

Isosceles triangles in Riemannian geometry – a characterization of the n -sphere

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Abstract. New characterization theorems for the linear Euclidean spaces and Euclidean spheres are given in terms of convexity and isosceles triangles in Riemannian manifolds.

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1 Introduction

In this work we give a classification of the linear Euclidean spaces and Euclidean spheres by using essentially the notion of convexity and isosceles triangles in Riemannian manifolds. The results obtained here generalize those ones obtained by the author in his doctoral thesis (see [6]) and published in [5]. In order to describe the main result we need first some notations and definitions.

In what follows, we will denote by $M = M^n$ a complete and connected Riemannian manifold with dimension $n \geq 2$. Let g be a geodesic in M such that $g(t_0) = p$ and $g(t_1) = q$, where $t_0 < t_1$. We will represent the segment $g([t_0, t_1])$ of g connecting the points p and q by $[p, q]_g$; If $g(t') = p'$ and $t_0 < t' < t_1$, we will say that p' occurs after p and before q along g .

In this work we will also assume every geodesic to be parameterized by the arc length and the word **segment** will mean a geodesic segment. The length of the segment $[p, q]_g$ will be denoted by $l([p, q]_g)$.

Three geodesic segments $[p, q]_\gamma$, $[q, r]_\sigma$ and $[r, p]_g$ connecting distinct points p , q and r in M make a figure that we call a geodesic triangle which will be simply represented by $\{[p, q]_\gamma [q, r]_\sigma [r, p]_g\}$. We say that a geodesic triangle is simple when the set made up of the union of their sides is a curve homeomorphic

to the circle S^1 or when their vertices belong to a unique segment without self-intersections.

A geodesic triangle is isosceles when it has two sides with same length. The third side, which has eventually different length, is called the **basis** of the isosceles triangle. We note that if r is the middle point of the segment $[p, q]_g$ then $\{[p, r]_g, [r, q]_g, [p, q]_g\}$ is an isosceles triangle whose basis is $[p, q]_g$.

We say that a subset $K \subset M$ is **strongly convex** if for every pair of points p and q in the closure \bar{K} there exists a unique minimal segment $[p, q]$ joining p to q and the interior of the segment $[p, q]$ is a subset of K . If K is strongly convex and its interior is non-empty, we say that K is a **strongly convex body**.

Let p be a point of M e let g be a geodesic passing through p . We reparameterize g such that $p = g(0)$. Let us consider the set $C_g(p) = \{t \in [0, \infty); d(p, g(t)) = t\}$. It is clear that $C_g(p)$ can be $[0, \infty)$ or $[0, t_0]$ for some $t_0 > 0$. If the latter case occurs we say that $g(t_0)$ is the cut point of g with respect to p . The set made up of all cut points with respect to p is called the **cut locus** of p and is represented by $C(p)$. A geodesic $g : \mathbb{R} \rightarrow M$ is said to be a **line** when g minimize the distance (arc length) between any two of its points. If the restriction of a geodesic g to the interval $[0, \infty)$ has the property of minimizing the distance between any two of its points we call this restriction of **geodesic ray**. We refer to [7] and [2] for details.

Definition 1. *Let g and γ geodesics of M parameterized by the arc length and having a common point $p \in M$. Without lost of generality we can assume $g(0) = p = \gamma(0)$ and the angle between the geodesics being the angle θ between the tangent vectors $g'(0)$ and $\gamma'(0)$.*

*The figure made up of the geodesic g and the geodesic segment of γ linking the point p to a point $q = \gamma(t)$ with $t > 0$ is called a **configuration**. If θ is the angle between g and γ in the point p then the configuration is represented by $\{g, \gamma, \theta\}_p$.*

First Isosceles Triangle Axiom – FITA

For every configuration $\{g, \gamma, \theta\}_p$ such that $0 \leq \theta < \frac{\pi}{2}$ and for every point $q = \gamma(s_0)$ with $s_0 > 0$, there exists a unique point $r = g(t_0)$ with $t_0 > 0$ and a unique geodesic segment $[q, r]_\sigma$ linking the point q to the point r in such a way that $\{[p, q]_\gamma, [q, r]_\sigma, [r, p]_g\}$ is the unique isosceles triangle whose basis is $[p, q]_\gamma$.

Remark 1. In this case it is easy to see that the segment $[q, r]_\sigma$ is minimal.

Second Isosceles Triangle Axiom – SITA

For every configuration $\{g, \gamma, \theta\}_p$ and for each point $q = \gamma(s) \neq p = g(0) = \gamma(0)$ there exist only two real numbers t_1 and t_2 with $t_2 < 0 < t_1$ such that the points $r_1 = g(t_1)$ and $r_2 = g(t_2)$ determine the segments $[q, r_1]_\sigma$ and $[q, r_2]_\tau$ in such a way that the triangles $\{[p, q]_\gamma [q, r_1]_\sigma [r_1, p]_g\}$ and $\{[p, q]_\gamma [q, r_2]_\tau [r_2, p]_g\}$ are isosceles triangles whose common basis is $[p, q]_\gamma$.

Remark 2. In the case of SITA the angle θ can be given arbitrarily, thus we our notation for a configuration will dismiss the angle θ , that is $\{g, \gamma\}_p$.

Theorem 1. *If M satisfies the first isosceles triangle axiom then M is isometric to the Euclidean space \mathbb{R}^n .*

Theorem 2. *If M satisfies the second isosceles triangle axiom then M is isometric to the Euclidean sphere S^n .*

2 Proof of Theorem 1

Lemma 1. *If M satisfies FITA then every metric ball is strongly convex.*

Proof. Let us suppose by contradiction that there exists a point $p_0 \in M$ and a real number $\rho > 0$ such that the open ball $B = B_\rho(p_0)$ is not strongly convex. Then there are points m_1 and m_2 such that the segment $[m_1, m_2]$ of the geodesic $\gamma(t)$ joining the points m_1 and m_2 has points outside the closure \overline{B} of the set B . Let p and q be the points in \overline{B} where $\gamma(t)$ get in and get out respectively.

Consider the configuration $\{g, \gamma, \theta\}_p$ given by the geodesics g and σ which get out of the point p_0 and pass through p and q respectively. We consider them parameterized so that $g(-\rho) = p_0 = \sigma(-\rho)$, $g(0) = p$ and $\sigma(0) = q$. The angle between $[p, q]_\gamma$ and g is θ .

By the Gauss lemma we have $\theta < \frac{\pi}{2}$, thus by FITA there is a point $r = g(t)$ with $t > 0$ such that $\{[p, q]_\gamma, [p, r]_g, [r, q]_\tau\}$ is an isosceles triangle with basis $[p, q]_\gamma$. On the other hand, $\{[p, q]_\gamma, [p_0, p]_g, [p_0, q]_\sigma\}$ is also an isosceles triangle and this contradicts the FITA. □

Lemma 2. *If M satisfies FITA then every geodesic of M realizes the distance between every pair of its points.*

Proof. If two points p and q could be joined by two distinct geodesics, then the ball centered in p and not containing q would not be strongly convex. This would contradict Lemma (1) and so every geodesic is a line. \square

The following lemmas are immediate consequences of Lemma (1).

Lemma 3. *If M satisfies FITA then for every $p \in M$ the exponential map $\exp_p : T_p M \rightarrow M$ is a homeomorphism. This means that M is diffeomorphic to \mathbb{R}^n and in particular M is simply connected and so is orientable.*

Lemma 4. *If M satisfies FITA then every geodesic of M cannot lie inside any compact set.*

Lemma 5. *Let M satisfies FITA. If distinct metric spheres S_1 and S_2 of M are tangent to each other then the set $S_1 \cap S_2$ is unitary.*

Proof. We denote by c_1, c_2, r_1, r_2 respectively the centers and the rays of the spheres S_1 and S_2 . Let p denote the tangential point of these spheres. As S_1 and S_2 are distinct and tangent to each other then $c_1 \neq c_2$.

Let us suppose by contradiction that there exists a point $q \in S_1 \cap S_2$ with $q \neq p$. Then there are segments $[c_1, p]_a$ and $[c_1, q]_b$ which join the point c_1 to the points p and q ; Likewise, there are segments $[c_2, p]_\alpha$ and $[c_2, q]_\beta$ which join the point c_2 to the points p and q . Moreover, we have

$$l([c_1, p]_a) = r_1 = l([c_1, q]_b)$$

$$l([c_2, p]_\alpha) = r_2 = l([c_2, q]_\beta)$$

Let us suppose that S_1 and S_2 are outer spheres. As they are tangent at p we get $d(c_1, c_2) = r_1 + r_2$. Moreover, the segments $[c_1, p]_a$ and $[c_2, p]_\alpha$ yield a new segment which we will denote by $[c_1, c_2]_g$.

This new segment is not minimal. Indeed, we have $p \neq q$ and the segments $[c_1, q]_b$ and $[c_2, q]_\beta$ meet at q and yield a broken geodesic whose length is

$$l([c_1, q]_b) + l([c_2, q]_\beta) = r_1 + r_2 = l([c_1, c_2]_g).$$

Therefore, there is another geodesic joining c_1 to c_2 whose length is smaller than $l([c_1, c_2]_g)$ and this contradicts the Lemma (2).

Let us assume now that S_1 and S_2 are inner spheres. As they are tangent at p we have $d(c_1, c_2) = |r_2 - r_1|$. Without loss of generality we can assume $r_1 < r_2$, i.e. c_1 lies inside the ball whose boundary is S_2 .

Let $[p, q]_\gamma$ be the segment joining p to q . The balls having S_1 and S_2 as boundaries are strongly convex, thus the angle θ between $[c_1, c_2]_g$ and $[p, q]_\gamma$ is smaller than $\frac{\pi}{2}$ and consequently we have the configuration $\{g, \gamma, \theta\}_p$. In this case we have two distinct isosceles triangle $\{[c_1, p]_a, [c_1, q]_b, [p, q]_\gamma\}$ and $\{[c_2, p]_\alpha, [c_2, q]_\beta, [p, q]_\gamma\}$ whose basis is $[p, q]_\gamma$. This contradicts FITA. \square

The following two lemmas are immediate consequences.

Lemma 6. *If M satisfies FITA then the closure of a strongly convex body in M is strongly convex.*

Lemma 7. *Let M satisfies FITA. If H and K are strongly convex intersecting subsets of M then $H \cap K$ is also strongly convex.*

2.1 Proof of Theorem 1

By using that M satisfies FITA and Lemma (2) we have that all geodesic of M are lines and consequently for every point $p \in M$, the exponential map $exp_p : T_pM \rightarrow M$ is a diffeomorphism (Lemma (3)).

Given an arbitrary point $p \in M$ and a unit vector $v \in T_pM$, let us consider the sets:

$$L_p = \{w \in T_pM; \langle w, v \rangle = 0\},$$

$$L_p^+ = \{w \in T_pM; \langle w, v \rangle \geq 0\},$$

$$L_p^- = \{w \in T_pM; \langle w, v \rangle \leq 0\}.$$

These sets allow us to define the following subsets of M :

$$\Sigma = exp_p(L_p),$$

$$H^+ = exp_p(L_p^+),$$

$$H^- = exp_p(L_p^-).$$

We consider the geodesic ray $r(t) = exp_p(tv)$ starting at p in the direction v and $B_r = \cup_{t>0} B_t(r(t))$, where $B_t(r(t))$ is the open ball centered at the point $r(t)$ and radius t .

The Lemma (5) assures that if $t_1 < t_2$ then $B_{t_1}(r(t_1)) \subset B_{t_2}(r(t_2))$. Moreover, as for each t the set $B_t(r(t))$ is strongly convex (Lemma (1)) and $B_{t_1}(r(t_1)) \subset B_{t_2}(r(t_2))$ when $t_1 < t_2$, we have that B_r is strongly convex (c.f. [4]).

Let us denote by $\overline{B_r}$ the closure of B_r . We will prove that $\overline{B_r} = H^+$. Using that for each $t > 0$, $\overline{B_t(r(t))} \subset H^+$, by convexity and the equality

$$\overline{B_r} = \overline{\cup_{t>0} B_t(r(t))} = \cup_{t>0} \overline{B_t(r(t))},$$

we conclude that $\overline{B_r} \subset H^+$.

Let $q \in H^+$ be an arbitrary point and let q_n be a convergent sequence made up of interior points in H^+ such that $\lim q_n = q$. Let ρ_n be the geodesic segment connecting the points p and q_n . As q_n is an interior point then $\langle r'(0), \rho'_n(0) \rangle > 0$. The manifold M satisfies FITA so there exists $r_n = r(t_n)$ in such a way that the geodesic triangle whose vertices are the points p , q_n , and r_n is an isosceles triangle with basis ρ_n . Therefore

$$q_n \in \overline{B_{t_n}(r(t_n))} \subset \overline{B_r}.$$

As $\overline{B_r}$ is closed we have $q \in \overline{B_r}$. Thus, $H^+ \subset \overline{B_r}$. This way we have proved that $\overline{B_r} = H^+$.

According to Lemma (6) the set B_r is strongly convex and consequently H^+ is also strongly convex.

By using a similar construction with the radius $s(t) = \exp_p t(-v)$, we obtain that the set $H^- = \overline{B_s}$ is strongly convex.

According to the Lemma (7) the set $\Sigma = H^+ \cap H^-$ is strongly convex. Since \exp_p is a diffeomorphism we have that Σ is a complete submanifold of M without boundary with dimension $n - 1$. This means that Σ is a totally geodesic submanifold of M .

Let us assume that $n \geq 3$. Since the points p and q are given arbitrarily, the manifold M satisfies p axiom of r -planes, for $r = n - 1 \geq 2$. It follows from the r -planes Theorem due to Cartan (see [3]) that M has constant sectional curvature (see [9]). As M is not compact it can only be isometric to the Euclidean space \mathbb{R}^n or to the hyperbolic space H^n .

Since the set $\Sigma = \partial \overline{B_r}$ is a horosphere in M and M is a space form, then all sectional curvatures of Σ vanish ([11]). On the other hand, the set Σ is totally geodesic and consequently all sectional curvatures of M must vanish. Therefore M is isometric to \mathbb{R}^n .

Let us assume now that $n = 2$. Let g be a geodesic (a line) in M . We say that a geodesic γ is an asymptote at g passing through the point $q = \gamma(0)$ if there exists a sequence of minimal geodesics $\sigma_n : [0, s_n] \rightarrow M$ such that for every real value s , the sequence $\sigma_n(s)$ converges to the restriction of γ to the interval $[0, \infty)$ and we have $\sigma_n(s_n) = g(t_n)$ with $t_n \rightarrow \infty$.

When there exists another sequence $\tau_n : [0, s_n] \rightarrow M$ such that for every real value s , the sequence $\tau_n(s)$ converges to the restriction of $\gamma(s)$ to the interval

$(-\infty, 0]$ and we have $\tau_n(s_n) = g(t_n)$ with $t_n \rightarrow -\infty$, we say that $\gamma(s)$ is a bi-asymptote at g passing through the point q .

Let g be a geodesic of M and $p = g(t_1)$ and $q = g(t_2)$ points on the geodesic g . By constructing horospheres Σ_p and Σ_q starting from the geodesic g , we notice that they both meet g orthogonally. On the other hand as FITA is satisfied we can immediately conclude that Σ_q is a bi-asymptote at Σ_p .

Eschenburg proves that there is an isometric immersion $F : [t_1, t_2] \times \mathbb{R} \rightarrow M$ such that $\Sigma_p = F|_{t_1 \times \mathbb{R}}$ and $\Sigma_q = F|_{t_2 \times \mathbb{R}}$ (see [8]). This implies that the region of M limited by Σ_p and Σ_q has curvature zero. Since M is simply connected, the curve g is an arbitrary geodesic and the points $g(t_1)$ and $g(t_2)$ are also arbitrary, we conclude that $M = \mathbb{R}^2$.

3 Proof of Theorem 2

Lemma 8. *If the Riemannian manifold M satisfies SITA than it is compact.*

Proof. To show the compactness, it suffices to find a point p_0 of M such that every geodesic $g : [0, \infty) \rightarrow M$, leaving p_0 , has a cut point with respect to p_0 , see [7], p. 135. Let us fix a point $p_0 \in M$. Choose $\delta > 0$ such that the open ball $B_\delta(p_0)$ with center p_0 and radius δ is convex. Denote by $S_r(p_0)$ the geodesic sphere of center p_0 and radius r , for some $\delta > r > 0$.

Let g be a geodesic leaving p_0 and p the point where g first crosses $S_r(p_0)$, we reparameterize g in such a way that $g(0) = p$ and $g(r) = p_0$. Finally, we construct a configuration $\{g, \gamma\}_p$ by choosing a point $q \neq p$ in $S_r(p_0)$ and γ a geodesic arc within $B_\delta(p_0)$ joining $p = \gamma(0) = g(0)$ to $q = \gamma(s)$. SITA assures the existence of exactly two real numbers $t_2 < 0 < t_1$ such that $r_j = g(t_j)$ determine geodesic segments $[q, r_1]_\sigma$ and $[q, r_2]_\tau$ that are the sides of two simple isosceles triangles whose common basis is $[p, q]_\gamma$. By construction, $t_1 = r$, i.e., $p_0 = g(r) = g(t_1) = r_1$, since p and q belong to the geodesic sphere $S_r(p_0)$. Therefore,

$$l([p_0, r_2]_g) = l([p_0, q]_\sigma) + l([q, r_2]_\tau).$$

Hence, the geodesic g has a cut point $p' = g(t')$ with respect to p_0 , which concludes the proof. □

Lemma 9. *If a Riemannian manifold M satisfies SITA then it has no geodesic loop.*

Proof. Let us suppose there exists a geodesic loop in M , that is, there exists a geodesic $g : \mathbb{R} \rightarrow M$ and points t_0 and \bar{t}_0 in \mathbb{R} , with $t_0 \neq \bar{t}_0$ such that $g(t_0) = p = g(\bar{t}_0)$ and $g'(t_0) \neq g'(\bar{t}_0)$.

We consider a strongly convex ball $B_\delta(p)$. Let $p_i = g(t_i)$ with $i = 1, 2, 3, 4$ be the points in the boundary $\partial B_\delta(p)$ where g gets in and gets out and afterwards gets in and gets out of $B_\delta(p)$.

Let $q = g(\tilde{t})$ with $\tilde{t} < t_2$ be points obtained in such a way that $d(q, p_2) = d(p_2, p_3)$. Joining q to p_3 using the segment $[q, p_3]_\gamma$ we obtain the configuration $\{g, \gamma, \theta\}_q$. According to SITA there is a point $r = g(\hat{t})$ with $\hat{t} < \tilde{t}$, in such a way that the triangles $\{p_2, q, p_3\}$ and $\{r, q, p_3\}$ are isosceles triangles whose basis is the segment $[q, p_3]$. Now, we observe that considering the medium point \bar{p} of the segment $[q, p_3]_g$, the triangle $\{q, \bar{p}, p_3\}$ is also a geodesic triangle distinct of the other two. This contradicts the SITA. \square

Lemma 10. *If M is a Riemannian manifold satisfying SITA then every geodesic is closed.*

Proof. We will prove that every geodesic of M is closed by showing the existence of geodesics which are not closed lead us to a contradiction to the SITA. Let $c(p)$ denote the function which associates to every point $p \in M$ the convexity radius of M at the point p , that is $c(p)$ is the greatest number such that the ball $B_r(p)$ centered at p and having radius $r < c(p)$ is strongly convex. According to the Whitehead Theorem $c(p)$ is a continuous function on M , (see [13]). As M is compact and c is continuous there exists a number $\delta > 0$ such that for every $p \in M$, the ball $B_\delta(p)$ is strongly convex.

Let us consider the family of open sets $\{B_r(p)\}_{p \in M}$ where $2r < \delta$. As such a family covers M and M is compact, we can find a finite cover of M , say $\{B_r(p_1), \dots, B_r(p_k)\}$.

Let us assume that there exists geodesic g which is not closed. In this case, as M is complete, either g gets in and gets out twice in the same ball of the family $\{B_r(p_1), \dots, B_r(p_k)\}$, or else there exists $t_0 \in \mathbb{R}$ such that for every $t > t_0$ the geodesic g is contained within open balls of the family $\{B_r(p_1), \dots, B_r(p_k)\}$.

In the first case, let us suppose that the geodesic g gets in and gets out twice in the ball $B_r(p)$, as there not exist geodesic loops (Lemma (9)), we know there exist four distinct points which we will denote by $p_i = g(t_i)$ in the boundary $\partial B_r(p)$ where g gets in and gets out and this geodesic gets in and gets out in $B_r(p)$.

Let us consider a point $q = g(t)$ with $t < t_2$ in such a way that $d(q, p_2) = d(p_2, p_3)$. Joining q to p_3 we obtain an isosceles triangle $\{q, p_2, p_3\}$ whose basis is $[q, p_3]$. Using SITA there exists a point $\bar{p} = g(\bar{t})$ with $\bar{t} < t_0$ such that $\{\bar{p}, q, p_3\}$ is an isosceles triangle whose basis is $[q, p_3]$. On the other hand, there is a point $g(\hat{t}) = \hat{p}$ in the segment $[p_2, p_3]_g$ so that $d(p_2, \hat{p}) = d(\hat{p}, p_3)$. From this we conclude that the triangle $\{p_2, \hat{p}, p_3\}$ is also isosceles whose basis is $[q, p_3]$. This contradicts SITA.

In the second case, let us assume there is a number $t_0 \in \mathbb{R}$ such that for every $t > t_0$, $g(t)$ is contained within the balls $B_r(p_i)$. Let us fix a point $q = g(\hat{t})$ in $B_r(p_i)$ and let us consider the strongly convex ball $B_\delta(q)$ which contains the set $B_r(p_i)$. This means that g passes through the center of the strongly convex ball $B_\delta(q)$ and that for $t \geq \bar{t}$, the number $g(t)$ is the radius of the ball $B_\delta(q)$. This is not possible for this ball has radius $\delta < \infty$. □

3.1 Proof of Theorem 2

Let p_0 an arbitrary point in M and let g be an arbitrary geodesic starting at p_0 and parameterized by the arc length. Using Lemma 3 we have that g is a simple closed geodesic, therefore, there exists $l \in \mathbb{R}$ satisfying $g(2l) = p_0 = g(-2l)$.

The point $\bar{p}_0 = g(l)$ will be called the antipode point of p_0 with respect to the geodesic g . In order to simplify our notation, we will denote by $\bar{g}(t) = g(-t)$ the geodesic satisfying $\bar{g}(0) = p_0$ and $\bar{g}'(0) = -g'(0)$.

We denote by $p'_0 = g(t_0)$ the cut point of g with respect to p_0 . It is clear that p'_0 cannot occur after the point \bar{p}_0 because $l([p_0, \bar{p}_0]_g) = l([p_0, \bar{p}_0]_{\bar{g}})$.

We shall prove now that $p_0 = \bar{p}_0$. Let us suppose by contradiction that $p'_0 \neq \bar{p}_0$. This means that $l([p'_0, \bar{p}_0]_g) > 0$ and consequently we can choose a real number \hat{t} such that $t_0 < \hat{t} < l$ and the point $\hat{p} = g(\hat{t})$ occurs after p'_0 and before \bar{p}_0 . The fact that g does not minimize the distance from p_0 to \hat{p} implies the existence of a minimal geodesic γ joining p_0 to \hat{p} and satisfying $l([p_0, \hat{p}]_g) > l([p_0, \hat{p}]_\gamma)$. Moreover, if we denote by $[\hat{p}, p_0]_g$ the segment joining \hat{p} to p_0 and passing through the point \bar{p}_0 , we have $l([\hat{p}, p_0]_g) > l([p_0, \hat{p}]_\gamma)$.

In this case, we can find a point p''_0 on g obtained from the point \hat{p} in such a way that $l([\hat{p}, p''_0]_g) = l([p_0, \hat{p}]_\gamma)$. Let us denote by $[p''_0, p_0]_\lambda$ the segment of the geodesic λ joining p''_0 to p_0 and let us consider the configuration $\{g, \lambda\}_p$. By construction we have the isosceles triangles

$$\{[p''_0, p_0]_\lambda, [p_0, \hat{p}]_\gamma, [\hat{p}, p_0]_g\}, \{[p''_0, p_0]_\lambda, [p_0, \check{p}]_g, [\check{p}, p''_0]_g\}$$

and their base is the segment $[p''_0, p_0]_\lambda$. Moreover, we also have the isosceles triangle $\{[p''_0, p_0]_\lambda, [p_0, \check{p}]_g, [\check{p}, p''_0]_g\}$ where \check{p} is the middle point of the segment

$[p_0, p_0'']_g$, joining the points p_0, p_0'' and passing through the point \bar{p}_0 , which contradicts the SITA. Therefore we have $p_0' = \bar{p}_0$.

Let us now consider a strongly convex ball $B_r(p_0)$ chosen so that the set $\overline{B_r(p_0)}$ be also strongly convex. We will denote by $\Sigma = \partial B_r(p_0)$ the boundary of $\overline{B_r(p_0)}$ and let us consider the points $p = g(r)$ and $\bar{p} = g(-r) = \bar{g}(r)$ where the geodesic g meets Σ .

We fix a point $q \in \Sigma$ given arbitrarily and different from the points p and \bar{p} ; We also consider the configuration $\{g, \gamma\}_p$ where γ is the geodesic joining p to q . According to the SITA, there exist segments $[p_0, q]_\sigma$ and $[\bar{p}_0, q]_\tau$ such that the triangles $\{[p_0, p]_g, [p_0, q]_\sigma, [p, q]_\gamma\}$ and $\{[p, \bar{p}_0]_g, [\bar{p}_0, q]_\tau, [p, q]_\gamma\}$ are simple and isosceles.

The fact

$$l([p_0, \bar{p}_0]_g) = l([p_0, p]_g) + l([p, \bar{p}_0]_g) = l([p_0, q]_\sigma) + l([q, \bar{p}_0]_\tau)$$

implies that $[p_0, q]_\sigma$ and $[q, \bar{p}_0]_\tau$ are segments of the same geodesic joining p_0 to \bar{p}_0 and passing through the point q , which we denote by ϕ . Besides, if $l([p_0, \bar{p}_0]_g) = l([p_0, \bar{p}_0]_\phi)$ and \bar{p}_0 is not the cut point of ϕ with respect to p_0 there would exist a geodesic segment with length smaller than the length of $[p_0, \bar{p}_0]_g$ joining p_0 to \bar{p}_0 and this contradicts what we have shown before.

Using the fact that the point q was taken arbitrarily we can conclude that \bar{p}_0 is the cut point of all geodesics passing through p_0 . Thus the cut locus $C(p_0)$ of p_0 is the set $\{\bar{p}_0\}$ and since p_0 is arbitrary we have that for every point p , the cut locus $C(p)$ is a unitary set and therefore M is a wiedersehen manifold.

If $n = 2$, the result follows from Green Theorem (see [10]) which says that M is isometric to the Euclidean sphere S^2 . If $n > 2$ and is an odd number, the result follows from the Yang Theorem (see [1]) which says that M is isometric to the Euclidean sphere S^n . If $n > 2$ and is an even number, the result follows from the Kazdan Theorem (see [12]) which says that M is isometric to the Euclidean sphere S^n .

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