Existence of Continuous Extension of Homeomorphisms in Uniform Spaces *

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Introduction. Some functions, like $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ given by $f(x) = tan\, x$, where $\mathbb{R}$ is the set of real numbers equipped with the usual metric topology and the domain is a subspace of $\mathbb{R}$, are homeomorphisms but cannot be extended continuously to the adherence of their domains. Nevertheless this extension is possible for other homeomorphisms, like $f: [0, +\infty) \longrightarrow [0, +\infty)$ given by $f(x) = x^2$, where the image is equipped with the usual metric topology and the domain is a subspace of $\mathbb{R}$ equipped with the topology of the pseudo-metric defined by: $d(x, y) = |x - y|$, if $x \geq 0$ and $y \geq 0$; $d(x, y) = 0$, if $x < 0$ and $y < 0$; $d(x, y) = x$, if $x \geq 0$ and $y < 0$; $d(x, y) = y$, if $x < 0$ and $y \geq 0$; a continuous extension of f to the adherence of $[0, +\infty)$, that is, to $\mathbb{R}$ is $g: \mathbb{R} \longrightarrow [0, +\infty)$ where $g(x) = 0$, for any $x < 0$.

The purpose of this note is to study necessary and sufficient conditions for the existence of continuous extension of homeomorphisms $f: A \longrightarrow F$ in uniform spaces, not necessarily uniform isomorphisms, where F is a complete locally compact Hausdorff uniform space and A is a subspace of a uniform space; and the result is presented in the theorem of **6.** in terms of uniform continuity of restrictions of f on subsets of a special class of subsets of A.

The preliminary propositions are established under weaker assumptions; among them we mention the proposition of 2 which gives a sufficient condition for the existence of a continuous extension of a mapping not necessarily uniformly continuous.

1. Definition. Let E be a topological space and let A and B be subsets of E such that $\overline{A} \supset B \supset A \neq \emptyset$. A family $\mathcal{G}$ of subsets of E is called a *cover of first kind of A adherent to B* if: (a) $G \neq \emptyset$ for any $G \in \mathcal{G}$; (b)

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$A = \cup\{G\colon G\in\mathscr{G}\}$; (c) is a partially ordered set directed to the right by $\subset$; (d) for any $x\in B$, there exists some $G\in\mathscr{G}$ such that $\overline{G}\cap B$ is a neighborhood of x in B.

If (c) is changed by: $\mathscr{G}$ is closed under the operation of taking union of a finite non-empty collection of its members, $\mathscr{G}$ is called a *cover of second kind of A adherent to B.*

Notice that if $\mathscr{G}$ is a cover of either first or second kind of A adherent to $\overline{A}$, (d) can be written as follows: for any $x\in\overline{A}$, there exists some $G\in\mathscr{G}$ such that $\overline{G}$ is a neighborhood of x in $\overline{A}$. Clearly, every cover of second kind is a cover of first kind.

2. Proposition. *Let E be a uniform space; A and B subspaces of E such that $\overline{A}\supset B\supset A\neq\varnothing$; F a complete Hausdorff uniform space; $f: A\longrightarrow F$ a mapping; and $\mathscr{G}$ a cover of first kind of A adherent to B such that $f\,|\,G$ is uniformly continuous for any $G\in\mathscr{G}$. Then there exists a unique continuous extension of f to B.*

PROOF. Since $f\,|\,G$ is uniformly continuous for any $G\in\mathscr{G}$, there exists a unique uniformly continuous extension f_G of $f\,|\,G$ to $\overline{G}$.

Let $g\colon B\longrightarrow F$ be the mapping defined in the following way: for any $x\in B$, there exists, by (d), some $G\in\mathscr{G}$ such that $x\in\overline{G}$ and let be set $g(x)=f_G(x)$. This mapping is well defined because if $x\in\overline{G}\cap\overline{H}$ with G, $H\in\mathscr{G}$, there exists, by (c), some $K\in\mathscr{G}$ such that $G\cup H\subset K$, then $f_K|\overline{G}$ is uniformly continuous and $(f_K|\overline{G})\,|\,G=f_K|G=(f_K|K)\,|\,G=(f\,|\,K)\,|\,G=f|G$; hence, by the uniqueness of the uniformly continuous extension, $f_K|\overline{G}=f_G$ and $f_G|\overline{G}\cap\overline{H}=(f_K|\overline{G})\,|\,\overline{G}\cap\overline{H}=f_K|\overline{G}\cap\overline{H}=(f_K|\overline{H})\,|\,\overline{G}\cap\overline{H}=f_H|\overline{G}\cap\overline{H}$.

The mapping g is continuous. Indeed, consider any $x\in B$ and any neighborhood V of $g(x)$ in F. Then there exists, by (d), $G\in\mathscr{G}$ such that $\overline{G}\cap B$ is a neighborhood of x in B; but $g(x)=f_G(x)$, therefore, V is also a neighborhood of $f_G(x)$ in F and, since f_G is continuous, there exists a neighborhood W of x in $\overline{G}$ such that for any $y\in W$, $f_G(y)\in V$ holds; in particular, for any $y\in W\cap B$, it follows that $g(y)\in V$. Let S and T be neighborhoods of x in E such that $W=S\cap\overline{G}$ and $\overline{G}\cap B=T\cap B$, but $W\cap B=S\cap\overline{G}\cap B=S\cap T\cap B$, hence $W\cap B$ is a neighborhood of x in B such that for any $y\in W\cap B$, $g(y)\in V$ holds.

The mapping g is an extension of f. In fact, for any $x\in A$, (b) implies that there exists $G\in\mathscr{G}$ such that $x\in G$, hence

$$g(x)=f_G(x)=(f_G\,|\,G)\,(x)=(f\,|\,G)\,(x)=f(x).$$

3. Example. Notice that f and consequently g are not necessarily uniformly continuous. Indeed, let $E=F=\mathbb{R}$ be the set of real numbers equipped with the usual metric topology; $A=\mathbb{Q}$, the set of rational numbers; $B=E$; $f\colon A\longrightarrow F$ given by $f(x)=x^2$ for any $x\in A$; and $\mathscr{G}$ the family of all subsets $G=(x-r,x+r)\cap\mathbb{Q}$ with $x\in\mathbb{R}$ and $r>0$. Then E is a uniform space; F is a complete Hausdorff uniform space; $A\neq\varnothing$; $B=\overline{A}=E$; and $\mathscr{G}$ is a cover of first kind of A adherent to B such that $f\,|\,G$ is uniformly continuous for any $G\in\mathscr{G}$. Hence, by the above proposition, there exists a unique continuous extension g of f to E.

4. Corollary. *Let E be a uniform space and let A and B be subspaces of E such that $\overline{A}\supset B\supset A\neq\varnothing$, being A a complete Hausdorff subspace. Then there exists a unique retraction of B in A.*

PROOF. It is sufficient to put, in the proposition of 2, $F=A$, $f=$ the identity mapping on A and $\mathscr{G}=\{A\}$; hence there exists a unique continuous extension g of f to B. Then g is the unique retraction of B in A.

5. Proposition. *Let E be a uniform space; A and B subspaces of E such that $\overline{A}\supset B\supset A\neq\varnothing$; F a locally compact uniform space; $f\colon A\longrightarrow F$ a homeomorphism; and g a continuous extension of f to B. Then there exists a cover of second kind $\mathscr{G}$ of A adherent to B such that $f\,|\,G$ is uniformly continuous for any $G\in\mathscr{G}$.*

PROOF. For any $x\in B$, let us choose a compact neighborhood V_x of $g(x)$. Let $\mathscr{G}$ be the family of subsets of E defined by

$$\mathscr{G}=\{G: G\text{ is a union of a finite non-empty collection of subsets }f^{-1}(V_x)\}.$$

The family $\mathscr{G}$ satisfies the conditions for a cover of second kind: (c) is immediate by the definition of $\mathscr{G}$; for (b), since $G \subset A$ for any $G \in \mathscr{G}$, it is sufficient to prove that $A \subset \cup \{G : G \in \mathscr{G}\}$, indeed if $x \in A$ then $g(x) \in V_x$ and $f(x) \in V_x$, hence $x \in f^{-1}(V_x)$; (d) is valid because, for any $x \in B$, $g^{-1}(V_x)$ is a neighborhood of x in B, hence $\overline{g^{-1}(V_x) \cap A} \cap B$ is a neighborhood of x in B, but $f^{-1}(V_x) = g^{-1}(V_x) \cap A$, then $\overline{f^{-1}(V_x)} \cap B$ is a neighborhood of x in B; (a) holds because $\overline{f^{-1}(V_x)} \cap B \neq \emptyset$.

The uniform continuity of $f \mid G$, for any $G \in \mathscr{G}$, follows from the facts that f^{-1} is continuous, $f^{-1}(V_x)$ is compact, a finite union of compact subsets is compact and the restriction of a continuous mapping to a compact subset is uniformly continuous in uniform spaces.

6. Theorem. *Let E be a uniform space; A and B subspaces of E such that $\overline{A} \supset B \supset A \neq \emptyset$; F a complete locally compact Hausdorff uniform space; and $f\colon A \longrightarrow F$ a homeomorphism. Then there exists a unique continuous extension of f to B if and only if there exists a cover of first kind $\mathscr{G}$ of A adherent to B such that $f \mid G$ is uniformly continuous for any $G \in \mathscr{G}$.*

PROOF. Sufficiency: Apply the proposition of 2. Necessity: Apply the proposition of 5.

7. Remark. The theorem of **6**. remains valid if $\mathscr{G}$ is a cover of second kind.

Reference

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