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Floquet torus for codimension-s Hopf bifurcation

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Abstract. Here we show the existence of Floquet invariant torus for a codimension s Hopf bifurcation. As a corollary is obtained the existence of Floquet invariant torus for generic perturbations of product of quadratic maps.

Keywords: Hopf, bifurcation, invariant torus, normally hyperbolic.

Mathematical subject classification: Primary: 37G10; Secondary: 34K14, 37D30.

1 Introduction

In this paper we consider a C^∞ family of mappings $f_{\boldsymbol{\mu}}\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$, with $n = 2s$ and such that for $\boldsymbol{\mu}_0 \in \mathbb{R}^p$ has a fixed point $\mathbf{x}_0$ i.e., $f_{\boldsymbol{\mu}_0}(\mathbf{x}_0) = \mathbf{x}_0$ with all of the eigenvalues of $D_{\mathbf{x}}(f_{\boldsymbol{\mu}_0})(\mathbf{x}_0)$ belonging to the unit circle. Let $\lambda_1, \dots, \lambda_s$, $\overline{\lambda}_1, \dots, \lambda_s, \overline{\lambda}_s$, be these eigenvalues. If the mapping $\boldsymbol{\mu} \mapsto (|\lambda_1(\boldsymbol{\mu})|, \dots, |\lambda_s|)$ is a submersion we can take the $|\lambda_i|$ as the first s-parameters. In [M] it was proved that, under generic conditions on the family $f_{\boldsymbol{\mu}}$, there exist open sets of parameters with $\boldsymbol{\mu}_0$ in the closure of these sets and such that for parameters $\boldsymbol{\mu}$ in these sets the map $f_{\boldsymbol{\mu}}$ exhibits invariant curves with irrational rotation number (even more, up to s of such an invariant curves can be exhibited). That result was used in order to show that certain high-codimension homoclinic bifurcation when unfolded exhibits invariant curves.

For the particular family of mappings $h_{(a,\mathbf{b})}(x_1, x_2, x_3, x_4) = (a - x_4^2 + \sum_1^3 b_i x_i, x_1, \dots, x_3)$, where the parameters $(a, \mathbf{b}) = (a, b_1, b_2, b_3) \in \mathbb{R}^4$, a numerical experiment, shows that for the case $n = 4$ and the value of the parameters $a = 0.745$, $b_1 = 0.01$, $b_2 = -0.01$, and $b_3 = 0.01$ the mapping exhibits a set which seems to be an invariant torus of dimension two and not simply an invariant curve. In Figure 1 we plot a picture made with Mathematica of this set.

This fact suggests that we could try to find invariant tori of higher dimensions, so here we address the following problem: There exist invariant tori

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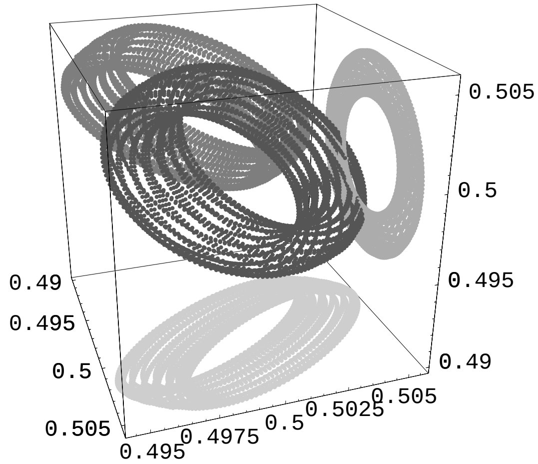

Figure 1: To plot this set we take the point $x_1 = .49$, $x_2 = .5$, $x_3 = 0.4995$ and $x_4 = .49$, and the piece of orbit $\{h^n_{(a,\mathbf{b})}(x_1, \dots, x_4)\colon n = 85000, \dots, 90000\}$. The plot shows the coordinates $x_1, x_2,$ and x_3.

for families $f_{\boldsymbol{\mu}}$ as above near the parameter $\boldsymbol{\mu}_0$? An invariant torus for a C^∞ map $h\colon \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a set V, which is invariant under h ($h(V) = V$) and C^k-diffeomorphic to $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$, the s-dimensional torus. The similar high dimensional notion of having rotational dynamics is the quasi-periodicity property and the most simple behavior of the dynamics in the normal direction of an invariant torus is the Floquet property. So we can ask even more, are these invariant tori of Floquet type with quasi-periodic dynamics? Here we say that the dynamics on the torus is *parallel* if it is conjugated to a rotation $R_{\boldsymbol{\phi}}\colon \boldsymbol{\theta} \in \mathbb{T}^s \mapsto \boldsymbol{\theta} + \boldsymbol{\phi} \in \mathbb{T}^s$. It is *quasi-periodic* if the components of $\boldsymbol{\phi}$ are rationally independent over $\mathbb{Z}^s$. A torus V with parallel dynamics is of *Floquet type* if there can be found coordinates $(\boldsymbol{\theta}, \mathbf{r})$ such that h can be written in a neighborhood of V as $h(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\theta} + \boldsymbol{\phi} + O(\mathbf{r}), \Lambda\mathbf{r} + O(\mathbf{r}))$, with Λ, a matrix of size $n - s$ does not depend on θ neither $\mathbf{r}$.

Theorem 1. *Let $f_{\boldsymbol{\mu}}\colon U \to \mathbb{R}^n$ be a family of C^∞-diffeomorphisms with $n = 2s$. Suppose that $f_{\boldsymbol{\mu}}$ satisfies:*

1. $f_{\boldsymbol{\mu}_0}(\mathbf{x}_0) = \mathbf{x}_0$ *;*

2. $Df_{\boldsymbol{\mu}_0}(\mathbf{x}_0)$ *has n complex eigenvalues* $\lambda_1(\boldsymbol{\mu}), \dots, \lambda_s(\boldsymbol{\mu}), \lambda_{s+1}(\boldsymbol{\mu}) = \overline{\lambda_1(\boldsymbol{\mu})}$, $\dots, \lambda_{2s}(\boldsymbol{\mu}) = \overline{\lambda_s(\boldsymbol{\mu})}$ *such that* $|\lambda_j(\boldsymbol{\mu})| = 1$ *for* $\boldsymbol{\mu} = \boldsymbol{\mu}_0$*;*

3. *If* $\lambda_j(\boldsymbol{\mu}) = |\lambda_j(\boldsymbol{\mu})| \exp(i\phi_j(\boldsymbol{\mu}))$*, then the map*

$$\boldsymbol{\mu} \mapsto (|\lambda_1(\boldsymbol{\mu})|, \dots, |\lambda_s(\boldsymbol{\mu})|, \phi_1(\boldsymbol{\mu}), \dots, \phi_1(\boldsymbol{\mu}))$$

is a submersion.

As well as we assume that the derivatives up to third order of $f_{\boldsymbol{\mu}}$ *accomplish generic conditions, then*

(i) *for each* $k \geq 1$*, there exists an open set* $\mathcal{U}_k$*, in the space the parameters, such that* $\mu_0 \in \overline{\mathcal{U}_k}$ *and for all* $\mu \in \mathcal{U}_k$ $f_{\boldsymbol{\mu}}$ *exhibits a* C^k *invariant torus of dimension* s *which is normally hyperbolic.*

(ii) *Inside* $\mathcal{U}$ *there exists another set* $\mathcal{T}_k$ *such that the invariant tori obtained for parameters there in is of Floquet type. This set has positive Lebesgue measure and* $\mu_0 \in \overline{\mathcal{T}_k}$

Hypothesis (3) implies that in the space of $C^\infty(\mathbb{R}^n)$ there exists a codimension s surface H given by the equations $|\lambda_i| = 1$ with $i = 1, \dots, s$. The generic conditions in the theorem implies that the map $\boldsymbol{\mu} \mapsto f_{\boldsymbol{\mu}}$ is transversal to the surface H at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

Now we consider the one parameter family of mappings $h_{(a)}(x_1, \dots, x_n) = (a - x_n^2, x_1, \dots, x_{n-1})$ which n-iterate h_a^n is the product $h_a^n(x_1, \dots, x_n) = (a - x_1^2, \dots, a - x_n^2)$. For $a = 3/4$ the map has a fixed point $P = (-1/2, \dots, -1/2)$ such that the eigenvalues of $Dh(P)$ satisfy the equation $\lambda^n = -1$.

Corollary 2. *Consider a generic family of* C^∞*-mappings* $h_{(a,\mathbf{b})} \colon \mathbb{R}^n \to \mathbb{R}^n$ *which is a* C^3 *perturbation of the family* h_a*, where* $n = 2s$*. Then for each* $k \geq 1$*, there exists an open set* $\mathcal{U}_k$ *with* $(3/4, \mathbf{0}) \in \overline{\mathcal{U}_k}$ *such that for each* $(a, \mathbf{b}) \in \mathcal{U}$ *the mapping* $h_{(a,\mathbf{b})}$ *exhibits a* C^k *invariant torus of dimension* s *which is normally hyperbolic. Moreover, there exits inside* $\mathcal{U}_k$ *a set* $\mathcal{T}_k$ *of positive Lebesgue measure, where the invariant tori are of Floquet type.*

This result is important because joined with the examples in [M] and [T] shows, on the best of our knowledge, the first example of a homoclinic tangency (higher codimension) which when unfolded, exhibits invariant torus of Floquet type. We have the hope that this example could be useful to find open sets in the space of diffeomorphisms which exhibits infinitely many invariant Floquet Torus in a persistent way, phenomenon which is widely known to hold for the conservative world.

When n is an odd integer it will be shown, elsewhere, that some kind of quasi-periodic bifurcation resembling the flip and the saddle-node bifurcations but now with invariant torus instead of fixed points happen.

The proof of Theorem 1 is done following the main steps for the codimension 1 case. First we prepare the family, bringing it to a normal form which up to terms of order three exhibits invariant torus normally hyperbolic. Then these invariant torus are shown to persist, using the contraction principle. We observe, like in the codimension 1 case, that the results of persistence of normally hyperbolic invariant manifolds can not be applied here since we don't know whether the size of the perturbation in the normal form match those in the theorems about the persistence of normally hyperbolic invariant manifolds. Once the existence of invariant torus is shown we apply the KAM dissipative quasi-periodic results.

This paper is structured as follows: in section 2 we prove the existence of invariant Floquet tori for perturbations of a family of mappings in normal form. In section 3 we recall normal forms for mappings having a fixed point whose all eigenvalues belong to the unit circle and show the proof of Theorem 1. In section 4 we show some properties of the family h_a and present the proof of Corollary 1. In the appendix we put some technical results used for proving Theorem 1 and a formula which defines the generic conditions satisfied for the derivatives up to third order of the family f_{μ}.

Finally we would like to acknowledge the referee for pointing out a mistake in the proof of Theorem 1 in a previous version of this work and his constructive and patient advice during the paper's review.

2 Invariant Tori

Let $\mathcal{O}$ be a neighborhood of $0 \in \mathbb{R}^n$ and let $\|.\|$ denote the standard Euclidean norm. Let us consider $g_{\boldsymbol{\mu}}: \mathbb{T}^n \times \mathcal{O} \to \mathbb{T}^n \times \mathbb{R}^n$, a family of C^q maps depending on the parameter $\boldsymbol{\mu} \in V \subset \mathbb{R}^p$ where $p \in \mathbb{N}$. Assume that $g_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) = N_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) + (O(\|\mathbf{r}\|^5), O(\|\mathbf{r}\|^4)) = (\boldsymbol{\Theta}, \mathbf{R})$ and $N_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\Theta}^N, \mathbf{R}^N)$ with

$$\begin{aligned} \mathbf{R}_j^N(\boldsymbol{\theta}, \mathbf{r}) =& \alpha_j(\boldsymbol{\mu}) r_j \left(1 + \sum_1^n c_{jl}(\boldsymbol{\mu}) r_l^2\right) \\ \boldsymbol{\Theta}_j^N(\boldsymbol{\theta}, \mathbf{r}) =& \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^n d_{jl}(\boldsymbol{\mu}) r_l^2. \end{aligned} \tag{1}$$

Proposition 3. *Let $g_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r})$ as above. Assume that $\Phi_1 \colon \boldsymbol{\mu} \mapsto (\alpha_1(\boldsymbol{\mu}), \ldots, \alpha_n(\boldsymbol{\mu}))$ is a submersion at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and that $C = C(\boldsymbol{\mu}_0) = (c_{ij}(\boldsymbol{\mu}_0))$ satisfies* *that* $\det(C) \neq 0$. *Then:*

1. *For each* $k \leq q$, *there exists an open set* $\mathcal{U}_{\boldsymbol{\mu}_0 k}$ *in the parameter space with* $\boldsymbol{\mu}_0 \in \overline{\mathcal{U}_{\boldsymbol{\mu}_0 k}}$ *such that if* $\boldsymbol{\mu} \in \mathcal{U}_{\boldsymbol{\mu}_0 k}$, *then has a* C^k *invariant torus of dimension n which is normally hyperbolic.*

2. *Moreover, if the map* $\Phi_2 \colon \boldsymbol{\mu} \mapsto (\phi_1(\boldsymbol{\mu}), \ldots, \phi_n(\boldsymbol{\mu}))$ *is a submersion at* $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, *and* $q \geq 4n+8$, *there exists* k_0 *and* τ *such that for each* $k \geq k_0$, *inside* $\mathcal{U}_{\boldsymbol{\mu}_0 k-\tau}$ *there exists a set* $\mathcal{T}$ *of positive Lebesgue measure such that the invariant tori obtained are of Floquet type. Also we get that* $\boldsymbol{\mu}_0 \in \overline{\mathcal{T}}$.

Proof. We assume without lose of generality that $\boldsymbol{\mu}_0 = \mathbf{0}$. Since Φ_1 is a submersion, we can take $\alpha_i(\boldsymbol{\mu}) - 1$ as the first coordinates of $\boldsymbol{\mu}$ for $i = 1, \ldots, n$. From now on we assume $\alpha_i(\boldsymbol{\mu}) = 1 + \mu_i$.

First of all, we look for an invariant torus of the mapping $N_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r})$. The simplest way to do this, is looking for solutions of the equations

$$\mathbf{R}_j^N(\boldsymbol{\theta}, \mathbf{r}) = \alpha_j r_j \left(1 + \sum_1^n c_{jl}(\boldsymbol{\mu}) r_l^2\right) = r_j. \tag{2}$$

We look them along the line $r_i = t a_i \neq 0$, with $t \neq 0$. The solution corresponding to $t = 0$ provides an invariant torus that could not be normally hyperbolic, for example if one have $\mu_i = 0$ for some i.[1] Since C is an invertible matrix, we can choose the vector $(a_1, \ldots, a_n)$ with $a_i \neq 0$ such that it satisfies that $m_j = \sum c_{jl} a_l^2 \neq 0$.

Then we need to solve the system of equations

$$\alpha_j(1 + m_j t^2) = 1 \quad \text{for } j = 1, \ldots, n. \tag{3}$$

To do that, we observe that $\det DF(1, \ldots, 1) = \frac{(-1)^n}{m_1 \ldots m_n} \neq 0$ for the mapping

$$F \colon (\alpha_1, \ldots, \alpha_n) \mapsto \left(\left(\frac{1}{\alpha_1} - 1\right)\frac{1}{m_1}, \ldots, \left(\frac{1}{\alpha_n} - 1\right)\frac{1}{m_n}\right).$$

From here, we get the existence of a curve $\gamma(t)$ in the parameter space $(\alpha_1, \ldots, \alpha_n)$, included in the set defined by $\left\{(\alpha_1, \ldots, \alpha_n) \colon \frac{m_j}{|m_j|}(\alpha_j - 1) \leq 0\right\}$ with $\gamma(t) \to (1, \ldots, 1)$ when $t \to 0$ and such that $(\alpha_1, \ldots \alpha_n) = \gamma(t)$ solves (2) with $r_j = t a_j$.

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[1] This proposition is used when $(\boldsymbol{\theta}, \mathbf{r})$ are considered as polar coordinates, in this case this solution does not give the expected result.

From now on, we restrict to the parameters given by $\boldsymbol{\mu}(t) = (\gamma(t), \mu_{n+1}, \ldots, \mu_p)$ where μ_i with $i = n+1, \ldots, p$, is assumed to be small enough. So we have a family of mappings

$$g_t = g_{\boldsymbol{\mu}(t)} \colon (\boldsymbol{\theta}, \mathbf{r}) \mapsto \big(\boldsymbol{\Theta}_t^N, \mathbf{R}_t^N\big) + \big(O(\|\mathbf{r}\|^5), O(\|\mathbf{r}\|^4)\big)$$

with $\boldsymbol{\Theta}_t^N, \mathbf{R}_t^N$ having the form in (1). Now we introduce the following change of coordinates: $z_l \mapsto r_l = ta_l(1+tz_l)$ and rename $\boldsymbol{\Theta}_t^N$ and $\mathbf{R}_t^N$ as $\overline{\boldsymbol{\Theta}}_t^N$ and $\overline{\mathbf{R}}_t^N$ in these new coordinates. $\overline{\boldsymbol{\Theta}}_t^N$ and$\overline{\mathbf{R}}_t^N$ have the following form:

$$\begin{aligned}
\overline{\mathbf{R}}_{tj}^N(\boldsymbol{\theta}, \mathbf{z}) &= \left(\alpha_j(1+tz_j)(1+\sum_1^n c_{jl}\, a_l^2\, t^2(1+tz_l)^2\,) - 1\right)\frac{1}{t} \\
&= \Bigg(\alpha_j(1+\sum_1^n c_{jl}\, a_l^2\, t^2\,) + 2\alpha_j \sum_1^n c_{jl}\, a_l^2\, t^3 z_l + \alpha_j \sum_1^n c_{jl}\, a_l^2\, t^4 z_l^2 + \\
&\qquad \alpha_j t z_j(1+\sum_1^n c_{jl}\, a_l^2\, t^2(1+tz_j)^2\,) - 1\Bigg)\frac{1}{t} \\
&= \Bigg(2\alpha_j \sum_1^n c_{jl}\, a_l^2\, t^3 z_l + \alpha_j \sum_1^n c_{jl}\, a_l^2\, t^4 z_l^2 \\
&\qquad + \alpha_j t z_j(1+\sum_1^n c_{jl}\, a_l^2\, t^2(1+tz_l)^2\,)\Bigg)\frac{1}{t} \\
&= \alpha_j z_j(1+\sum_1^n c_{jl}\, a_l^2\, t^2\,) + 2\alpha_j \sum_1^n c_{jl}\, a_l^2\, t^2 z_l + O(t^3) \\
&= z_j + 2\alpha_j \sum_1^n c_{jl}\, a_l^2\, t^2 z_l + O(t^3)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\boldsymbol{\Theta}}_{tj}^N(\boldsymbol{\theta}, \mathbf{z}) &= \theta_j + \phi_j(t) + \sum_1^n d_{jl}\, a_l^2\, t^2(1+tz_l)^2 \\
&= \theta_j + \phi_j(t) + \left(\sum_1^n d_{jl}\, a_l^2\right) t^2 + O(t^3).
\end{aligned} \tag{4}$$

Now, let us first observe that $D_{\mathbf{z}}\overline{\mathbf{R}}_t^N(\boldsymbol{\theta}, \mathbf{0})$ has eigenvalues outside of the unit circle. That is so, since its eigenvalues have the form $1 + t^2\sigma + \cdots$ where

$\sigma = \eta + i\zeta \neq 0$ is an eigenvalue of $2AC\widetilde{A}$ and where A, $\widetilde{A}$ are the diagonal matrices with diagonal entries $A_{ii} = \alpha_i$, $\widetilde{A}_{ii} = a_i^2$ respectively; in consequence for t small enough the absolute value of these eigenvalues have the form $1+t^2\eta+bt^3+t^4(\eta^2+\zeta^2)+\cdots$ and from here we conclude that they have absolute value not equal to 1. So we can write for each small enough t, $\mathbf{z} = \mathbf{z}_{ut} + \mathbf{z}_{st}$ with $\mathbf{z}_{ut(st)}$ in the subspace $E_{ut(st)}$ which is the eigenspace associated with the eigenvalues with norm bigger (less) than 1 of $I + 2t^2AC\widetilde{A}$. In these new coordinates we get

$$\begin{aligned} \overline{\mathbf{R}}_t^N(\boldsymbol{\theta}, \mathbf{z}_{ut}, \mathbf{z}_{st})_{ut} &= \mathbf{z}_{ut} + 2t^2AC\widetilde{A}\,\mathbf{z}_{ut} + O(t^3) \\ \overline{\mathbf{R}}_t^N(\boldsymbol{\theta}, \mathbf{z}_{ut}, \mathbf{z}_{st})_{st} &= \mathbf{z}_{st} + 2t^2AC\widetilde{A}\,\mathbf{z}_{st} + O(t^3). \end{aligned} \tag{5}$$

Secondly, we introduce a new norm $\|z\|_{t*} = \max\{\|z_{ut}\|_{ut*}, \|z_{st}\|_{st*}\}$ in $\mathbb{R}^n$ where $\|.\|_{ut*(st*)}$ is a norm where $\|D_{\mathbf{z}_{ut}(\mathbf{z}_{st})}\overline{\mathbf{R}}_t^N|E_{ut(st)}\|_{ut*(st*)} > 1+O(t^2)(< 1-O(t^2))$. This can be done since from (5) the eigenvalues of $D_{\mathbf{z}_{ut}(\mathbf{z}_{st})}\overline{\mathbf{R}}_t^N|E_{ut(st)}$ are of the form $1 + O(t^2)(1 - O(t^2))$. Finally, we put the following norm on $\mathbb{T}^n \times \mathbb{R}^n$: $\|(\boldsymbol{\theta}, \mathbf{z})\|' = \max\{\|\boldsymbol{\theta}\|, \|z\|_{t*}\}$ and $\|T\|'$ would mean $\sup_{(\boldsymbol{\theta},\mathbf{z})\in\mathcal{O}}\{\|T(\boldsymbol{\theta}, \mathbf{z})\|'\}$ when T is a vector-valued or matrix valued function. It follows that $\mathbf{z} = \mathbf{0}$ is an invariant torus for the map $T_t(\boldsymbol{\theta}, \mathbf{z}) = (T_{t\boldsymbol{\theta}}, T_{t\mathbf{z}})$

$$\begin{aligned} T_{t\boldsymbol{\theta}\,j} &= \theta_j + \phi_j(t) + \left(\sum_1^n d_{jl}\,a_l^2\right)t^2 \\ T_{t\mathbf{z}\,j} &= z_j + 2\alpha_j\sum_1^n c_{jl}\,a_l^2\,t^2 z_l, \end{aligned}$$

which is normally hyperbolic since

$$\begin{aligned} \|\pi_s DT_t|E_{st}\|' < 1 - O(t^3) < 1 - O(t^2) &< \|DT_t|\mathbb{R}^n \times \{\mathbf{0}\}\|' \\ &< 1 + O(t^2) < 1 + O(t^3) < \|\pi_u DT_t|E_{ut}\|', \end{aligned} \tag{6}$$

where π_s and π_u are the projections on E_{st} and E_{ut} respectively. (It does not matter if the subspaces $E_{s(u)}$ are not invariant, the conditions above imply the normal hyperbolicity.) The invariant torus for $g_t(\theta, \mathbf{z}_{ut}, \mathbf{z}_{st}) = (\boldsymbol{\Theta}_t, \mathbf{R}_{ut}, \mathbf{R}_{st})$ will be obtained as a perturbation of the last one. We are going to get the invariant torus we are looking for, as the intersection of two invariant manifolds (center-stable and center-unstable). In order to get the center unstable invariant manifold we first must to write the map g_t in a cross form as follows:

$$\begin{aligned} \mathbf{R}_{st} &= F(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}, \mathbf{z}_{st}) \\ \mathbf{z}_{ut} &= G_1(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}, \mathbf{z}_{st}) \\ \boldsymbol{\theta} &= G_2(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}, \mathbf{z}_{st}). \end{aligned} \tag{7}$$

This can be done since the map $\boldsymbol{\theta} \mapsto \boldsymbol{\Theta}_t$ is a diffeomorphims: the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \boldsymbol{\phi}(0)$, with $\boldsymbol{\phi}(t) = (\phi_1(t), \ldots, \phi_n(t))$, is a diffeomorphism and for t small enough it follows from (4) that the mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\Theta}_t$ is so close to $\boldsymbol{\theta} \mapsto \boldsymbol{\theta} + \boldsymbol{\phi}(0)$ as we want; and $\mathbf{R}_{ut}$ is an expansion, as easily follows from (5).

The facts: $\|D_{\mathbf{z}_{st}}\boldsymbol{\Theta}_t\|' = O(t^3)$, $\|(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut})^{-1}\|' = 1 + O(t^3)$, $\|D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}\mathbf{R}_{st}\|' = O(t^3)$ and $\|D_{\mathbf{z}_{st}}\mathbf{R}_{st}\| = 1 - O(t^2)$, allow us to conclude that

$$\begin{aligned}
&2\sqrt{\|(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|' \, \|D_{\mathbf{z}_{st}}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut})\|' \, \|D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}\mathbf{R}_{st}.(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|'} \\
&\quad + \|(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, , \mathbf{R}_{ut}))^{-1}\|' \, \|D_{\mathbf{z}_{st}}\mathbf{R}_{st}\|' \leq 2O(t^3) + 1 - O(t^2) < 1.
\end{aligned}$$

and

$$\begin{aligned}
&\sqrt[k+1]{\big(\|D_{\mathbf{z}_{st}}\mathbf{R}_{st}\|' + \|D_{\mathbf{z}_{st}}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut})\|'.\|D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}\mathbf{R}_{st}.(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|'\big) \, \|(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|'^k} \\
&+ \sqrt{\|(D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|' \, \|D_{\mathbf{z}_{st}}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut})\|' \, \|D_{(\boldsymbol{\theta},\mathbf{z}_{ut})}\mathbf{R}_{st}.(D_{(\boldsymbol{\theta},\mathbf{z}_{ut}}(\boldsymbol{\Theta}_t, \mathbf{R}_{ut}))^{-1}\|'} \\
&\leq \sqrt[k+1]{\big[(1 - O(t^2)) + O(t^3)\, O(t^3)\, (1 + O(t^3))\big] \, (1 + O(t^3))^k} \\
&+ \sqrt{(1 + O(t^3))\, O(t^3)\, O(t^3)\, (1 + O(t^3))} \\
&\leq \sqrt[k+1]{(1 - O(t^2))\, (1 + O(t^3))^k} + O(t^3) \\
&\leq 1 - O(t^2) + O(t^3) < 1.
\end{aligned}$$

In the proof of Theorem 4.2 of [SSTCh, pag. 243], it is shown that the last two inequalities imply that

$$\sqrt{\|D_{\mathbf{z}_{st}}F\|' \, \|D_{(\boldsymbol{\Theta},\mathbf{R}_{ut})}(G_1, G_2)\|'} + \sqrt{\|D_{(\boldsymbol{\Theta},\mathbf{R}_{ut})}F\|' \, \|D_{\mathbf{z}_{st}}(G_1, G_2)\|'} < 1,$$

$$\|D_{\mathbf{z}_{st}}F\|' + \sqrt{\|D_{(\boldsymbol{\Theta},\mathbf{R}_{ut})}F\|' \|D_{\mathbf{z}_{st}}(G_1, G_2)\|'} < 1,$$

and

$$\sqrt[k+1]{\{\|D_{\mathbf{z}_{st}F}\|' \, \|D_{(\boldsymbol{\Theta},\mathbf{R}_{ut})}(G_1, G_2)\|'^k\}} + \sqrt{\|D_{(\boldsymbol{\Theta},\mathbf{R}_{ut})}F\|' \, \|D_{\mathbf{z}_{st}}(G_1, G_2)\|'} < 1.$$

All of these together say that we can apply

Proposition 4. *Let the C^q-mapping $g\colon (\boldsymbol{\theta}, \mathbf{z}_u, \mathbf{z}_s) \mapsto (\boldsymbol{\Theta}, \mathbf{R}_u, \mathbf{R}_s)$ be defined in $\mathcal{D} = \mathbb{T}^n \times \mathcal{O}_u \times \mathcal{O}_s$ with $s + u = n$ which can be written in the cross form:*

$$\begin{aligned} \mathbf{R}_s &= F(\boldsymbol{\Theta}, \mathbf{R}_u, \mathbf{z}_s), \\ \mathbf{z}_u &= G_1(\boldsymbol{\Theta}, \mathbf{R}_u, \mathbf{z}_s), \\ \boldsymbol{\theta} &= G_2(\boldsymbol{\Theta}, \mathbf{R}_u, \mathbf{z}_s), \end{aligned}$$

and such that it holds the following conditions for $k \le q$:

(1) $\sqrt{\|D_{\mathbf{z}_s}F\|' \, \|D_{(\boldsymbol{\Theta},\mathbf{R}_u)}(G_1, G_2)\|'} + \sqrt{\{\|D_{(\boldsymbol{\Theta},\mathbf{R}_u)}F\|' \, \|D_{\mathbf{z}_s}(G_1, G_2)\|'\}} < 1;$

(2) $\|D_{\mathbf{z}_s}F\|' + \sqrt{\{\|D_{(\boldsymbol{\Theta},\mathbf{R}_u)}F\|'\|D_{\mathbf{z}_s}(G_1, G_2)\|'\}} < 1;$

(3) $\sqrt[k+1]{\{\|D_{\mathbf{z}_s F}\|'\|D_{(\boldsymbol{\Theta},\mathbf{R}_u)}(G_1, G_2)\|'^k\}} + \sqrt{\|D_{(\boldsymbol{\Theta},\mathbf{R}_u)}F\|'\|D_{\mathbf{z}_s}(G_1, G_2)\|'} < 1,$

where $\|T\|' = \sup_{(\boldsymbol{\Theta},\mathbf{R}_u,\mathbf{z}_s)\in\mathcal{D}}\{\|T(\boldsymbol{\Theta}, \mathbf{R}_u, \mathbf{z}_s)\|\}$ for T a vector-valued or matrix-valued function. Then the mapping g has an invariant manifold M of dimension $n + u$ in $\mathbb{T}^n \times \mathcal{O} = \mathbb{T}^n \times \mathcal{O}_u \times \mathcal{O}_s$ which is normally attracting. This manifold is given as the graph of a function $h\colon \mathbb{T}^n \times \mathcal{O}_u \to \mathcal{O}_s$ which is a C^k map.

which is a consequence of Theorem 4.3 and Theorem 4.4 of [SSTCh, pag. 252 and pag. 255] to get the center unstable manifold $\mathbb{T}^{cu} = M$ as the graph of a function h_u. In order to get the center stable manifold $\mathbb{T}^{cs}$ we consider g_t^{-1}. Since $\mathbf{r} = \mathbf{0}$ is an invariant torus for the map (1), and this map is a diffeomorphism in a neighborhood of this torus, then we can conclude that $g_{\boldsymbol{\mu}}^{-1}$ has the form: $g_{\boldsymbol{\mu}}^{-1}(\boldsymbol{\Theta}, \mathbf{R}) = \tilde{N}_{\boldsymbol{\mu}}(\boldsymbol{\Theta}, \mathbf{R}) + (O(\|\mathbf{R}\|^5), O(\|\mathbf{R}\|^4))$, and $\tilde{N}_{\boldsymbol{\mu}}(\boldsymbol{\Theta}, \mathbf{R}) = (\boldsymbol{\theta}^{\tilde{N}}, \mathbf{r}^{\tilde{N}})$ with

$$\begin{aligned} \mathbf{r}_j^{\tilde{N}}(\boldsymbol{\Theta}, \mathbf{R}) &= \frac{R_j}{\alpha_j(\boldsymbol{\mu})}\left(1 - \sum_1^n \frac{c_{jl}(\boldsymbol{\mu})}{\alpha_l(\boldsymbol{\mu})^2} R_l^2\right) \\ \boldsymbol{\theta}_j^{\tilde{N}}(\boldsymbol{\Theta}, \mathbf{R}) &= \Theta_j - \phi_j(\boldsymbol{\mu}) - \sum_1^n \frac{d_{jl}(\boldsymbol{\mu})}{\alpha_l(\boldsymbol{\mu})^2} R_l^2. \end{aligned} \tag{8}$$

So we can apply the same arguments used with g_t to g_t^{-1} to get the existence of the center stable manifold.

Now we get the invariant torus we are looking for as: $\mathbb{T}^{cs} \cap \mathbb{T}^{cu}$. That this is a torus follows since for each $\boldsymbol{\theta}$ there exist a unique point of intersection of $\mathbb{T}^{cu}$ and $\mathbb{T}^{cs}$ with the points of $\{\boldsymbol{\theta}\} \times \mathcal{O}$. The normal hyperbolicity follows since the subspaces $\{(\mathbf{0}, v_u, D_{\mathbf{z}_u}h_u v_u)\}$ and $\{(\mathbf{0}, D_{\mathbf{z}_s}h_s v_s, v_s)\}$ are near to the subspaces E_u and E_s respectively and it holds estimates as in (6), which can be checked readily.

So we had obtained for each parameter t near enough to 0 the existence of an invariant torus C^k-normally hyperbolic. So we can conclude the existence of a set $\mathcal{U}_{\mu_0 k}$ as stated in the item i) of the proposition because for each t as above we get an open set around $\boldsymbol{\mu}(t)$ for which the invariant torus persists.

For the second part, we choose q in item $i)$ so the Central Manifold Theorem allows us to write the invariant tori $\mathbb{T}_{\boldsymbol{\mu}}$ found in item i) for each μ around $\boldsymbol{\mu}(t)$ as the graph of a C^q function $\psi(\boldsymbol{\theta}, \boldsymbol{\mu})$ and, in turn, from the form of $g_{\boldsymbol{\mu}}$ we get that $g_{\boldsymbol{\mu}}$ can be written in a neighborhood of each $\mathbb{T}_{\boldsymbol{\mu}}$ as follows

$$\begin{aligned} \theta_j &\mapsto \theta_j + \phi_j(\boldsymbol{\mu}) + O(\|\psi(\boldsymbol{\theta}, \boldsymbol{\mu})\|) + O(\|\mathbf{r}\|), \\ r_j &\mapsto \alpha_j r_j + O(\|\psi\|\, \|\mathbf{r}\|) + O(\|\mathbf{r}\|^2). \end{aligned}$$

We observe that for each $\boldsymbol{\mu}(t)$, $\|\psi(\boldsymbol{\theta}, \boldsymbol{\mu})\|$ can be done small enough uniformly in θ, just taking $\boldsymbol{\mu}$ near enough $\boldsymbol{\mu}(t)$. This fact implies the map above can be viewed as a perturbation of

$$\begin{aligned} \theta_j &\mapsto \theta_j + \phi_j(\boldsymbol{\mu}) + O(\|\mathbf{r}\|), \\ r_j &\mapsto \alpha_j r_j + O(\|\mathbf{r}\|^2), \end{aligned}$$

for which we have that the map $\boldsymbol{\mu} \mapsto (\alpha_1, \ldots, \alpha_n, \phi_1, \ldots, \phi_n)$ is a submersion as well as $|\alpha_j| \neq 1$ for each $\boldsymbol{\mu}(t)$ small. Since the form of the r_j's part of the map we can apply the diffeomorphism's version of the quasi-periodic stability, appendix's Theorem 5.2, in order to get the conclusion of the item ii). ☐

Remark 5. *In this proof I would like to thank the referee for pointing me out the fact the an additional condition on matrix C, requested in the previous version of this paper, could be ruled out.*

3 Normal forms

In this section we recall normal forms for mappings f which have a fixed point P with all of eigenvalues of $Df(P)$ belongs to the unit circle. So, consider $f_{\boldsymbol{\mu}}\colon U \to \mathbb{R}^n$, $n = 2s$, a family of C^∞-diffeomorphisms defined on an open set $U \subset \mathbb{R}^n$ with the parameter $\boldsymbol{\mu}$ varying in the open set $V \subset \mathbb{R}^p$. Suppose that $P = \mathbf{0} \in U$ and $f_{\boldsymbol{\mu}}(\mathbf{0}) = \mathbf{0}$ for all $\boldsymbol{\mu}$. The Taylor expansion of $f_{\boldsymbol{\mu}}$ around $\mathbf{0}$ is given by

$$f_{\boldsymbol{\mu}}(\mathbf{x}) = A(\boldsymbol{\mu})\mathbf{x} + f^2_{\boldsymbol{\mu}} + f^3_{\boldsymbol{\mu}} + O(\|\mathbf{x}\|^4),$$

with $A(\boldsymbol{\mu}) = Df_{\boldsymbol{\mu}}(\mathbf{0})$ and where $f^2_{\boldsymbol{\mu}}$, $f^3_{\boldsymbol{\mu}}$ are mappings whose components are homogeneous polynomials of degree 2 and 3 respectively.

Assume that $A(\boldsymbol{\mu})$ has n simple complex eigenvalues $\{\lambda_1(\boldsymbol{\mu}), \dots, \lambda_{2s}(\boldsymbol{\mu})) = (\lambda_1(\boldsymbol{\mu}), \dots, \lambda_s(\boldsymbol{\mu}), \overline{\lambda}_1(\boldsymbol{\mu}), \dots, \overline{\lambda}_s(\boldsymbol{\mu}))\}$, which for $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ belong to the unit circle. After a basis change, we have that $A(\boldsymbol{\mu})$ at $\boldsymbol{\mu}_0$ is in its real Jordan normal form, i.e., if $\lambda_j(\boldsymbol{\mu}) = \alpha_j(\boldsymbol{\mu}) + i\beta_j(\boldsymbol{\mu})$, then

$$A(\boldsymbol{\mu}) = \left(\begin{array}{cccc|cccc}
\alpha_1(\boldsymbol{\mu}) & 0 & \cdots & 0 & -\beta_1(\boldsymbol{\mu}) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_s(\boldsymbol{\mu}) & 0 & 0 & \cdots & -\beta_s(\boldsymbol{\mu}) \\
\hline
\beta_1(\boldsymbol{\mu}) & 0 & \cdots & 0 & \alpha_1(\boldsymbol{\mu}) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta_s(\boldsymbol{\mu}) & 0 & 0 & \cdots & \alpha_s(\boldsymbol{\mu})
\end{array}\right)$$

In order to work the normal form, we introduce the following change of coordinates $P\mathbf{z} = \mathbf{x}$ with $\mathbf{z} = (z_1, \dots, z_{2s})$, where

$$P = \frac{1}{2}\left(\begin{array}{c|c}
I_s & I_s \\
\hline
-iI_s & iI_s
\end{array}\right)$$

with I_s the identity matrix of size s.

Now we assume that $\mathbf{x}$, $\mathbf{z}$ have complex coordinates. We remark that when $\mathbf{x}$ is restricted to get real values we have that $z_{s+j} = \overline{z}_j$. In these new coordinates f_μ is written as

$$f_\mu(\mathbf{z}) = J\mathbf{z} + f_\mu^2(\mathbf{z}) + f_\mu^3(\mathbf{z}) + O(\|\mathbf{z}\|^4),$$

where J is a diagonal matrix with diagonal given by $(\lambda_1(\boldsymbol{\mu}), \dots, \lambda_{2s}(\boldsymbol{\mu}))$. Here, $f_\mu^2(\mathbf{z})$, $f_\mu^3(\mathbf{z})$ are again mappings whose coordinates are polynomials of degree 2 and 3 respectively.

We recall that the eigenvalues $\{\lambda_i\}_1^{2s}$ are resonant of order N if it happens that $\lambda_i = \prod_1^{2s} \lambda_l^{k_l}$ with $k_l \in \mathbb{N} \setminus \{0\}$ and $\sum_1^{2s} |k_l| = N$. A resonance will be *non-avoidable* if it happens that $\prod_1^{2s} \lambda_l^{k_l} = \lambda_i \prod |\lambda_{l_r}|^{\alpha_r}$.

We assume now, that nonresonant conditions of order $N \geq 5$ hold for the eigenvalues $\lambda_j(\boldsymbol{\mu})$, exception done with the non-avoidable ones. So with a polynomial change of variables we get that, in these new variables the j-th component of the map f_μ can be written as

$$f_{\mu j}(\mathbf{z}) = \lambda_j(\boldsymbol{\mu}) z_j + \sum_1^s \gamma_{jl} z_j |z_l|^2 + z_j\, G_j(|z_1|, \dots, |z_s|) + O(\|\mathbf{z}\|^N),$$

where G_j is a polynomial of order $N-1$ and all of its terms of at least degree four. Also remember, that the normal form theory says to us that $f_{\mu j}(\mathbf{z}) = \overline{f_{\mu j+s}}(\mathbf{z})$ for $j = 1, \ldots, s$.

Now let $k \geq 5$ be fixed. The Belitskii-Samovol theorem [IL, Theorem 1.6, pag 238] on normal forms allows us to find a C^k change of coordinates such that $f_{\mu j}$ has now the following form

$$f_{\mu j}(\mathbf{z}) = \lambda_j(\boldsymbol{\mu})z_j + \sum_1^s \gamma_{j\,l} z_j |z_l|^2 + z_j\ O(\|\mathbf{z}\|^4), \tag{9}$$

if we take $N(k)$ bigger enough.

To find the normal form for f_{μ}, which we are going to use, we introduce polar coordinates in each coordinate z_j. So let θ_j, r_j be such that $z_j = r_j \exp(i\,\theta_j)$. If $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)$, $\mathbf{r} = (r_1, \ldots, r_s)$ and $f_{\mu j}(\boldsymbol{\theta}, \mathbf{r}) = \mathbf{R}_j \exp(i\,\boldsymbol{\Theta}_j)$ and $f_{\mu j+s}(\boldsymbol{\theta}, \mathbf{r}) = \mathbf{R}_j \exp(-i\,\boldsymbol{\Theta}_j)$ for $j = 1, \ldots, s$.

As

$$\begin{aligned}
|f_{\mu j}|^2 &= f_{\mu j}\overline{f_{\mu j}} \\
&= \left(\lambda_j(\boldsymbol{\mu})z_j + \sum_1^s \gamma_{j\,l} z_j|z_l|^2 + O(|z_j|\|\mathbf{z}\|^4)\right) \\
&\quad \times \overline{\left(\lambda_j(\boldsymbol{\mu})z_j + \sum_1^s \gamma_{j\,l} z_j|z_l|^2 + O(|z_j|\|\mathbf{z}\|^4)\right)} \\
&= |\lambda_j(\boldsymbol{\mu})|^2|z_j|^2 + \lambda_j(\boldsymbol{\mu})\sum_1^s \overline{\gamma_{j\,l}}\,|z_j|^2\,|z_l|^2 \\
&\quad + \overline{\lambda_j(\boldsymbol{\mu})}\sum_1^s \gamma_{j\,l}|z_j|^2\,|z_l|^2 + O(|z_j|^2\|\mathbf{z}\|^4) \\
&= |\lambda_j(\boldsymbol{\mu})|^2|z_j|^2\left(1 + \sum_1^s 2\Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2\right) + O(|z_j|^2\|\mathbf{z}\|^4),
\end{aligned}$$

we obtain for the absolute value of $f_{\mu j}$ that

$$\begin{aligned}
|f_{\mu j}| &= |\lambda_j(\boldsymbol{\mu})||z_j|\sqrt{1 + \sum_1^s 2\Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2 + O\left(\|\mathbf{z}\|^4\right)} \\
&= |\lambda_j(\boldsymbol{\mu})||z_j|\left(1 + \sum_1^s \Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2\right) + O(|z_j|\|\mathbf{z}\|^4).
\end{aligned}$$

Using this expression we can write

$$\begin{aligned} f_{\boldsymbol{\mu} j} &= |f_{\boldsymbol{\mu} j}| \exp(i\,\boldsymbol{\Theta}_j) \\ &= |\lambda_j(\boldsymbol{\mu})||z_j| \left(1 + \sum_1^s \Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2\right) \exp(i\,\boldsymbol{\Theta}_j) + O(|z_j|\|\mathbf{z}\|^4). \end{aligned}$$

Now, the previous equality and the following fact:

$$\exp\left(\sum_1^s \frac{\gamma_{j\,l}}{\lambda_j}|z_l|^2\right) = 1 + \sum_1^s \frac{\gamma_{j\,l}}{\lambda_j}|z_l|^2 + O(\|\mathbf{z}\|^4),$$

implies that

$$\begin{aligned} f_{\boldsymbol{\mu} j} &= |\lambda_j(\boldsymbol{\mu})|\,|z_j| \left(1 + \sum_1^s \Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2 + O(\|\mathbf{z}\|^4)\right) \\ &\quad \times \exp\left(i\left(\phi_j(\boldsymbol{\mu}) + \theta_j + \sum_1^s \Im\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)\right)|z_l|^2\right) \\ &\qquad \Downarrow \\ \exp(i\,\boldsymbol{\Theta}_j) &= \exp\left(i\left(\phi_j(\boldsymbol{\mu}) + \theta_j + \sum_1^s \Im\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right)|z_l|^2\right)\right) + O(\|\mathbf{z}\|^4), \end{aligned}$$

so we obtain that $\boldsymbol{\Theta}_j, \mathbf{R}_j$ have the following expression

$$\begin{aligned} \mathbf{R}_j(\boldsymbol{\theta}, \mathbf{r}) &= |\lambda_j(\boldsymbol{\mu})| r_j \left(1 + \sum_1^s \Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right) r_l^2\right) + O(\|\mathbf{r}\|^5), \\ \boldsymbol{\Theta}_j(\boldsymbol{\theta}, \mathbf{r}) &= \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^s \Im\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right) r_l^2 + O(\|\mathbf{r}\|^4), \end{aligned}$$

where $\lambda_j(\boldsymbol{\mu}) = |\lambda_j(\boldsymbol{\mu})| \exp(i\phi_j(\boldsymbol{\mu}))$ and $j = 1, \ldots, s$.

Summing up, we get the following proposition

Proposition 6. *Let $f_{\boldsymbol{\mu}} : U \to \mathbb{R}^n$ be a family of C^∞ diffeomorphisms. Suppose that this family satisfy the following conditions:*

(1) $f_{\boldsymbol{\mu}}(\mathbf{0}) = \mathbf{0}$*;*

(2) $Df_{\boldsymbol{\mu}}(\mathbf{0})$ *has n complex eigenvalues* $\lambda_1(\boldsymbol{\mu}), \ldots, \lambda_s(\boldsymbol{\mu}), \lambda_{s+1}(\boldsymbol{\mu}) = \overline{\lambda_1(\boldsymbol{\mu})}$, $\ldots, \lambda_{2s}(\boldsymbol{\mu}) = \overline{\lambda_s(\boldsymbol{\mu})}$ *such that* $|\lambda_j(\boldsymbol{\mu})| = 1$ *for* $\boldsymbol{\mu} = \boldsymbol{\mu}_0$.

Let $k \geq 5$, then there exists $N(k)$ such that if the eigenvalues accomplish non-resonant conditions of order $N(k)$ (exception done with the non-avoidable ones) there exist new C^k-coordinates $(\boldsymbol{\theta}, \mathbf{r})$, such that

$$f_\mu(\boldsymbol{\theta}, \mathbf{r}) = g_\mu(\boldsymbol{\theta}, \mathbf{r}) + (O(\|\mathbf{r}\|^5), O(\|\mathbf{r}\|^4)),$$

with $g_\mu(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\Theta}, \mathbf{R})$ with the following form

$$\begin{aligned}\mathbf{R}_j(\boldsymbol{\theta}, \mathbf{r}) &= |\lambda_j(\boldsymbol{\mu})| r_j (1 + \sum_1^s c_{jl}(\boldsymbol{\mu}) r_l^2),\\ \boldsymbol{\Theta}_j(\boldsymbol{\theta}, \mathbf{r}) &= \theta_j + \phi_j(\boldsymbol{\mu}) + \sum_1^s d_{jl}(\boldsymbol{\mu}) r_l^2,\end{aligned}$$

with $c_{jl}(\boldsymbol{\mu}), d_{jl}(\boldsymbol{\mu})$ C^k functions of $\boldsymbol{\mu}$ which are defined as follows:

$$\begin{aligned}c_{jl}(\boldsymbol{\mu}) &= \Re\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right),\\ d_{jl}(\boldsymbol{\mu}) &= \Im\left(\frac{\gamma_{j\,l}}{\lambda_j(\boldsymbol{\mu})}\right),\end{aligned}$$

where the $\gamma_{j\,l}$ are given as in (9) and, in turn, these are defined by terms which are built up with terms up to order three of the mapping f_μ in the initial coordinates.

As an immediate consequence of the previous proposition and the Proposition (2), we get the proof of the main theorem.

Theorem 1. *Let $f_\mu \colon U \to \mathbb{R}^n$ be a family of C^∞-diffeomorphisms with $n = 2s$. Suppose that f_μ satisfies:*

(1) $f_{\mu_0}(\mathbf{x}_0) = \mathbf{x}_0$ *;*

(2) $Df_{\mu_0}(\mathbf{x}_0)$ *has n complex eigenvalues* $\lambda_1(\boldsymbol{\mu}), \ldots, \lambda_s(\boldsymbol{\mu}), \lambda_{s+1}(\boldsymbol{\mu}) = \overline{\lambda_1(\boldsymbol{\mu})}, \ldots, \lambda_{2s}(\boldsymbol{\mu}) = \overline{\lambda_s(\boldsymbol{\mu})}$ *such that* $|\lambda_j(\boldsymbol{\mu})| = 1$ *for* $\boldsymbol{\mu} = \boldsymbol{\mu}_0$*;*

(3) *If* $\lambda_j(\boldsymbol{\mu}) = |\lambda_j(\boldsymbol{\mu})| \exp(i\phi_j(\boldsymbol{\mu}))$*, then the map*

$$\boldsymbol{\mu} \mapsto (|\lambda_1(\boldsymbol{\mu})|, \ldots, |\lambda_s(\boldsymbol{\mu})|, \phi_1(\boldsymbol{\mu}), \ldots, \phi_1(\boldsymbol{\mu}))$$

is a submersion.

As well as we assume that the derivatives up to third order of f_μ accomplish generic conditions, then

(i) *for each $k \geq 1$, there exists an open set $\mathcal{U}_k$, in the space the parameters, such that $\mu_0 \in \overline{\mathcal{U}_k}$ and for all $\mu \in \mathcal{U}_k$ $f_{\boldsymbol{\mu}}$ exhibits a C^k invariant torus of dimension s which is normally hyperbolic.*

(ii) *Inside $\mathcal{U}$ there exists another set $\mathcal{T}_k$ such that the invariant tori obtained for parameters there in is of Floquet type. This set has positive Lebesgue measure and $\mu_0 \in \overline{\mathcal{T}_k}$*

Proof. Just make the translation $\mathbf{x}_0 \mapsto \mathbf{0}$, then the new family is in the hypotheses of Proposition (3) thus obtaining a new family in new coordinates such that now we can apply Proposition (2) in order to get the thesis of the Theorem. □

4 The quadratic family $h_a(x_1, \ldots, x_n) = (a - x_n^2, x_1, \ldots, x_{n-1})$

In this section we want to use the results of the last section in order to show the existence of invariant tori of Floquet type for generic perturbations $h_{(a,\mathbf{b})}$ of the family h_a when the parameters a is near enough to $3/4$ and $\mathbf{b} \approx \mathbf{0}$.

Proof of Corollary 2. We observe that the family h_a has two fixed points $P = (x_+, \ldots, x_+)$ and $Q = (x_-, \ldots, x_-)$ for $a \geq -1/4$ where $x_\pm = \frac{-1\pm\sqrt{(1+4a}}{2}$. At $a = 3/4$, $Dh_a(P)$ has n simple eigenvalues which are solutions of $\lambda^n = -1$. This situation persists for families $h_{(a,\mathbf{b})}$ which are C^2 perturbations of h_a. So let $\lambda_i(a, \mathbf{b}) = |\lambda_i(a, \mathbf{b})| \exp(i\theta_i(a, \mathbf{b}))$, in [M] it is proved that $\{\nabla|\lambda_i|(3/4, \mathbf{0}), \nabla\theta_i(3/4, \mathbf{0})\}_{i=1}^{s}$ is a linearly independent set of vectors. This imply that $S = \{(a, \mathbf{b})\colon |\lambda_1(a, \mathbf{b})| = \cdots = |\lambda_s(a, \mathbf{b})| = 1\}$ is a codimension s surface. On this surface, the set of parameters $(a, \mathbf{b})$ where the hypothesis (3) in Theorem 1 is an open and dense set containing $(3/4, \mathbf{0})$ in its closure. Generically, for parameters $(a, \mathbf{b})$ in this set the matrix $C(a, \mathbf{b})$ has all its eigenvalues outside the imaginary axe. So we can apply Theorem 1 in order to get the thesis.

Now consider the family $h_{(a,\mathbf{b})}(x_1, x_2, x_3, x_4) = (a - x_4^2 + \sum_1^3 b_i x_i, x_1, \ldots, x_3)$ with $\mathbf{b} = (b_1, b_2, b_3, b_4)$. We would like to mention that for this family the results of Corollary 2 holds.

Theorem 7. *For each $k \geq 1$, there exists an open set $\mathcal{U}_k$ with $(3/4, \mathbf{0}) \in \overline{\mathcal{U}_k}$ such that for each $(a, \mathbf{b}) \in \mathcal{U}_k$ the mapping $h_{(a,\mathbf{b})}$ exhibits a C^k invariant torus of dimension 2 which is normally hyperbolic. Moreover, there exits inside $\mathcal{U}_k$ a set $\mathcal{T}_k$ of positive Lebesgue measure, where the invariant tori are of Floquet type.*

Proof. We just presents the results for the matrix $C(a, \mathbf{b})$ and its eigenvalues, in order to check the generic hypothesis imposed on them in Corollary 2. The lengthy and straightforward computations needed to compute the matrix $C(a, \mathbf{b})$ and its eigenvalues can be done by hand or by the use of computational tools as Mathematica or Maple. Now Lemma 5.1 allows us to bring $h_{(a,\mathbf{b})}$ into the form

$$h_{i(a,\mathbf{b})} = \lambda_i z_i + \sum p_{i\,j}\, z_i |z_j|^2 + h.o.t., \tag{10}$$

and then Proposition 3 says that we can bring (10) into the form

$$h_{(a,\mathbf{b})}(\boldsymbol{\theta}, \mathbf{r}) = g_{(a,\mathbf{b})}(\boldsymbol{\theta}, \mathbf{r}) + (O(\|\mathbf{r}\|^5), O(\|\mathbf{r}\|^4)),$$

with $g_{(a,\mathbf{b})}(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\Theta}, \mathbf{R})$ written as

$$\begin{aligned} \mathbf{R}_j(\boldsymbol{\theta}, \mathbf{r}) &= |\lambda_j(a, \mathbf{b})| r_j (1 + \sum_1^s c_{j\,l}(a, \mathbf{b}) r_l^2) \\ \boldsymbol{\Theta}_j(\boldsymbol{\theta}, \mathbf{r}) &= \theta_j + \phi(a, \mathbf{b}) j + \sum_1^s d_{j\,l}(a, \mathbf{b}) r_l^2 \end{aligned}$$

So using that $c_{i\,j} = \Re\left(\frac{p_{i\,j}}{\lambda_i}\right)$, the formula for $p_{i\,j}$ provided by Lemma 5.1 and after an straightforward calculation, for $(a, \mathbf{b}) \approx (3/4, \mathbf{0})$, it is obtained that

$$\begin{aligned} c_{1\,1} &= \frac{3}{16} - \frac{3}{16}\left(a - \frac{3}{4}\right) - \frac{\left(-4 + 3\sqrt{2}\right)}{128} b_1 + \frac{32 - 9\sqrt{2}}{256} b_2 \\ &+ \frac{-4 + 3\sqrt{2}}{128} b_3 + h.o.t, \\ c_{2\,2} &= \frac{3}{16} - \frac{3}{16}\left(a - \frac{3}{4}\right) - \frac{4 + 3\sqrt{2}}{128} b_1 + \frac{32 + 9\sqrt{2}}{256} b_2 \\ &- \frac{4 + 3\sqrt{2}}{128} b_3 + h.o.t, \\ c_{1\,2} &= \frac{3}{16} - \frac{3}{16}\left(a - \frac{3}{4}\right) - \frac{12 + 17\sqrt{2}}{128} b_1 + \frac{20 - 9\sqrt{2}}{256} b_2 \\ &+ \frac{-12 + 23\sqrt{2}}{128} b_3 + h.o.t., \\ c_{2\,1} &= \frac{3}{16} - \frac{3}{16}\left(a - \frac{3}{4}\right) + \frac{-12 + 17\sqrt{2}}{128} b_1 + \frac{20 + 9\sqrt{2}}{256} b_2 \\ &- \frac{12 + 23\sqrt{2}}{128} b_3 + h.o.t.. \end{aligned}$$

And from here, the eigenvalues of C, χ_1, χ_2 are given by

$$\begin{aligned}\chi_1(a,b_1,b_2,b_3) &= \frac{1}{16}b_1+\frac{3}{64}b_2+\frac{1}{16}b_3+h.o.t \quad \text{and}\\ \chi_2(a,b_1,b_2,b_3) &= \frac{3}{8}-\frac{3}{8}\left(a-\frac{3}{4}\right)-\frac{1}{8}b_1+\frac{13}{64}b_2-\frac{1}{8}b_3+h.o.t.\end{aligned}$$

Now its follows easily that the matrix $C(a,\mathbf{b})$ is hyperbolic for parameters $(a,\mathbf{b}) \approx (3/4,\mathbf{0})$ and hence $\det(C) \neq 0$. ☐

Remark 8. *Observe that*

$$C\left(\frac{3}{4},\mathbf{0}\right)=\left(\frac{3}{16}\right)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},$$

which does not satisfy the hypothesis in Proposition 2*. When this hypothesis is missing what we obtain are central manifolds which we can not assure that contain invariant non normally hyperbolic torus.*

5 Appendix

In this appendix we present the formula to compute the third order resonant terms used in section 3 and present the diffeomorphisms version of the quasi-periodic stability theorem of [BHS].

5.1 Computation of third order resonant terms

We consider a map

$$h(Z,\overline{Z})=(E,\overline{E})$$

where $(Z,\overline{Z})=(z_1,\dots,z_s,\overline{z}_1,\dots,\overline{z}_s)$, $E=(e_1,\dots,e_s)$ y $\overline{E}=(\overline{e}_1,\dots,\overline{e}_s)$.

We suppose that the map e_i has the following form

$$e_i(Z,\overline{Z})=\lambda_i z_i - B_i(Z,\overline{Z})M\begin{pmatrix}Z\\ \overline{Z}\end{pmatrix}$$

where M is the matrix with entries $M_{\alpha,\beta}=\frac{1}{4}$ for $\alpha,\beta=1,\dots,2s$, and $B_i\in\mathbb{C}$. As well as suppose that $\lambda_i,\overline{\lambda}_i$ satisfy nonresonant conditions of order two and three as in Proposition 3. Then it can be eliminated with a change of variables the quadratic terms and the nonresonant cubic terms. The next lemma give us a formula to compute the resonant cubic terms.

Lemma 9. *In the above conditions, there exists a change of coordinates* $\Phi(Z, \overline{Z})$ *such that if* $H(Z, \overline{Z}) = (c_1, \ldots, c_s, \overline{c}_1, \ldots, \overline{c}_s)$ *is the expression in these new coordinates of* h, *then*

$$c_i = \lambda_i z_i + \sum p_{i\,j}\, z_i |z_j|^2 + h.o.t.$$

with

$$\begin{aligned} p_{i\,j} \;=\; & \frac{3}{4} B_i \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i\, \lambda_j^{-1}} + \frac{B_k}{\lambda_k - \lambda_i\, \overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} \right. \\ & \qquad \left. + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\, \lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\, \overline{\lambda}_j^{-1}} \right) \\ & - \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - \lambda_j\, \overline{\lambda}_j} + \frac{B_k}{\lambda_k - \lambda_i\, \overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\, \overline{\lambda}_j} \right. \\ & \qquad \left. + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\, \overline{\lambda}_j} + \frac{B_k}{\lambda_k - \lambda_j\, \lambda_i} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\, \lambda_i} \right) \end{aligned}$$

Proof. We want to do a change of coordinates which kill all quadratic terms. This change of coordinates has the following form:

$$\Phi(Z, \overline{Z}) = (\Psi_1(Z, \overline{Z}), \ldots, \Psi_s(Z, \overline{Z}), \overline{\Psi}_1(Z, \overline{Z}), \ldots, \overline{\Psi}_s(Z, \overline{Z})),$$

where

$$\Psi_i(Z, \overline{Z}) = z_i + (Z, \overline{Z}) N^i \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix}$$

with $N^i = (n^i_{kj})$ a symmetrical matrix of size $2s$. In order to write Φ^{-1} up to third order terms, we will need the following lemma which is an easy application of the formula for the third derivative of a composition of two maps as presented in [N, pag. 6]. $\square$

Lemma 10. *Let* $f : V \subset \mathbb{R}^N \to \mathbb{R}^N$ *be a* C^3*-diffeomorphism with* $f(0) = 0$ *and* $Df(0) = Id$, *then* f^{-1} *can be written as:*

$$f^{-1}(y) = y - \frac{1}{2} D^2 f(0) y^2 + \frac{1}{3!} \left[3D^2 f(0)(D^2 f(0) y^2, y) - D^3 f(0) y^3 \right] + h.o.t,$$

where $y^2 = (y, y)$ *and* $y^3 = (y, y, y)$.

From the above formula we conclude that for

$$\Phi^{-1}(W,\overline{W}) = (\Upsilon_1(W,\overline{W}),\dots,\Upsilon_s(W,\overline{W}),\overline{\Upsilon}_1(W,\overline{W}),\dots,\overline{\Upsilon}_s(W,\overline{W})),$$

we have that

$$\Upsilon_i(W,\overline{W}) = w_i - (W,\overline{W})N^i\begin{pmatrix} W \\ \overline{W}\end{pmatrix} + 2(\Xi,\overline{\Xi})N^i\begin{pmatrix} W \\ \overline{W}\end{pmatrix} + h.o.t.,$$

where

$$(\Xi,\overline{\Xi}) = (\Xi_1,\dots,\Xi_s,\overline{\Xi_1},\dots,\overline{\Xi_s}) \quad \text{with} \quad \Xi_j = (W,\overline{W})N^j\begin{pmatrix} W \\ \overline{W}\end{pmatrix}$$

for $j = 1,\dots,s$. Let $H(Z,\overline{Z})$ be the map $\Phi^{-1}\circ h\circ\Phi(Z,\overline{Z})$, and put $(\Psi,\overline{\Psi}) = (\Psi_1,\dots,\Psi_s,\overline{\Psi}_1,\dots,\overline{\Psi}_s)$, $h\circ\Phi(Z,\overline{Z}) = (d_1,\dots,d_s,\overline{d}_1,\dots,\overline{d}_s) = (D,\overline{D})$ and $H(Z,\overline{Z}) = (c_1,\dots,c_s,\overline{c}_1,\dots,\overline{c}_s)$, then

$$\begin{aligned} d_i &= \lambda_i\Psi_i - B_i(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix}, \\ c_i &= \Upsilon_i(D,\overline{D}) = d_i - (D,\overline{D})N^i\begin{pmatrix} D \\ \overline{D}\end{pmatrix} + 2(\Xi,\overline{\Xi})N^i\begin{pmatrix} D \\ \overline{D}\end{pmatrix} + \Pi, \end{aligned}$$

where Π represents terms of order four or greater not included in the remaining part of the formula and $\Xi = \Xi(D,\overline{D})$. Now letting

$$D = \Lambda\Psi - B(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix},$$

where $\Lambda\Psi = (\lambda_1\Psi_1,\dots,\lambda_s\Psi_s)$ and $B = (B_1,\dots,B_s)$, we get that

$$\begin{aligned} c_i = {} & \lambda_i\Psi_i - B_i(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix} \\ & - \left(\Lambda\Psi - B(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix}, \overline{\Lambda\Psi} - \overline{B}(\overline{\Psi},\Psi)M\begin{pmatrix}\overline{\Psi} \\ \Psi\end{pmatrix}\right) \\ & \quad N^i\begin{pmatrix} \Lambda\Psi - B(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix} \\ \overline{\Lambda\Psi} - \overline{B}(\overline{\Psi},\Psi)M\begin{pmatrix}\overline{\Psi} \\ \Psi\end{pmatrix}\end{pmatrix} \\ & + 2(\Xi(D,\overline{D}),\overline{\Xi(D,\overline{D})})N^i\begin{pmatrix} \Lambda\Psi - B(\Psi,\overline{\Psi})M\begin{pmatrix}\Psi \\ \overline{\Psi}\end{pmatrix} \\ \overline{\Lambda\Psi} - \overline{B}(\overline{\Psi},\Psi)M\begin{pmatrix}\overline{\Psi} \\ \Psi\end{pmatrix}\end{pmatrix} + \Pi. \end{aligned}$$

So we can conclude that:

i) The lineal term of c_i is: $\lambda_i\, z_i$.

ii) The quadratic terms of c_i are given by:

$$\begin{aligned} c_i^2 \;=\; & \lambda_i(Z,\overline{Z})N^i \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix} - B_i(Z,\overline{Z})M \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix} \\ & - (\Lambda Z, \overline{\Lambda Z})N^i \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix}. \end{aligned} \tag{11}$$

iii) The cubic terms of c_i are:

$$\begin{aligned} c_i^3 \;=\; & -2B_i(Z,\overline{Z})M \begin{pmatrix} \Psi^2 \\ \overline{\Psi}^2 \end{pmatrix} \\ & - \left(\Lambda\Psi^2, \overline{\Lambda\Psi}^2\right) N^i \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix} - (\Lambda Z, \overline{\Lambda Z})N^i \begin{pmatrix} \Lambda\Psi^2 \\ \overline{\Lambda\Psi}^2 \end{pmatrix} \\ & + (\Lambda Z, \overline{\Lambda Z})N^i \begin{pmatrix} B \\ \overline{B} \end{pmatrix} \cdot (Z,\overline{Z})M \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix} \\ & + (B,\overline{B})N^i \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix} \cdot (Z,\overline{Z})M \begin{pmatrix} Z \\ \overline{Z} \end{pmatrix} \\ & + 2\left(\Xi(\Lambda Z, \overline{\Lambda Z}), \overline{\Xi(\Lambda Z, \overline{\Lambda Z})}\right) N^i \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix}, \end{aligned} \tag{12}$$

where Ψ^2 denotes the quadratic part of Ψ. We get, from the vanishing condition on the quadratic term c_i^2 and its formula, that the matrix N^i has the following form:

$$\begin{aligned} n^i_{k\alpha} &= \frac{B_i}{4(\lambda_i - \lambda_k\,\lambda_\alpha)} \\ n^i_{k\,s+\alpha} &= \frac{B_i}{4(\lambda_i - \lambda_k\,\overline{\lambda}_\alpha)} = n^i_{s+\alpha\,k} \\ n^i_{s+k\,s+\alpha} &= \frac{B_i}{4(\lambda_i - \overline{\lambda}_k\,\overline{\lambda}_\alpha)} \end{aligned}$$

where $\alpha, k = 1, \dots, s$.

As for the cubic term c_i^3, we get

$$
\begin{aligned}
c_i^3 = &- \frac{B_i}{2} \sum_{\alpha} z_\alpha \, (\Psi_1^2 + \cdots + \Psi_s^2 + \overline{\Psi}_1^2 + \cdots + \overline{\Psi}_s^2) \\
&- \frac{B_i}{2} \sum_{\alpha} \overline{z}_\alpha \, (\Psi_1^2 + \cdots + \Psi_s^2 + \overline{\Psi}_1^2 + \cdots + \overline{\Psi}_s^2) \\
&- \lambda_1 \Psi_1^2 \sum_{\alpha} (n^i_{1\alpha} \, \lambda_\alpha \, z_\alpha + n^i_{1\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha) \\
&\qquad\qquad \vdots \quad (1) \\
&- \overline{\lambda}_s \overline{\Psi}_s^2 \sum_{\alpha} (n^i_{2s\, \alpha} \, \lambda_\alpha \, z_\alpha + n^i_{2s\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha) \\
&- \lambda_1 z_1 \sum_{\alpha} (n^i_{1\alpha} \, \lambda_\alpha \Psi_\alpha^2 + n^i_{1\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{\Psi}_\alpha^2) \\
&\qquad\qquad \vdots \quad (2) \\
&- \overline{\lambda}_s \overline{z}_s \sum_{\alpha} (n^i_{2s\, \alpha} \, \lambda_\alpha \Psi_\alpha^2 + n^i_{2s\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{\Psi}_\alpha^2) \\
&+ \lambda_1 z_1 \sum_{\alpha} (n^i_{1\alpha} \, B_\alpha + n^i_{1\, s+\alpha} \, \overline{B}_\alpha) \left(\frac{z_1 + \overline{z}_1}{2} + \frac{z_2 + \overline{z}_2}{2} + \cdots + \frac{z_s + \overline{z}_s}{2} \right)^2 \\
&\qquad\qquad \vdots \quad (3) \\
&+ \overline{\lambda}_s \overline{z}_s \sum_{\alpha} (n^i_{2s\, \alpha} \, B_\alpha + n^i_{2s\, s+\alpha} \, \overline{B}_\alpha) \left(\frac{z_1 + \overline{z}_1}{2} + \frac{z_2 + \overline{z}_2}{2} + \cdots + \frac{z_s + \overline{z}_s}{2} \right)^2 \\
&+ B_1 \sum_{\alpha} (n^i_{1\, \alpha} \, \lambda_\alpha \, z_\alpha + n^i_{1\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha) \left(\frac{z_1 + \overline{z}_1}{2} + \frac{z_2 + \overline{z}_2}{2} + \cdots + \frac{z_s + \overline{z}_s}{2} \right)^2 \\
&\qquad\qquad \vdots \quad (4) \\
&+ \overline{B}_s \sum_{\alpha} (n^i_{2s\, \alpha} \, \lambda_\alpha \, z_\alpha + n^i_{2s\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha) \left(\frac{z_1 + \overline{z}_1}{2} + \frac{z_2 + \overline{z}_2}{2} + \cdots + \frac{z_s + \overline{z}_s}{2} \right)^2 \\
&+ 2 \left(\Lambda Z, \overline{\Lambda Z} \right) N^1 \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix} \left(\sum_{\alpha} n^i_{1\alpha} \, \lambda_\alpha \, z_\alpha + n^i_{1\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha \right) \\
&\qquad\qquad \vdots \\
&+ 2 \left(\Lambda Z, \overline{\Lambda Z} \right) N^s \begin{pmatrix} \Lambda Z \\ \overline{\Lambda Z} \end{pmatrix} \left(\sum_{\alpha} n^i_{s\alpha} \, \lambda_\alpha \, z_\alpha + n^i_{s\, s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha \right) \\
&+ 2 \left(\overline{\Lambda Z}, \Lambda Z \right) \overline{N}^1 \begin{pmatrix} \overline{\Lambda Z} \\ \Lambda Z \end{pmatrix} \left(\sum_{\alpha} n^i_{s+1\, \alpha} \, \lambda_\alpha \, z_\alpha + n^i_{s+1\ s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha \right) \\
&\qquad\qquad \vdots \\
&+ 2 \left(\overline{\Lambda Z}, \Lambda Z \right) \overline{N}^s \begin{pmatrix} \overline{\Lambda Z} \\ \Lambda Z \end{pmatrix} \left(\sum_{\alpha} n^i_{2s\, \alpha} \, \lambda_\alpha \, z_\alpha + n^i_{2s\ s+\alpha} \, \overline{\lambda}_\alpha \, \overline{z}_\alpha \right).
\end{aligned}
$$

We observe that the sum of terms in (1) equals the sum of terms in (2) and the same happens with the sum of terms in (3) and (4). So we write the above coefficient as

$$
\begin{aligned}
c_i^3 \;=\; & -\frac{B_i}{2}\sum_{\alpha}(z_\alpha+\overline{z_\alpha})(\Psi_1^2+\cdots+\Psi_s^2+\overline{\Psi}_1^2+\cdots+\overline{\Psi}_s^2)\\
& - 2\sum_{\alpha\,k}\lambda_k\Psi_k^2(n^i_{k\alpha}\lambda_\alpha\, z_\alpha+n^i_{k\,s+\alpha}\overline{\lambda}_\alpha\,\overline{z}_\alpha)-2\sum_{\alpha\,k}\overline{\lambda}_k\overline{\Psi}_k^2(n^i_{k+s\alpha}\lambda_\alpha\, z_\alpha+n^i_{k+s\,s+\alpha}\overline{\lambda}_\alpha\,\overline{z}_\alpha)\\
& + 2\sum_{\alpha\,k}\lambda_k z_k(n^i_{k\alpha}B_\alpha+n^i_{k\,s+\alpha}\overline{B}_\alpha)\left(\frac{z_1+\overline{z}_1}{2}+\frac{z_2+\overline{z}_2}{2}+\cdots+\frac{z_s+\overline{z}_s}{2}\right)^2\\
& + 2\sum_{\alpha\,k}\overline{\lambda}_k\overline{z}_k(n^i_{k+s\,\alpha}B_\alpha+n^i_{k+s\,s+\alpha}\overline{B}_\alpha)\left(\frac{z_1+\overline{z}_1}{2}+\frac{z_2+\overline{z}_2}{2}+\cdots+\frac{z_s+\overline{z}_s}{2}\right)^2\\
& + 2\left(\Lambda Z,\overline{\Lambda Z}\right)N^1\left(\frac{\Lambda Z}{\overline{\Lambda Z}}\right)\left(\sum_\alpha n^i_{1\alpha}\,\lambda_\alpha\, z_\alpha+n^i_{1\,s+\alpha}\,\overline{\lambda}_\alpha\,\overline{z}_\alpha\right)\\
& \qquad\qquad\vdots\\
& + 2\left(\Lambda Z,\overline{\Lambda Z}\right)N^s\left(\frac{\Lambda Z}{\overline{\Lambda Z}}\right)\left(\sum_\alpha n^i_{s\alpha}\,\lambda_\alpha\, z_\alpha+n^i_{s\,s+\alpha}\,\overline{\lambda}_\alpha\,\overline{z}_\alpha\right)\\
& + 2\left(\overline{\Lambda Z},\Lambda Z\right)\overline{N}^1\left(\frac{\overline{\Lambda Z}}{\Lambda Z}\right)\left(\sum_\alpha n^i_{s+1\,\alpha}\,\lambda_\alpha\, z_\alpha+n^i_{s+1\;s+\alpha}\,\overline{\lambda}_\alpha\,\overline{z}_\alpha\right)\\
& \qquad\qquad\vdots\\
& + 2\left(\overline{\Lambda Z},\Lambda Z\right)\overline{N}^s\left(\frac{\overline{\Lambda Z}}{\Lambda Z}\right)\left(\sum_\alpha n^i_{2s\,\alpha}\,\lambda_\alpha\, z_\alpha+n^i_{2s\;s+\alpha}\,\overline{\lambda}_\alpha\,\overline{z}_\alpha\right)
\end{aligned}
$$

Let $p_{i\,j}$ be the coefficient of the monomial $z_i z_j \overline{z}_j$ in c_i^3. Then we get from the last equation

$$
\begin{aligned}
p_{i\,j} \;=\; & -\; B_i\sum_k\left(n^k_{s+j\;j}+n^k_{s+j\,i}+\overline{n}^k_{s+j\;j}+\overline{n}^k_{s+i\;j}+\,n^k_{i\;j}+\overline{n}^k_{s+i\,s+j}\right)\\
& - 4\sum_k\left(\lambda_k\,\lambda_i\,n^i_{k\,i}\,n^k_{s+j\;j}+\lambda_k\,\lambda_j\,n^i_{k\;j}\,n^k_{s+j\,i}+\lambda_k\,\overline{\lambda}_j\,n^i_{k\,s+j}\,n^k_{i\;j}\right)\\
& - 4\sum_k\left(\overline{\lambda}_k\,\lambda_i\,n^i_{k+s\,i}\,\overline{n}^k_{s+j\;j}+\overline{\lambda}_k\,\lambda_j\,n^i_{k+s\;j}\overline{n}^k_{s+i\;j}+\overline{\lambda}_k\,\overline{\lambda}_j\,n^i_{k+s\,s+j}\,\overline{n}^k_{s+i\,s+j}\right)\\
& + \sum_k\left(\lambda_i\,n^i_{i\,k}\,B_k+\lambda_j\,n^i_{j\,k}\,B_k+\lambda_i\,n^i_{i\,s+k}\,\overline{B}_k+\lambda_j\,n^i_{j\,s+k}\,\overline{B}_k\right)\\
& + \sum_k\left(\overline{\lambda}_j\,n^i_{j+s\,k}\,B_k+\overline{\lambda}_j\,n^i_{j+s\,s+k}\,\overline{B}_k\right)\\
& +4\sum_k\lambda_i\,\lambda_j\,\overline{\lambda}_j\left(n^i_{k\;j}\,n^k_{s+j\;i}+n^i_{ki}\,n^k_{j\,s+j}+n^i_{k\,s+j}\,n^k_{ij}\right)\\
& + 4\sum_k\lambda_i\,\lambda_j\,\overline{\lambda}_j\left(n^i_{s+k\,i}\,\overline{n}^k_{j\,s+j}+n^i_{s+k\;j}\,\overline{n}^k_{s+i\;j}+n^i_{s+k\,s+j}\,\overline{n}^k_{s+i\,s+j}\right)
\end{aligned}
$$

and using the formula for the matrix N^i we get

$$\begin{aligned}
p_{i\,j} \;=\; -\,\frac{B_i}{4}\sum_k & \left(\frac{B_k}{\lambda_k-\lambda_j\,\overline{\lambda}_j}+\frac{B_k}{\lambda_k-\lambda_i\,\overline{\lambda}_j}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_j\,\overline{\lambda}_j}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_i\,\overline{\lambda}_j}\right.\\
&\left.\quad+\frac{B_k}{\lambda_k-\lambda_j\,\lambda_i}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_j\,\lambda_i}\right)\\
&-\sum_k\left(\frac{\lambda_i\,\lambda_k\,B_k\,B_i}{(\lambda_i-\lambda_k\,\lambda_i)(\lambda_k-\lambda_j\,\overline{\lambda}_j)}+\frac{\lambda_j\,\lambda_k\,B_k\,B_i}{(\lambda_i-\lambda_k\,\lambda_j)(\lambda_k-\lambda_i\,\overline{\lambda}_j)}\right.\\
&\left.\quad+\frac{\overline{\lambda}_j\,\lambda_k\,B_k\,B_i}{(\lambda_k-\lambda_i\,\lambda_j)(\lambda_i-\lambda_k\,\overline{\lambda}_j)}\right)\\
&-\sum_k\left(\frac{\lambda_i\,\overline{\lambda}_k\,\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\lambda_i)(\overline{\lambda}_k-\lambda_j\,\overline{\lambda}_j)}+\frac{\lambda_j\,\overline{\lambda}_k\,\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\lambda_j)(\overline{\lambda}_k-\lambda_i\,\overline{\lambda}_j)}\right.\\
&\left.\quad+\frac{\overline{\lambda}_j\,\overline{\lambda}_k\,\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\overline{\lambda}_j)(\overline{\lambda}_k-\lambda_i\,\lambda_j)}\right)\\
&+\frac{1}{4}\sum_k\left(\frac{\lambda_i\,B_i\,B_k}{\lambda_i-\lambda_i\,\lambda_k}+\frac{\lambda_j\,B_i\,B_k}{\lambda_i-\lambda_j\,\lambda_k}+\frac{\lambda_i\,B_i\,\overline{B}_k}{\lambda_i-\lambda_i\,\overline{\lambda}_k}+\frac{\lambda_j\,B_i\,\overline{B}_k}{\lambda_i-\lambda_j\,\overline{\lambda}_k}\right)\\
&+\frac{1}{4}\sum_k\left(\frac{\overline{\lambda}_j\,B_i\,B_k}{\lambda_i-\lambda_k\,\overline{\lambda}_j}+\frac{\overline{\lambda}_j\,B_i\,\overline{B}_k}{\lambda_i-\overline{\lambda}_k\,\overline{\lambda}_j}\right)\\
&+\sum_k\lambda_i\,\lambda_j\,\overline{\lambda}_j\left(\frac{B_k\,B_i}{(\lambda_i-\lambda_k\,\lambda_j)(\lambda_k-\lambda_i\,\overline{\lambda}_j)}+\frac{B_k\,B_i}{(\lambda_i-\lambda_k\,\lambda_i)(\lambda_k-\lambda_j\,\overline{\lambda}_j)}\right.\\
&\left.\quad+\frac{B_k\,B_i}{(\lambda_i-\lambda_k\,\overline{\lambda}_j)(\lambda_k-\lambda_i\,\lambda_j)}\right)\\
&+\sum_k\lambda_i\,\lambda_j\,\overline{\lambda}_j\left(\frac{\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\lambda_i)(\overline{\lambda}_k-\lambda_j\,\overline{\lambda}_j)}+\frac{\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\lambda_j)(\overline{\lambda}_k-\lambda_i\,\overline{\lambda}_j)}\right.\\
&\left.\quad+\frac{\overline{B}_k\,B_i}{(\lambda_i-\overline{\lambda}_k\,\overline{\lambda}_j)(\overline{\lambda}_k-\lambda_j\,\lambda_i)}\right).
\end{aligned}$$

Now, putting together the terms of the first, fourth and fifth sums above, we obtain

$$\begin{aligned}
p_{i\,j} \;=\; -\,\frac{B_i}{4}\sum_k & \left(\frac{B_k}{\lambda_k-\lambda_j\,\overline{\lambda}_j}+\frac{B_k}{\lambda_k-\lambda_i\,\overline{\lambda}_j}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_j\,\overline{\lambda}_j}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_i\,\overline{\lambda}_j}\right.\\
&\left.\quad+\frac{B_k}{\lambda_k-\lambda_j\,\lambda_i}+\frac{\overline{B}_k}{\overline{\lambda}_k-\lambda_j\,\lambda_i}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i \lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i \lambda_j^{-1}} \right. \\
&\qquad \left. + \frac{B_k}{\lambda_k - \lambda_i \overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i \overline{\lambda}_j^{-1}} \right) \\
&+ B_i \sum_k \left(\frac{\lambda_k B_k}{(\lambda_k - 1)(\lambda_k - \lambda_j \overline{\lambda}_j)} + \frac{\lambda_k B_k}{(\lambda_k - \lambda_i \overline{\lambda}_j)(\lambda_k - \lambda_i \lambda_j^{-1})} \right. \\
&\qquad \left. + \frac{\lambda_k B_k}{(\lambda_k - \lambda_i \lambda_j)(\lambda_k - \lambda_i \overline{\lambda}_j^{-1})} \right) \\
&+ B_i \sum_k \left(\frac{\overline{\lambda}_k \overline{B}_k}{(\overline{\lambda}_k - 1)(\overline{\lambda}_k - \lambda_j \overline{\lambda}_j)} + \frac{\overline{\lambda}_k \overline{B}_k}{(\overline{\lambda}_k - \lambda_i \lambda_j^{-1})(\overline{\lambda}_k - \lambda_i \overline{\lambda}_j)} \right. \\
&\qquad \left. + \frac{\overline{\lambda}_k \overline{B}_k}{(\overline{\lambda}_k - \lambda_i \lambda_j)(\overline{\lambda}_k - \lambda_i \overline{\lambda}_j^{-1})} \right) \\
&- B_i \sum_k \left(\frac{\lambda_i \overline{\lambda}_j B_k}{(\lambda_k - \lambda_i \lambda_j^{-1})(\lambda_k - \lambda_i \overline{\lambda}_j)} + \frac{\lambda_j \overline{\lambda}_j B_k}{(\lambda_k - 1)(\lambda_k - \lambda_j \overline{\lambda}_j)} \right. \\
&\qquad \left. + \frac{\lambda_i \lambda_j B_k}{(\lambda_k - \lambda_i \overline{\lambda}_j^{-1})(\lambda_k - \lambda_i \lambda_j)} \right) \\
&- B_i \sum_k \left(\frac{\lambda_j \overline{\lambda}_j \overline{B}_k}{(\overline{\lambda}_k - 1)(\overline{\lambda}_k - \lambda_j \overline{\lambda}_j)} + \frac{\lambda_i \overline{\lambda}_j \overline{B}_k}{(\overline{\lambda}_k - \lambda_i \lambda_j^{-1})(\overline{\lambda}_k - \lambda_i \overline{\lambda}_j)} \right. \\
&\qquad \left. + \frac{\lambda_i \lambda_j \overline{B}_k}{(\overline{\lambda}_k - \lambda_i \overline{\lambda}_j^{-1})(\overline{\lambda}_k - \lambda_j \lambda_i)} \right) \\
= &- \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - \lambda_j \overline{\lambda}_j} + \frac{B_k}{\lambda_k - \lambda_i \overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j \overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i \overline{\lambda}_j} \right. \\
&\qquad \left. + \frac{B_k}{\lambda_k - \lambda_j \lambda_i} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j \lambda_i} \right) \\
&- \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i \lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i \lambda_j^{-1}} \right. \\
&\qquad \left. + \frac{B_k}{\lambda_k - \lambda_i \overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i \overline{\lambda}_j^{-1}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\ B_i \sum_k \left(\frac{(\lambda_k - \lambda_j\,\overline{\lambda}_j)\,B_k}{(\lambda_k - 1)(\lambda_k - \lambda_j\,\overline{\lambda}_j)} + \frac{(\lambda_k - \lambda_i\,\overline{\lambda}_j)\,B_k}{(\lambda_k - \lambda_i\,\overline{\lambda}_j)(\lambda_k - \lambda_i\,\lambda_j^{-1})} \right.\\
&\qquad\qquad \left. + \frac{(\lambda_k - \lambda_i\,\lambda_j)\,B_k}{(\lambda_k - \lambda_i\,\lambda_j)(\lambda_k - \lambda_i\,\overline{\lambda}_j^{-1})} \right)\\
&+\ B_i \sum_k \left(\frac{(\overline{\lambda}_k - \lambda_j\,\overline{\lambda}_j)\overline{B}_k}{(\overline{\lambda}_k - 1)(\overline{\lambda}_k - \lambda_j\,\overline{\lambda}_j)} + \frac{(\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j)\overline{B}_k}{(\overline{\lambda}_k - \lambda_i\,\lambda_j^{-1})(\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j)} \right.\\
&\qquad\qquad \left. + \frac{(\overline{\lambda}_k - \lambda_i\,\lambda_j)\overline{B}_k}{(\overline{\lambda}_k - \lambda_i\,\lambda_j)(\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j^{-1})} \right)\\
= &-\ \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - \lambda_j\,\overline{\lambda}_j} + \frac{B_k}{\lambda_k - \lambda_i\,\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\,\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j} \right.\\
&\qquad\qquad \left. + \frac{B_k}{\lambda_k - \lambda_j\,\lambda_i} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\,\lambda_i} \right)\\
&-\ \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i\,\lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\lambda_j^{-1}} \right.\\
&\qquad\qquad \left. + \frac{B_k}{\lambda_k - \lambda_i\,\overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j^{-1}} \right)\\
&+\ B_i \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i\,\lambda_j^{-1}} + \frac{B_k}{\lambda_k - \lambda_i\,\overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} \right.\\
&\qquad\qquad \left. + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j^{-1}} \right)\\
= &\,\frac{3}{4} B_i \sum_k \left(\frac{B_k}{\lambda_k - 1} + \frac{B_k}{\lambda_k - \lambda_i\,\lambda_j^{-1}} + \frac{B_k}{\lambda_k - \lambda_i\,\overline{\lambda}_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - 1} \right.\\
&\qquad\qquad \left. + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\lambda_j^{-1}} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j^{-1}} \right)\\
&-\ \frac{B_i}{4} \sum_k \left(\frac{B_k}{\lambda_k - \lambda_j\,\overline{\lambda}_j} + \frac{B_k}{\lambda_k - \lambda_i\,\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\,\overline{\lambda}_j} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_i\,\overline{\lambda}_j} \right.\\
&\qquad\qquad \left. + \frac{B_k}{\lambda_k - \lambda_j\,\lambda_i} + \frac{\overline{B}_k}{\overline{\lambda}_k - \lambda_j\,\lambda_i} \right)
\end{aligned}
$$

□

5.2 Statement of the quasi-periodicity stability result

In order to state the result let us introduce first the context where the result lives. We will follow very closely [BHS, section 2.3.1]. In this section $\|.\|$ will denote any Euclidean norm.

Consider a C^q family of maps $g_{\boldsymbol{\mu}}\colon \mathbb{T}^n \times \mathcal{O} \to \mathbb{T}^n \times \mathbb{R}^n$, where $\mathcal{O}$ is a neighborhood of $0 \in \mathbb{R}^n$, the parameter $\boldsymbol{\mu} \in V \subset \mathbb{R}^p$ and $q \geq 4n+8$. We assume that

$$g_{\boldsymbol{\mu}}(\boldsymbol{\theta}, \mathbf{r}) = (\boldsymbol{\theta} + \omega(\boldsymbol{\mu}) + \alpha(\boldsymbol{\theta}, \mathbf{r}, \boldsymbol{\mu}), \Omega(\boldsymbol{\mu})\mathbf{r} + \beta(\boldsymbol{\theta}, \mathbf{r}, \boldsymbol{\mu})), \tag{13}$$

where $\alpha = O(\|\mathbf{r}\|)$ and $\beta = O(\|\mathbf{r}\|^2)$. For each $\boldsymbol{\mu} \in \Gamma \subset V$ (Γ is diffeomorphic to a closed ball in $\mathbb{R}^p$) assume that

(1) all the eigenvalues

$$\delta_1, \ldots, \delta_{N_1}, \rho_1 e^{\pm i\phi_1}, \ldots, \rho_{N_2} e^{\pm i\phi_{N_2}}$$

($\phi_j \in (0, \pi)$) of matrix $\Omega(\boldsymbol{\mu})$ are simple and other than 1 ($N_1 + 2N_2 = n$);

(2) the mapping

$$\mathbb{R}^p \ni \boldsymbol{\mu} \mapsto (\omega, \delta, \rho, \phi) \in \mathbb{R}^{2n}$$

is submersive.

Fix $\tau > n$. Set $\omega^N = \phi \in \mathbb{R}^r$, where $r = N_2$. For $l \in \mathbb{Z}^r$ let $|l| = \sum_1^r l_i$. By Γ_γ, where $\gamma > 0$, denote the set

$$\Gamma_\gamma = \Big\{\boldsymbol{\mu} \in \Gamma\colon \forall k \in \mathbb{Z}^r \setminus \{0\}\ \forall k_0 \in \mathbb{Z}\ \forall l \in \mathbb{Z}^r, |l| \leq 2, \\ |\langle \omega, k\rangle + \langle \omega^N, l\rangle + 2\pi k_0| \geq \frac{\gamma}{|k|^\tau}\Big\}.$$

In this conditions we have

Theorem A. *There exist constants k_0 and η which only depend on τ, such that the following holds. For any $\gamma > 0$ and for $k_0 \leq k \leq q$, let g_μ a C^k family of mappings* (13) *satisfying the conditions above, then there exists a C^k neighborhood $\mathcal{D}$ of g_μ such that for all $\tilde{g}_\mu$ and $\boldsymbol{\mu} \in \Gamma_\gamma$ one has two maps*

(1) $\Phi_{\boldsymbol{\mu}}\colon (\boldsymbol{\Theta}, \mathbf{R}) \mapsto (\boldsymbol{\Theta} + U(\boldsymbol{\Theta}, \boldsymbol{\mu}), \mathbf{R} + A(\boldsymbol{\Theta}, \boldsymbol{\mu}) + B(\boldsymbol{\Theta}, \boldsymbol{\mu})\mathbf{R})$, *where U, A, B are functions on $\mathbb{T}^n \times \Gamma_\gamma$ and $U(\boldsymbol{\Theta}, \boldsymbol{\mu}) \in \mathbb{T}^n$, $A(\boldsymbol{\Theta}, \boldsymbol{\mu}) \in \mathbb{R}^n$ and $B(\boldsymbol{\Theta}, \boldsymbol{\mu})$ is a square matrix of size n.*

(2) $\Lambda\colon \Gamma_\gamma \to \mathbb{R}^p$;

which are $C^{k-\eta}$ *and which bring the family* $\tilde{g}_{\boldsymbol{\mu}}$ *to the following form*

$$\Phi_{\boldsymbol{\mu}}^{-1} \circ \tilde{g}_{\boldsymbol{\mu}+\Lambda(\boldsymbol{\mu})} \circ \Phi_{\boldsymbol{\mu}}(\boldsymbol{\Theta}, \mathbf{R}) = (\boldsymbol{\Theta} + \omega(\boldsymbol{\mu}) + O(\|\mathbf{R}\|), \Omega(\boldsymbol{\mu})\mathbf{R} + O(\|\mathbf{R}\|^2)).$$

This theorem says that for each $\mu \in \Gamma_\gamma$, $\Phi_{\boldsymbol{\mu}}(\mathbb{T}^n \times \{0\})$ is an invariant Floquet Torus of the map $\tilde{g}_{\boldsymbol{\mu}+\Lambda(\boldsymbol{\mu})}$ with parallel dynamics and with its frequency vector $\omega(\boldsymbol{\mu})$ satisfying a Diophantine condition. We remark that the set Γ_γ has Lebesgue positive measure in $\mathbb{R}^p$.

As stated, this theorem does not appear proved in the literature on the best of my known. But its proof follows the same lines as for the vector field case. See the remarks in [BHS, chapter 5] and [BHTS, Appendix, pag. 79]

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