Smooth Dependence of Solutions of Differential Equations on Initial Data: A Simple Proof*

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ABSTRACT. A simple proof of smooth dependence of solutions of ordinary differential equations, with respect to initial conditions, is given.

The proof uses the Fibre Contraction Theorem.

A proof of the following theorem can be found in [1, 2].

1. FIBRE CONTRACTION THEOREM. *Let (X, d) and (X', d') be complete metric spaces and let $\hat{F}: X \times X' \longrightarrow X \times X'$ be a map of the form*

$$\hat{F}(x, x') = (F(x),\ F'(x, x')).$$

Assume that

a) $F: X \longrightarrow X$ has an attracting fixed point p, that is:

$$F(p) = p, \text{ and } \lim_{n\to\infty} F^n(x) = p, \text{ for every } x \in X.$$

b) The map $x \longrightarrow F'(x, x')$ is continous in X, for every $x' \in X'$.

c) For every $x \in X$ the map $F'_x : X' \longrightarrow X'$ defined by $F'_x(x') = F'(x, x')$ is a λ-contraction, with $\lambda < 1$.

This means that

$$d'(F'_x(x'),\ F'_x(y')) \leq \lambda d'(x', y')$$

for all $x \in X$ and $x', y' \in X'$.

Then if p' denotes the unique attracting fixed point of F'_p, the point $\hat{p} = (p, p') \in X \times X'$ is an attracting fixed point of $\hat{F}$.

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REMARKS

1) Condition a) above is satisfied if, for instance, F is a λ-contraction with $\lambda < 1$. This is the well known shrinking Lemma. The proof of Theorem 1 is essentially elementary, although slightly technical.

2) In [1, 2] Theorem 1 was used to prove the smoothness of the invariant manifolds associated to hyperbolic fixed points.

2. THEOREM ON SMOOTH DEPENDENCE ON INITIAL CONDITIONS. *Let f be a map of class C^1 (continuous with continuous first partial derivatives) in an open set $A \subset \mathbb{R}^n$, with values in $\mathbb{R}^n$.*

For each point $x_0 \in A$ there are associated positive numbers α, β and a unique map ϕ of class C^1 in

$$I_\alpha \times B_\beta = \{(t, x);\ |t| < \alpha, |x - x_0| < \beta\},$$

with values in A, such that

$$(\sigma) \qquad \frac{\partial \phi}{\partial t}(t, x) = f(\phi(t, x)), \qquad \phi(0, x) = x$$

for all $(t, x) \in I_\alpha \times B_\beta$.

PROOF. Let $b > 0$ be such that $\overline{B_b} = \{x;\ |x - x_0| \leq b\} \subset A$ and let $m = sup\,|f(x)|$, $l = sup\,\|Df(x)\|$, for $x \in \overline{B_b}$.

Take α and β such that $\alpha m + \beta < b$ and $\lambda = l\alpha < 1$. Denote by X the space of bounded continuous maps of $I_\alpha \times B_\beta$, endowed with the metric

$$d(\phi, \gamma) = sup\{|\phi(t, x) - \Psi(t, x)|;\ (t, x) \in I_\alpha \times B_\beta\}.$$

Denote by $\mathcal{L}$ the space of linear endomorphisms of $\mathbb{R}^n$, endowed with the norm $\|L\| = \{sup\,|L|;\ |x| = 1\}$. Let X' be the vector space of bounded continuous maps of $I_\alpha \times B_\beta$ in $\mathcal{L}$, endowed with the metric

$$d'(\phi', \Psi') = sup\{\|\phi'(t, x) - \Psi'(t, x)\|;\ (t, x) \in I_\alpha \times B_\beta\}.$$

Define F: $X \longrightarrow X$ by

$$F(\phi)(t, x) = x + \int_0^t f(\phi(s, x))\, ds,$$

and F': $X \times X' \longrightarrow X'$ by

$$F'(\phi, \phi')(t, x) = E + \int_0^t Df(\phi(s, x))\, \phi'(s, x)\, ds,$$

where E is the identiy element of $\mathcal{L}$.

The map $\hat{F} = (F, F')$ satisfies the hypothesis of Theorem 1. In fact:

a) F is a λ-contraction, since, by the mean value Theorem:

$$d(F(\phi), F(\Psi)) = sup\left|\int_0^t |f(\phi(s, x)) - f(\Psi(s, x))|\, ds\right|$$

$$\leq sup\left|\int_0^t l\,|\phi(s, x) - \Psi(s, x)|\, ds\right| < \alpha l d(\phi, \Psi) = \lambda\, d(\phi, \Psi).$$

Hence F has a (unique) attracting fixed point $\gamma \in X$.

b) Is immediate since Df is uniformly continuous in $\overline{B}_b$.

c) $d'(F'_\phi(\phi') \cdot F'_\phi(\Psi')) = sup\left\|\int_0^t Df(\phi(s, x))\,|\phi'(s, x) - \Psi'(s, x)|\, ds\right\| \leq \lambda d'(\phi', \Psi')$.

The (unique) attracting fixed point of $\hat{F}$ is of the form $\hat{\phi} = (\phi, \phi')$, where $F(\phi) = \phi$ and $F'(\phi') = \phi'$. Relation (σ) is obtained differentiating with respect to t both membrs of $F(\phi) = \phi$.

Continuity of ϕ is $I_\alpha \times B_\beta$ is immediate since $\phi \in X$. To prove that ϕ is of class C^1 it is enough to verify that $D_2\phi$ is equal to ϕ' which is continuous in $I_\alpha \times B_\beta$, since $D_1\phi = f \circ \phi$ is continuous in $I_\alpha \times B_\beta$. The sequence $(\phi_n, \phi'_n) = \hat{F}^n(\phi_0, \phi'_0)$, where $\phi_0(t, x) = x$ and $\phi' \equiv E$, satisfies $\phi_n \longrightarrow \phi$ and $\phi'_n \longrightarrow \phi'$, uniformly in $I_\alpha \times B_\beta$; moreover, every ϕ_n is of class C^1 and, for every n, $D_2\phi_n = \phi'_n$. This follows by induction.

Therefore, since $\phi'_n = D_2\phi_n$ is continuous because it is an element of X', it follows, by the theorem on the exchange of the order of taking uniform limits and differentiating, that $D_2\phi$ exists and is equal to ϕ'. This ends the proof.

REMARKS

1) The same arguments in the proof above lead to the somoothness of solutions of "non autonomous" differential equations $x' = f(t, x)$, $x(t_0) = x$, where f is continuous with $D_2 f$ continuous in an open set of the (t, x)-space.

2) Other classical theorems of Analysis, like the *Inverse Function Theorem*, can be proved using Theorem 1. We outline such proof in what follows. The interested reader can fill in details.

If g is of class C^1 in an open set of $\mathbb{R}^n$ with $L = Dg(x_0)$ non singular, it can be assumed, by translating x_0 and $g(x_0)$ to $0 \in \mathbb{R}^n$ and composing with L^{-1}, that g is of the form $y = g(x) = x + \Delta(x)$, with $\Delta(0) = 0$ and $D\Delta(0) = 0$. Let B_δ be a ball centered at 0 with radius δ, where $\|D\Delta\| \leq \frac{1}{2}$. Call X the space of continuous maps γ of $B_{\delta/2}$ with values in $\overline{B_{\delta/2}}$, and let X' the space of bounded continuous maps γ' from $B_{\delta/2}$ to $\mathscr{L}$, the space of linear endomorphisms of $\mathbb{R}^n$. Apply Theorem 1 to the map $\hat{F} = (F, F')$, where $F\colon X \longrightarrow X$ is given by

$$F(\gamma)(y) = -\Delta(y + \gamma(y)),$$

and $F'\colon X \longrightarrow X'$ is defined by

$$F(\gamma, \gamma')(y) = -D\Delta(y + \gamma(y)) \circ [E + \gamma'(y)],$$

and obtain the existence of the inverse of g, g^{-1}, in the form $x = y + \gamma(y)$, with $\gamma \in X$, and its differentiability. Actually, going backwards, this form of g^{-1} motivates the definition of F, since it follows, by substitution in g, that

$$y + \gamma(y) + \Delta(y + \gamma(y)) = y, \quad \text{and} \quad \gamma(y) = -\Delta(y + \gamma(y)).$$

Also, the definition of F' is obtained heuristically differentiating this last equation.

REFERENCES

[1] HIRSH, PUGH, *Stable manifolds for hyperbolic sets*, Proc. Symp. in Pure Math., vol. XIV, AMS, 1970.

[2] I. C. DE OLIVEIRA, Variedades Invariantes de Pontos Fixos Hiperbólicos, São Paulo, 1971.

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