\mathbb{Z}_Q$-module. Thus $H_n Y$ is finitely generated by Theorem I.3.10 of [3]. Thus Y is of finite type; and similarly Z is of finite type. If $H_n X = 0$, $H_n W = 0$, then $(H_n Y)_P = 0$, $(H_n Y)_Q = 0$, so $H_n Y = 0$. Thus Y is quasifinite if X and W are quasifinite; so similarly is Z. The converse implications are proved in exactly similar fashion.
We now give some examples of (P, Q)-squares.

EXAMPLE 2.4. We obtain an example of a (P, Q)-square by starting with any nilpotent space X and localizing at P and Q. Thus

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_P \arrow[d] \\
X_Q \arrow[r] & X_{P \cap Q}
\end{tikzcd}$$

EXAMPLE 2.5. We consider the genus $G(X)$ of a quasifinite H-space X. Let $Y, Z \in G(X)$, so that, by Theorem III.1.14 of [3], Y and Z are quasifinite H-spaces. As shown in the proof of that theorem, we can then certainly find maps $\phi: X \longrightarrow Y$, $\psi: X \longrightarrow Z$ such that ϕ is a P-equivalence, ψ is a Q-equivalence, where $P \cup Q = \Pi$. Use Proposition 2.2 to embed $\{\phi, \psi\}$ in a (P, Q)-square

(2.12)
$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, "\psi"'] & Y \arrow[d, "\rho"] \\
Z \arrow[r, "\sigma"'] & W
\end{tikzcd}$$

Then W is quasifinite by Proposition 2.3 and obviously $W \in G(X)$ since ρ is a Q-equivalence and σ is a P-equivalence. Thus W is a quasifinite H-space in the genus of X and it is natural to ask if W is uniquely determined by X, Y, Z (this would be the case if we put some extra restriction on ϕ or ψ).

In the argument proving Theorem III.1.14 of [3], the pair $\{\phi, \psi\}$ was constructed so that $\{\phi, \psi\}: X \longrightarrow Y \times Z$ admitted a retraction $\kappa: Y \times Z \longrightarrow X$, $\kappa\{\phi, \psi\} = 1$. If $\lambda: W \longrightarrow Y \times Z$ is the fibre of κ, then $X \times W \simeq Y \times Z$. Moreover, we may find a homotopy equivalence $\omega: Y \times Z \longrightarrow X \times W$ such that, if $\omega = \{\omega', \omega''\}$, $\omega': Y \times Z \longrightarrow X$, $\omega'': Y \times Z \longrightarrow W$, then

$$X \xrightarrow{\{\phi, \psi\}} Y \times Z \xrightarrow{\omega''} W \tag{2.13}$$

is a fibration. From this it readily follows that W is in the genus of X. However, it is not immediately clear that we may convert (2.13) into a (P, Q)-square (2.12). This would be the case if we could arrange for ω'' to be an H-map when $Y \times Z$ has a direct product H-structure.

On the other hand, given (2.12), we may construct a fibration

$$X \xrightarrow{\{\phi, \psi\}} Y \times Z \xrightarrow{\tau} W,$$

where $\tau(y, z) = \rho y \,.\, \sigma z^{-1}$ (confusing maps and homotopy classes!) Here it is not at all plain that the fibration has a cross-section.

Of course, we may use Proposition 2.1 instead of Proposition 2.2 to construct (P, Q)-squares having three given quasifinite H-spaces of the same genus as three of their vertices.

REMARK. Obviously if we apply a homology functor H_n or a homotopy functor π_n to a (P, Q)-square (2.10) we obtain a (P, Q)-square in Ab, except in the case of π_1, and a (P, Q)-square in N in the case of π_1. These squares are then bicartesian as shown in [2]. We may also apply a homology functor H_n to the weak pushout (2.4) and it follows immediately from the Mayer-Vietoris theorem that we obtain a (P, Q)-square in Ab, which is consequently bicartesian. This shows that, although we may have choice in the construction of a map θ with $\theta\bar{\rho} = \rho$, $\theta\bar{\sigma} = \sigma$, the induced homomorphism $\theta_*: H_*\overline{W} \longrightarrow H_*W$ (which is an isomorphism) is uniquely determined.

3. The local expansion and its rationalization.

We revert to the square (1.4)

$$\begin{tikzcd}
X \arrow[r, "\check{e}"] \arrow[d, "r"] & \check{X} \arrow[d, "r"] \\
X_0 \arrow[r, "\check{e}_0"'] & \check{X}_0
\end{tikzcd} \tag{3.1}$$

and proceed to prove Theorem B, asserting that this is square is weakly bicartesian in NH. Since it was shown in [2] that (3.1) is a weak pullback, it remains only to show that it is a weak pushout in NH.

LEMMA 3.1. *The map* $\check{e}: X \longrightarrow \check{X}$ *induces an injection*

$$\check{e}(G): H_n(X; G) \longrightarrow H_n(\check{X}; G),$$

for any coefficient group G.

PROOF. Let $\tau_p: \check{X} \longrightarrow X_p$ be the projection, so that $\tau_p\check{e} = e_p: X \longrightarrow X_p$. Then $\tau_p(G)\check{e}(G) = e_p(G): H_n(X; G) \longrightarrow H_n(X_p; G)$ is the p-localization map, and hence p-localizes to an isomorphism. Thus $\check{e}(G)$ p-localizes to an injection for each prime p, and hence is itself an injection.

Now form the weak pushout of $\{\check{e}, r\}$ in H,

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "r"'] & X \arrow[d, "\rho"] \\
X_0 \arrow[r, "\sigma"'] & W
\end{tikzcd} \tag{3.2}$$

There is then a map $\theta: W \longrightarrow \check{X}_0$ with $\theta\rho = r$, $\theta\sigma = \check{e}_0$. We will show that θ is a homology equivalence.

PROPOSITION 3.2. *The diagram (3.2) gives rise to a bicartesian square*

$$\begin{tikzcd}
H_n(X;G) \arrow[r, "\check{e}(G)"] \arrow[d, "r(G)"'] & H_n(\check{X};G) \arrow[d, "\rho(G)"] \\
H_n(X_0;G) \arrow[r, "\sigma(G)"'] & H_n(W;G)
\end{tikzcd} \tag{3.3}$$

in Ab, for any coefficient group G.

PROOF. The diagram (3.2) certainly gives rise to a Mayer-Vietoris sequence in homology with coefficients in G. However, since $\check{e}(G)$ is injective by Lemma 3.1, the Mayer-Vietoris sequence breaks up into a collection of short exact sequence, each of which attests that (3.3) is bicartesian for some value of n.

PROPOSITION 3.3. $\check{e}(\mathbb{Z}/p): H_n(X;\mathbb{Z}/p) \longrightarrow H_n(\check{X};\mathbb{Z}/p)$ *is an isomorphism.*

PROOF. The proof exactly parallels that of (3.5) of [2]. We observe that $\check{X} = X_p \times X(p)$, where $X(p) = \prod_{q \neq p} X_q$. Since each X_q is p'-local, so is $X(p)$. Thus $H_n(X(p);\mathbb{Z}/p) = 0$, $n \geq 1$. It follows, by applying the the Künneth formula, that $\pi_p: \check{X} \longrightarrow X_p$ induces an isomorphism $\tau_p(\mathbb{Z}/p): H_n(\check{X};\mathbb{Z}/p) \cong H_n(X_p;\mathbb{Z}/p)$. Of course $e_p(\mathbb{Z}/p)$ is an isomorphism so the proposition follows from the relation $\tau_p\check{e} = e_p$.

We now complete the proof of Theorem *B*; again our argument is modelled on that given in [2]. Since (3.3) is bicartesian, it follows that $\sigma(\mathbb{Z}/p)$ is an isomorphism. But $H_n(X_0;\mathbb{Z}/p) = 0$, $n \geq 1$, so

$$H_n(W;\mathbb{Z}/p) = 0,\ n \geq 1,\ \textit{all } p. \tag{3.4}$$

Of course,

$$H_n(\check{X}_0;\mathbb{Z}/p) = 0,\ n \geq 1,\ \textit{all } p. \tag{3.5}$$

Now $r(\mathbb{Q}): H_n(X;\mathbb{Q}) \cong H_n(X_0;\mathbb{Q})$, so that Proposition 3.2 implies that

$$\rho(\mathbb{Q}): H_n(\check{X};\mathbb{Q}) \cong H_n(W;\mathbb{Q}),\ n \geq 0. \tag{3.6}$$

Of course

$$r(\mathbb{Q}): H_n(\check{X};\mathbb{Q}) \cong H_n(\check{X}_0;\mathbb{Q}),\ n \geq 0. \tag{3.7}$$

Thus, since $\theta\rho = r$, it follows from (3.6) and (3.7) that

$$\theta(\mathbb{Q}): H_n(W;\mathbb{Q}) \cong H_n(\check{X}_0;\mathbb{Q}),\ n \geq 0. \tag{3.8}$$

Now (3.4), (3.5) and (3.8) imply that θ induces

$$\theta_*: H_nW \cong H_n\check{X}_0,\ n \geq 0,$$

and Theorem *B* is proved.

REMARK. Since (3.3) is bicartesian, it follows that every $\theta: W \longrightarrow \check{X}_0$, such that $\theta\rho = r$, $\theta\sigma = \check{e}_0$, induces the *same* $\theta_*: H_*W \longrightarrow H_*\check{X}_0$.

In order to apply Theorem *B* to the study of *H*-spaces we need a preliminary result. We first prove a lemma.

LEMMA 3.4. *Let X be a connected H-space which is a rational space (that is, $X = X_0$), and, for any space Y, let*

$$\delta: [Y, X] \longrightarrow \mathrm{Hom}\,(H_*(Y;\mathbb{Q}),\ H_*(X;\mathbb{Q}))$$

be the evident map. Then δ is injective.

PROOF. Since X is a connected H-space which is rational⁽¹⁾, it follows that

$$X \simeq \prod_{n \geq 1} K(R_n, n),$$

where R_n is a $\mathbb{Q}$-vector space. Thus there are natural isomorphisms

$$[Y, X] \cong \prod [Y, K(R_n, n)] \cong \prod H^n(Y; R_n) \cong \prod \mathrm{Hom}(H_n(Y;\mathbb{Q}), R_n)$$

and we write $\varepsilon: [Y, X] \cong \prod \mathrm{Hom}(H_n(Y;\mathbb{Q}), R_n)$ for the composite isomorphism. On the other hand, the family of projections $X \longrightarrow K(R_n, n)$ induces

$$\kappa: \mathrm{Hom}(H_*(Y;\mathbb{Q}),\ H_*(X;\mathbb{Q})) \longrightarrow \prod \mathrm{Hom}(H_n(Y;\mathbb{Q}),\ R_n),$$

where we have identified $H_n(K(R_n, n);\mathbb{Q})$ with R_n. It is plain that $\kappa\delta = \varepsilon$, so that δ is injective.

(1) We avoid the phrase 'rational *H*-space', since this refers to a nilpotent space X such that X_0 admits an *H*-structure.

COROLLARY 3.5. *Let X, Z be connected H-spaces with X rational, and let $f\colon Z \longrightarrow X$. Then f is an H-map if and only if*

$$f_*\colon H_*(Z;\mathbb{Q}) \longrightarrow H_*(X;\mathbb{Q})$$

is an algebra map.

PROOF. Note first that, if we identify $H_*(Y_1 \times Y_2;\mathbb{Q})$ with $H_*(Y_1;\mathbb{Q}) \otimes H_*(Y_2;\mathbb{Q})$, then, under the map δ of Lemma 3.4,

$$\delta(\alpha_1 \times \alpha_2) = \delta\alpha_1 \otimes \delta\alpha_2.$$

Now let $m\colon X \times X \longrightarrow X$, $m\colon Z \times Z \longrightarrow Z$ be the multiplications. Then, by Lemma 3.4, the diagram

$$\begin{tikzcd} Z \times Z \arrow[r, "m"] \arrow[d, "f \times f"] & Z \arrow[d, "f"] \\ X \times X \arrow[r, "m"] & X \end{tikzcd} \tag{3.9}$$

(homotopy) commutes if and only if the diagram

$$\begin{tikzcd} H_*(Z;\mathbb{Q}) \otimes H_*(Z;\mathbb{Q}) \arrow[r, "\delta m"] \arrow[d, "\delta f \otimes \delta f"] & H_*(Z;\mathbb{Q}) \arrow[d, "\delta f"] \\ H_*(X;\mathbb{Q}) \otimes H_*(X;\mathbb{Q}) \arrow[r, "\delta m"] & H_*(X;\mathbb{Q}) \end{tikzcd} \tag{3.10}$$

commutes. But the commutativity of (3.9) asserts that f is an H-map; and the commutativity of (3.10) asserts that $f_* = \delta f$ is an algebra homomorphism.

We may now prove the following theorem (compare Theorem III.1.8 of [3]):

THEOREM 3.6. *Let $X \in NH$ and assume that, for each $p \in \Pi \cup \{0\}$, X_p is endowed with an H-structure $\mu(p)$. Let $\check{X}$ and $\check{X}_0$ be given the induced H-structures. Suppose that $\check{e}_0\colon X_0 \longrightarrow \check{X}_0$ induces an algebra homomorphism*

$$\check{e}_{0*}\colon H_*(X_0, \mathbb{Q}) \longrightarrow H_*(\check{X}_0;\mathbb{Q}).$$

Then X admits an H-structure μ such that the p-localization maps $e_p\colon X \longrightarrow X_p$ are H-maps.

PROOF. In the diagram (3.1) r is an H-map by construction and $\check{e}_0$ is an H-map by Corollary 3.5. Since (3.1) is a weak pullback in H, it follows that X admits an H-structure μ such that $r\colon X \longrightarrow X_0$ and $\check{e}\colon X \longrightarrow \check{X}$ are H-maps. Thus the theorem is proved.

It is easy to see that the condition that $\check{e}_{0*}$ be an algebra map implies that

$$r_{p*}\colon H_*(X_p;\mathbb{Q}) \longrightarrow H_*(X_0;\mathbb{Q}),$$

where $r_p\colon X_p \longrightarrow X_0$ is the canonical map, is an *algebra* isomorphism for each p; that is, in the light of Corollary 3.5, that $r_p\colon X_p \longrightarrow X_0$ is an H-map for every p. This condition is also manifestly equivalent to the requirement that the algebra structure on $H_*(X;\mathbb{Q})$ induced by $e_p\colon X \longrightarrow X_p$ from that on $H_*(X_p;\mathbb{Q})$ is independent of p. Conversely, we may prove the following

PROPOSITION 3.7. *Let $X \in NH$ and assume that, for each $p \in \Pi$, X_p is endowed with an H-structure $\mu(p)$. If the algebra structure on $H_*(X;\mathbb{Q})$ induced by $e_p\colon X \longrightarrow X_p$ from that on $H_*(X_p;\mathbb{Q})$ is independent of p, then X_0 may be given a unique H-structure $\mu(0)$ such that $r_p\colon X_p \longrightarrow X_0$ is an H-map for each p. Further $\check{e}_0\colon X_0 \longrightarrow \check{X}_0$ is an H-map for the induced H-structure on $\check{X}_0$, provided that, for each i, the group $\pi_i X$ has p-torsion for only finitely many primes.*

PROOF. By hypothesis, we get a unique algebra structure on $H_*(X_0;\mathbb{Q})$ such that r_{p*} is an algebra isomorphism for each p. But this algebra structure is induced by the H-structure on X_0 making r_p an H-map, so that this H-structure is, by Lemma 3.4, independent of p.

The final assertion of the Proposition is proved exactly as in the proof of the corresponding part of Theorem III.1.8 of [3] (note that, in [3], our map $\check{e}_0$ is called ρ); however, we now apply Theorem I.3.8 of [3] in its strong form, since we are no longer assuming X to be of finite type.

Theorem 3.6 and Proposition 3.7 together provide the improvement of the first assertion of Theorem III.1.8 of [3] promised in the Introduction.

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