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Precise asymptotics in the law of the iterated logarithm*

Yuexu Zhao

Abstract. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$ and positive, finite variance σ^2, and set $S_n = X_1 + \ldots + X_n$. For any $\alpha > -1, \beta > -1/2$ and for $\kappa_n(\epsilon)$ a function of ϵ and n such that $\kappa_n(\epsilon)\log\log n \to \lambda$ as $n \uparrow \infty$ and $\epsilon \downarrow \sqrt{\alpha+1}$, $EX_1^2(\log|X_1|)^{\alpha+1}(\log\log|X_1|)^{\beta+1} < \infty$, we prove that

$$\lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\epsilon^2 - (\alpha+1))^{\beta+1/2} \sum_{n\geq 3} \frac{(\log n)^\alpha (\log\log n)^\beta}{n} P(|S_n| \geq \sigma\sqrt{2n\log\log n}(\epsilon + \kappa_n(\epsilon))) = (1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2).$$

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1 Introduction and main results

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows: a sequence of random variables $\xi_1, \xi_2, \ldots$ is said to converge completely to a constant C, if $\sum_{n\geq 1} P(|\xi_n - C| > \epsilon) < \infty$ for all $\epsilon > 0$. Hsu and Robbins (1947) proved that the sequence of arithmetic means converges completely to the expected value if the variance of the summands is finite. Let us adopt the following conventions: assume that $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$ and positive, finite variance, and set $S_n = X_1 + \ldots + X_n$, Baum and Katz (1965) obtained the following result.

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Theorem A. *Let $p < 2, r \geq p$. Then*

$$\sum_{n\geq 1} n^{r/p-2} P(|S_n| \geq \epsilon n^{1/p}) < \infty, \qquad \epsilon > 0 \tag{1.1}$$

if and only if $E|X_1|^r < \infty$ and, as $r \geq 1$, $EX_1 = 0$.

It is obvious that (1.1) converges under certain conditions, a natural question is that how large the convergence rate is when $\epsilon \to 0$? As we know the limiting behavior is fairly well understood by Heyde (1975), he presented an interesting and beautiful result.

Theorem B. *If $EX_1 = 0$ and $EX_1^2 < \infty$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \sum_{n\geq 1} P(|S_n| \geq \epsilon n) = EX_1^2. \tag{1.2}$$

This is a precise estimate for the convergence rate of probability series, the result has been generalized and extended in several directions. Remaining values of r and p were considered by Chen (1978), Spătaru (1999), Gut and Spătaru (2000a), Gut and Spătaru (2000b) took care of the result as follows.

Theorem C. *Suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \sum_{n\geq 3} \frac{1}{n\log n} P(|S_n| \geq \epsilon\sqrt{n\log\log n}) = \sigma^2. \tag{1.3}$$

Note that the convergence of the series in (1.3) implies that

$$\lim_{\epsilon\downarrow 0} \epsilon^2 \sum_{n\geq 3} \frac{1}{n\log n} P(|S_n| \geq \epsilon\sqrt{n\log\log n} + a_n) = \sigma^2,$$

whenever $a_n = o(\sqrt{n\log\log n})$.

We are also interested in the limiting behavior of tail probability series, the results in this paper extend those of Gut and Spătaru (2000b) in the following aspect: while they work with the function $f(x) = 1/x\log x$, our purpose is to show that it suffices to work with more general function

$$f(x) = (\log x)^\alpha (\log\log x)^\beta / x \quad \text{or} \quad (\log\log x)^\beta/(x\log x)$$

and the methods in our paper are different from Gut's. Within the arguments used in the paper, there are three which turn out to be main tools: probability and moment inequality, central limit theorem and truncation technique. The paper is organized as follows: we first introduce our main results, after which the proofs of theorems are exposed in Section 2 and Section 3.

Throughout this paper, we adopt the following notations: let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$, and set $S_n = X_1 + \ldots + X_n$, write log for the natural logarithm, $\log x = \log_e(x \vee e)$ and $\log\log x = \log(\log x)$, $[z]$ denotes the largest integer $\le z$, $x_n \sim y_n$ means that $\lim_{n\to\infty} x_n/y_n = 1$. We are now ready to state our results.

Theorem 1.1. *Let $\alpha > -1$, $\beta > -1/2$ and let $\kappa_n(\epsilon)$ be a function of ϵ and n such that $\kappa_n(\epsilon)\log\log n \to \lambda$ as $n \uparrow \infty$ and $\epsilon \downarrow \sqrt{\alpha+1}$, where λ is a real number, $EX_1^2(\log|X_1|)^{\alpha+1}(\log\log|X_1|)^{\beta+1} < \infty$. Then*

$$\begin{aligned} \lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\epsilon^2 - (\alpha+1))^{\beta+1/2} \sum_{n\ge 3} \frac{(\log n)^\alpha (\log\log n)^\beta}{n} \\ P(|S_n| \ge \sigma\sqrt{2n\log\log n}(\epsilon + \kappa_n(\epsilon))) \\ = (1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2), \end{aligned} \tag{1.4}$$

where $\Gamma(\cdot)$ is the Gamma function.

Remark 1. Compare with the results of Gut and Spătaru (2000b), we require $f(x) = (\log x)^\alpha(\log\log x)^\beta/x$ which is more general function, our results include those of Gut and Spătaru as special cases, furthermore if we take $\alpha = 0$ or $\beta = 0$, the results follows immediately.

Corollary 1.1. *Let $\beta > -1/2$ and $EX_1^2(\log\log|X_1|)^{\beta+3/2} < \infty$. Then*

$$\begin{aligned} \lim_{\epsilon\downarrow 1}(\epsilon^2 - 1)^{\beta+1/2} \sum_{n\ge 3} \frac{(\log\log n)^\beta}{n} P(|S_n| \ge \sigma\sqrt{2n\log\log n}(\epsilon + \kappa_n(\epsilon))) \\ = (1/\sqrt{\pi})\exp(-2\lambda)\Gamma(\beta+1/2). \end{aligned}$$

Corollary 1.2. *Let $\alpha > -1$ and $EX_1^2(\log|X_1|)^{\alpha+1} < \infty$. Then*

$$\begin{aligned} \lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\epsilon^2 - (\alpha+1))^{1/2} \sum_{n\ge 3} \frac{(\log n)^\alpha}{n} P(|S_n| \ge \sigma\sqrt{2n\log\log n}(\epsilon + \kappa_n(\epsilon))) \\ = (\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1}). \end{aligned}$$

Theorem 1.2. *Let $\beta > -1$ and $EX_1^2(\log\log|X_1|)^\beta < \infty$. Then*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\sum_{n\geq 3}\frac{(\log\log n)^\beta}{n\log n}P(|S_n|\geq \epsilon\sigma\sqrt{2n\log\log n}) = 2^{-(\beta+1)}(\beta+1)^{-1}E|N|^{2(\beta+1)}, \tag{1.5}$$

where random variable $N\sim N(0,1)$.

Remark 2. Theorem 1.1 does not include the case of $\alpha=-1$, but by modifying the limit trend, that is, let ϵ tend to zero, we establish the relationship between tail probability series and Gaussian moment, which is an interesting result. Clearly taking $\beta = 0$, Theorem 2 of Gut and Spătaru (2000b) is obtained.

The proofs of (1.4) and (1.5) consist of two stages, respectively. We first assume $X_1, X_2, \ldots$ are Gaussian random variables, after which the general case is considered, without loss of generality, throughout the paper, we assume $\sigma^2 = 1$.

2 Proof of Theorem 1.1

2.1 Gaussian case

Let $X_1, X_2, \ldots$ be nondegenerate Gaussian sequences, write $\Psi(x) = P(|X_1|\geq x) = 1-\Phi(x)+\Phi(-x)$, $x\geq 0$, where $\Phi(x)$ is function of standard Gaussian distribution.

Proposition 2.1. *Let $\alpha > -1$, $\beta > -1/2$ and let $\kappa_n(\epsilon)$ be a function of ϵ such that $\kappa_n(\epsilon)\log\log n\to\lambda$ as $n\uparrow\infty$ and $\epsilon\downarrow\sqrt{\alpha+1}$, where λ is a real number. Then*

$$\lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^\beta}{n} P(|S_n|\geq\sqrt{2n\log\log n}(\epsilon+\kappa_n(\epsilon))) = (1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2). \tag{2.1}$$

Proof. Note that Feller's (1971) result as follows

$$P(X_1 > y)\sim(\sqrt{2\pi}y)^{-1}\exp(-y^2/2),\qquad y\to\infty,$$

hence the following formula holds

$$P(|X_1| > y) \sim 2(\sqrt{2\pi}\,y)^{-1}\exp(-y^2/2) \overset{\triangle}{=} \theta y^{-1}\exp(-y^2/2),$$

where $\theta = 2/\sqrt{2\pi}$, let $y = \sqrt{2\log\log n}(\epsilon + \kappa_n)$ in (2.1), it follows that

$$\begin{aligned} &P(|X_1| > \sqrt{2\log\log n}(\epsilon+\kappa_n)) \\ &\quad\sim\ \theta(\sqrt{2\log\log n}(\epsilon+\kappa_n))^{-1}\exp(-(\sqrt{2\log\log} n(\epsilon+\kappa_n))^2/2) \\ &\quad\sim\ \theta(\sqrt{2\log\log n}(\epsilon+\kappa_n))^{-1}\exp(-\epsilon^2\log\log n)\exp(-2\epsilon\kappa_n\log\log n) \\ &\quad\sim\ \theta(\epsilon\sqrt{2\log\log n})^{-1}\exp(-\epsilon^2\log\log n)\exp(-2\epsilon\lambda), \end{aligned} \tag{2.2}$$

as $n \uparrow \infty$, for some $\delta > 0$ uniformly in $\epsilon \in (\sqrt{\alpha+1}, \sqrt{\alpha+1}+\delta)$, therefore for any $0 < \beta < 1$, there exist $\delta > 0$ and N, such that for all $n \geq N$ and the above ϵ. Then

$$\begin{aligned} &\theta(\sqrt{\alpha+1}\sqrt{2\log\log n})^{-1}\exp(-\epsilon^2\log\log n)\exp(-2\lambda\sqrt{\alpha+1}-\beta) \\ &\quad\leq P(|X_1| > \sqrt{2\log\log n}(\epsilon+\kappa_n)) \\ &\quad\leq \theta(\sqrt{\alpha+1}\sqrt{2\log\log n})^{-1}\exp(-\epsilon^2\log\log n)\exp(-2\lambda\sqrt{\alpha+1}+\beta). \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), one can show that

$$\begin{aligned} &\lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^\beta}{n}P(|X_1| \geq \sqrt{2\log\log n}(\epsilon+\kappa_n)) \\ &= \lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{\theta(\log n)^\alpha(\log\log n)^\beta}{n}(\sqrt{2\log\log} n(\epsilon+\kappa_n))^{-1} \\ &\quad\times\exp(-(\sqrt{2\log\log} n(\epsilon+\kappa_n))^2/2) \\ &= \lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{\theta(\log n)^\alpha(\log\log n)^\beta}{n}(\epsilon\sqrt{2\log\log} n)^{-1} \\ &\quad\times\exp(-\epsilon^2\log\log n)\exp(-2\epsilon\lambda) \\ &= \lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}(\sqrt{2})^{-1}\theta(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1}) \\ &\quad\times\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^{\beta-\frac{1}{2}}}{n}\exp(-\epsilon^2\log\log n) \\ &= \lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}(\sqrt{2})^{-1}\theta(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1}) \end{aligned}$$

$$
\begin{aligned}
&\times \int_e^\infty \frac{(\log x)^\alpha (\log\log x)^{\beta-\frac{1}{2}}}{x} \exp(-\epsilon^2 \log\log x)dx \\
&= \lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\epsilon^2-(\alpha+1))^{\beta+1/2}(\sqrt{2})^{-1}\theta(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1}) \\
&\times \int_0^\infty y^{\beta-1/2}\exp(-(\epsilon^2-(\alpha+1))y)dy \\
&= \lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\sqrt{2})^{-1}\theta(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1}) \int_0^\infty z^{\beta-1/2}\exp(-z)dz \\
&= (1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2).
\end{aligned}
$$

Remark 3. We complete the proof via an equivalent relation, note, in particular, that the moment condition is not necessary, while it is necessary in the following case.

2.2 General case

Let $X_1, X_2, \ldots$ be i.i.d. random variable sequences, we need the truncation technique as follows, set $X_{nk} = X_k I(|X_k| < \epsilon\sqrt{n})$, $S_{nn} = \sum_{k=1}^n X_{nk}$, $\sigma_n^2 = EX_{n1}^2 - (EX_{n1})^2$.

Proposition 2.2. *Under the conditions in Theorem 1.1. We have*

$$
\lim_{\epsilon\downarrow\sqrt{\alpha+1}} (\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^\beta}{n}|\Psi((\epsilon+\kappa_n(\epsilon))\sqrt{2\log\log n}) \\
-\Psi((\epsilon+\kappa_n(\epsilon))\sigma_n^{-1}\sqrt{2\log\log n})| = 0.
$$

Proof. Note that $EX_1 = 0$ and $\epsilon > \sqrt{\alpha+1}$, it is easy to get

$$
\begin{aligned}
|EX_{nk}| &= |EX_k I(|X_k| < \epsilon\sqrt{n})| \leq E|X_k|I(|X_k| \geq \sqrt{(\alpha+1)n}) \\
&\leq C\frac{EX_1^2(\log|X_1|)^{\alpha+1}}{\sqrt{n}(\log\sqrt{n})^{\alpha+1}} \leq \frac{C}{\sqrt{n}(\log\sqrt{n})^{\alpha+1}}.
\end{aligned}
\tag{2.4}
$$

Since $\sigma_n^2 \to 1$, similar to Gut et al. (2000b), there exists $N \geq 4$, such that $\sigma_n^2(1+\sigma_n^2) \geq 1$ for $n \geq N$, by Lagrange´s theorem, it follows that

$$
\begin{aligned}
&\Psi((\epsilon+\kappa_n(\epsilon))\sqrt{2\log\log n}) - \Psi((\epsilon+\kappa_n(\epsilon))\sigma_n^{-1}\sqrt{2\log\log n}) \\
&= C(\epsilon+\kappa_n(\epsilon))\frac{1-\sigma_n^2}{\sigma_n^2(1+\sigma_n^2)}\sqrt{2\log\log n}\exp(-\xi^2/2),
\end{aligned}
\tag{2.5}
$$

where

$$(\epsilon + \kappa_n(\epsilon))\sqrt{2\log\log n}) \le \xi \le (\epsilon + \kappa_n(\epsilon))\sigma_n^{-1}\sqrt{2\log\log n}),$$

note that

$$-\xi^2/2 \le -\epsilon^2 \log\log n - 2\epsilon\lambda \le -\epsilon^2 \log\log n + 2\epsilon|\lambda|,$$

thus (2.5) shows that

$$\begin{aligned} &\Psi((\epsilon + \kappa_n(\epsilon))\sqrt{2\log\log n}) - \Psi((\epsilon + \kappa_n(\epsilon))\sigma_n^{-1}\sqrt{2\log\log n}) \\ &\qquad \le C(1-\sigma_n^2)\sqrt{2\log\log n}(\log n)^{-\epsilon^2}. \end{aligned} \tag{2.6}$$

Recalling $EX_1^2 = 1, |EX_{n1}| \le 1/\sqrt{n}$ and the moment condition, by Fubini´s theorem, we can obtain

$$\begin{aligned} &\sum_{n\ge 3} \frac{(\log n)^\alpha(\log\log n)^\beta}{n} |\Psi((\epsilon + \kappa_n(\epsilon))\sqrt{2\log\log n}) \\ &\quad - \Psi((\epsilon + \kappa_n(\epsilon))\sigma_n^{-1}\sqrt{2\log\log n})| \\ &\le N + C\sum_{n\ge N} \frac{(\log n)^{\alpha-\epsilon^2}(\log\log n)^{\beta+1/2}}{n} EX_1^2 I(|X_1| \ge \epsilon\sqrt{n}) \\ &\quad + C\sum_{n\ge N} \frac{(\log n)^{\alpha-\epsilon^2}(\log\log n)^{\beta+1/2}}{n^2} \\ &\le C + CE(X_1^2 I(|X_1| \ge 3) \sum_{n=4}^{[X^2/2]} \frac{(\log n)^{\alpha-\epsilon^2}(\log\log n)^{\beta+1/2}}{n}) \\ &\le C + CE(X_1^2 I(|X_1| \ge 3) \int_3^{[X^2/2]} \frac{(\log x)^{\alpha-\epsilon^2}(\log\log x)^{\beta+1/2}}{x} dx) \\ &\le C + CE(X_1^2 I(|X_1| \ge 3) \int_{(\epsilon^2-\alpha-1)\log\log 3}^{(\epsilon^2-\alpha-1)\log\log[X^2/2]} (\epsilon^2-\alpha-1)^{-\beta-3/2} \exp(-z) z^{\beta+1/2} dz) \\ &\le C + CE(X_1^2 I(|X_1| \ge 3) \int_{(\epsilon^2-\alpha-1)\log\log 3}^{(\epsilon^2-\alpha-1)\log\log[X^2/2]} (\epsilon^2-\alpha-1)^{-\beta-3/2} z^{\beta+1/2} dz) \\ &\le C + CE(X_1^2(\log\log X_1^2)^{\beta+3/2}) < \infty, \end{aligned}$$

thus the proof of Proposition 2.2 is completed. The following proposition will be used for proving Theorem 1.2, since it is in the same spirit as Proposition 2.2, we state it here.

Proposition 2.3. *Set $S'_{nn} = S_{nn} - ES_{nn}$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2(\alpha+1)} \sum_{n\geq 3} \frac{(\log n)^{\alpha}(\log\log n)^{\beta}}{n} |P(|S'_{nn}/\sqrt{n}| \geq \sqrt{2\log\log n}(\epsilon + \kappa_n(\epsilon))) \\ -\Psi(\sqrt{2\log\log n}(\epsilon+\kappa_n(\epsilon)))| = 0.$$

In order to prove this proposition, we use the Berry-Essen inequality [see, e.g., Petrov (1995), page 149].

Lemma 2.1. *Suppose that $X_1, X_2, \ldots$ be i.i.d. random variable sequences with $EX_1 = 0$, $EX_1^2 = \sigma^2 > 0$, and $E|X_1|^3 < \infty$, let $\rho = E|X_1|^3/\sigma^3$, $S_n = X_1 + \ldots + X_n$. Then*

$$\sup_{x\in R} |P\left(\frac{S_n}{\sigma\sqrt{n}} \leq x\right) - \Phi(x)| \leq \frac{A\rho}{\sqrt{n}},$$

where $\Phi(x)$ is function of standard normal distribution and A is a positive constant.

Proof of Proposition 2.3. By Lemma 2.1, it is easy to get

$$|P(|S'_{nn}/\sqrt{n}| \geq \sqrt{2\log\log n}(\epsilon+\kappa_n(\epsilon))) - \Psi(\sqrt{2\log\log n}(\epsilon+\kappa_n(\epsilon)))| \\ \leq \frac{A}{\sqrt{n}} E|X_{nk} - EX_{nk}|^3 \leq \frac{C}{\sqrt{n}} E|X_{nk}|^3 + \frac{C}{n^2}.$$

By $EX_1^2(\log|X_1|)^{\alpha+1}(\log\log|X_1|)^{\beta+1} < \infty$, we now consider the following series

$$\sum_{n\geq 3} \frac{(\log n)^{\alpha}(\log\log n)^{\beta}}{n} |P(|S'_{nn}/\sqrt{n}| \geq \sqrt{2\log\log n}(\epsilon+\kappa_n(\epsilon))) \\ - \Psi(\sqrt{2\log\log n}(\epsilon+\kappa_n(\epsilon)))| \\ \leq C + C\sum_{n\geq N} \frac{(\log n)^{\alpha}(\log\log n)^{\beta}}{n^{3/2}} E|X_1|^3 I(|X_1| \leq \epsilon\sqrt{2}) \\ + C\sum_{n\geq N} \frac{(\log n)^{\alpha}(\log\log n)^{\beta}}{n^{3/2}} E|X_1|^3 I(\epsilon\sqrt{2} \leq |X_1| < \epsilon\sqrt{n}) \\ \leq C + CE(|X_1|^3 I(|X_1| \geq \sqrt{2}\epsilon) \sum_{n > X_1^2/\epsilon^2} \frac{(\log n)^{\alpha}(\log\log n)^{\beta}}{n^{3/2}}) \\ \leq C + C\epsilon EX_1^2(\log|X_1|)^{\alpha+1}(\log\log|X_1|)^{\beta+1} < \infty.$$

Proof of Theorem 1.1. From Propositions 2.1∼2.3, it follows that

$$\lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^\beta}{n}$$
$$\times P(|S_{nn}-ES_{nn}|\geq\sqrt{2n\log\log n}(\epsilon+\kappa_n(\epsilon)))$$
$$=(1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2).$$

Note that $\kappa_n(\epsilon)\log\log n\to\lambda, n\uparrow\infty, \epsilon\downarrow\sqrt{\alpha+1}$, on account of (2.4), we have

$$|ES_{nn}|+|\kappa_n\sqrt{2n\log\log n}|\leq C\sqrt{n}/\sqrt{\log\log n}.$$

Furthermore, we can obtain

$$P(|S_{nn}-ES_{nn}|\geq\epsilon\sqrt{2n\log\log n}+C\sqrt{n}/\sqrt{\log\log n})$$
$$\leq P(|S_{nn}-ES_{nn}|\geq(\epsilon+\kappa_n)\sqrt{2n\log\log n}+|ES_{nn}|)$$
$$\leq P(|S_{nn}|\geq(\epsilon+\kappa_n)\sqrt{2n\log\log n})$$
$$\leq P(|S_{nn}-ES_{nn}|\geq\epsilon\sqrt{2n\log\log n}-C\sqrt{n}/\sqrt{\log\log n}),$$

therefore it follows that

$$\lim_{\epsilon\downarrow\sqrt{\alpha+1}}(\epsilon^2-(\alpha+1))^{\beta+1/2}\sum_{n\geq 3}\frac{(\log n)^\alpha(\log\log n)^\beta}{n}$$
$$\times P(|S_{nn}|\geq\sqrt{2n\log\log n}(\epsilon+\kappa_n(\epsilon)))$$
$$=(1/\sqrt{\pi})(\alpha+1)^{-1/2}\exp(-2\lambda\sqrt{\alpha+1})\Gamma(\beta+1/2).$$

Applying $EX_1^2(\log|X_1|)^{\alpha+1}(\log\log|X_1|)^{\beta+1}<\infty$, one can get

$$\sum_{n\geq 3}(1/n)(\log n)^\alpha(\log\log n)^\beta P(S_n\neq S_{nn})<\infty,$$

then from the above discussion, the proof of Theorem 1.1 is completed.

3 Proof of Theorem 1.2

3.1 Gaussian case

Let $N, N_1, N_2, \ldots$ be nondegenerate (i.i.d.) Gaussian sequences, write $\Psi(x)=P(|N|\geq x)$.

Proposition 3.1. *We have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\sum_{n\geq 3}\frac{(\log\log n)^{\beta}}{n\log n}P(|N|\geq \epsilon\sqrt{2\log\log n})$$
$$=2^{-(\beta+1)}(\beta+1)^{-1}E|N|^{2(\beta+1)}.$$

Proof. By integral formula and transformation, it is enough to show that for any $\beta > -1$,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\sum_{n\geq 3}\frac{(\log\log n)^{\beta}}{n\log n}P(|N|\geq \epsilon\sqrt{2\log\log n})\\
&=\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\sum_{n\geq 3}\int_n^{n+1}\frac{(\log\log x)^{\beta}}{x\log x}P(|N|\geq \epsilon\sqrt{2\log\log x})dx\\
&=\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\int_e^{\infty}\frac{(\log\log x)^{\beta}}{x\log x}P(|N|\geq \epsilon\sqrt{2\log\log x})dx\\
&=\lim_{\epsilon\downarrow 0}2^{-(\beta+1)}\epsilon^{2(\beta+1)}\int_e^{\infty}\epsilon^{-2(\beta+1)}y^{2\beta+1}P(|N|\geq y)dy\\
&=2^{-(\beta+1)}(\beta+1)^{-1}E|N|^{2(\beta+1)}.
\end{aligned}$$

3.2 General case

Let $X_1, X_2, \ldots$ be i.i.d. random variable sequences, $N \sim N(0,1)$, in this section, moment inequality, the convergence rate of central limit theorem and truncation technique are used, especially we divide the probability series into two parts, we will explain that the truncation technique for random variables and probability series is important in proving Theorem 1.2, we first give several propositions which are technical steps, the following assumption is most convenient to avoid complex questions: suppose that $\varphi(x) = (\log\log x)^{\frac{\beta+1}{2}}$, $\varphi^{-1}(x)$ is the inverse function of $\varphi(x)$, write $H(x) = \epsilon\sqrt{2x\log\log x}$. We are ready to state the propositions.

Proposition 3.2. *For large enough constant η. We have*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2(\beta+1)}\sum_{\epsilon^{\beta+1}\varphi(n)\leq\eta}\frac{(\log\log n)^{\beta}}{n\log n}P(|S_n|\geq \epsilon\sqrt{2n\log\log n})$$
$$=2^{-(\beta+1)}(\beta+1)^{-1}E|N|^{2(\beta+1)}.$$

Proof. Note that $\varphi(x) = (\log\log x)^{\frac{\beta+1}{2}}$, by $\epsilon^{\beta+1}\varphi(n) \leq \eta$, it is easy to get $n \leq \varphi^{-1}(\eta\epsilon^{-(\beta+1)})$, write $M(\epsilon) = \varphi^{-1}(\eta\epsilon^{-(\beta+1)})$. We have

$$\sum_{\epsilon^{\beta+1}\varphi(n)\leq\eta} \frac{(\log\log n)^\beta}{n\log n} P(|S_n| \geq \epsilon\sqrt{2n\log\log n})$$
$$\leq \sum_{n\leq[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} |P(|S_n| \geq \epsilon\sqrt{2n\log\log n}) - \Psi(\epsilon\sqrt{2\log\log n})|$$
$$+ \sum_{n\leq[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} \Psi(\epsilon\sqrt{2\log\log n}).$$

From Proposition 3.1, observe that $M(\epsilon) \to \infty$ as $\epsilon \downarrow 0$, one can easily get

$$\lim_{\epsilon\downarrow 0} \epsilon^{2(\beta+1)} \sum_{n\leq[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} \Psi(\epsilon\sqrt{2\log\log n})$$
$$= 2^{-(\beta+1)}(\beta+1)^{-1}E|N|^{2(\beta+1)}.$$

Note that $S_n/\sqrt{n} \xrightarrow{\mathfrak{D}} N(0,1)$, let $\Delta_n = \sup_x |P(|S_n/\sqrt{n}| \geq \epsilon\sqrt{2\log\log x}) - \Psi(\epsilon\sqrt{2\log\log x})|$, since $\Psi(x)$ is continuous function, hence $\lim_{n\to\infty} \Delta_n = 0$, write $\Delta_n = o(1)$, it follows that

$$\epsilon^{2(\beta+1)} \sum_{n\leq[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} |P(|S_n/\sqrt{n}| \geq \epsilon\sqrt{2\log\log n}) - \Psi(\epsilon\sqrt{2\log\log n})|$$
$$= \epsilon^{2(\beta+1)} \sum_{n\leq[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} \times o(1)$$
$$\leq C\epsilon^{2(\beta+1)} \times o(1) \int_e^{n\leq M(\epsilon))} d(\log\log x)^{\beta+1}$$
$$\leq C\eta^2 \times o(1) \to 0.$$

Proposition 3.3. *Suppose that $EX_1^2(\log\log|X_1|)^\beta < \infty$. Then for large enough constant η, we have*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2(\beta+1)} \sum_{\epsilon^{\beta+1}\varphi(n)>\eta} \frac{(\log\log n)^\beta}{n\log n} P(|S_n| \geq \epsilon\sqrt{2n\log\log n}) = 0.$$

To prove Proposition 3.3, we need a simple lemma [see, e.g., Petrov (1995), page 72].

Lemma 3.1. *Let $X_1, X_2, \ldots$ be independent random variables with $EX_1 = 0$, $p \geq 2$. We have*

$$E|\sum_{k=1}^{n} X_k|^p \leq C(p)\left(\sum_{k=1}^{n} E|X_k|^p + (\sum_{k=1}^{n} EX_k^2)^{p/2}\right),$$

where $C(p)$ is a positive constant depending only on p.

Assume $X_1, X_2, \ldots$ be i.i.d. random variables, using Lemma 3.1, the following holds.

$$E|\sum_{k=1}^{n} X_k|^p \leq C(p)(nE|X_1|^p + (nEX_1^2)^{p/2}). \tag{3.1}$$

Proof of Proposition 3.3. By probability inequality, one can show that

$$\begin{aligned}
&\sum_{\epsilon^{\beta+1}\varphi(n)>\eta} \frac{(\log\log n)^\beta}{n\log n} P(|S_n| \geq \epsilon\sqrt{2n\log\log n}) \\
&\quad \leq \sum_{n>[M(\epsilon)]} \frac{n(\log\log n)^\beta}{n\log n} P(|X_1| \geq H(n)) \\
&\qquad + \sum_{n>[M(\epsilon)]} \frac{(\log\log n)^\beta}{n\log n} P\left(|\sum_{i=1}^{n} X_{ni}| \geq H(n) - n|EX_{ni}|\right) \\
&\quad \triangleq I_1 + I_2,
\end{aligned}$$

where $X_{ni} = X_i I(|X_i| \leq \epsilon\sqrt{2n\log\log n})$. Applying Fubini´s theorem we get

$$\begin{aligned}
\epsilon^{2(\beta+1)} I_1 &= \epsilon^{2(\beta+1)} \sum_{n>[M(\epsilon)]} \frac{(\log\log n)^\beta}{\log n} \sum_{k=n}^{\infty} P(H(k) \leq |X_1| < H(k+1)) \\
&= \epsilon^{2(\beta+1)} \sum_{k>[M(\epsilon)]} P(H(k) \leq |X_1| < H(k+1)) \sum_{n=[M(\epsilon)]}^{k} \frac{(\log\log n)^\beta}{\log n} \\
&\leq C\epsilon^{2(\beta+1)} \sum_{k>[M(\epsilon)]} k(\log\log k)^{\beta+1} P(H(k) \leq |X_1| < H(k+1)) \qquad (3.2) \\
&\leq C\epsilon^{2(\beta+1)} \sum_{k>[M(\epsilon)]} \epsilon^{-2}(\log\log k)^\beta (\epsilon^2 k\log\log k) \\
&\quad P(H(k) \leq |X_1| < H(k+1)) \\
&\leq C\epsilon^{2\beta(1-\delta)} EX_1^2(\log\log X_1^2)^\beta I(|X_1| \geq H([M(\epsilon)])),
\end{aligned}$$

the last inequality follows from the fact that $\log\log x$ is a slowly changeable function which means for a function $L(x)$ that

(a_1) For any $M \geq 0$. Then $\lim_{x\to\infty} L(x+M)/L(x) = 1$;
(a_2) For any $\delta > 0$. Then $\lim_{x\to\infty} x^{\delta} L(x) = \infty$; $\lim_{x\to\infty} x^{-\delta} L(x) = 0$;
(a_3) As $k \to \infty$. Then $\sup_{2^k \leq t \leq 2^{k+1}} L(t)/L(2^k) \to 1$,

see [10]. Therefore for any $\delta > 0$, we have $x^{-\delta} \log\log x \downarrow 0$, using

$$(\epsilon^{-2}k^2)^{-\delta} \log\log(\epsilon^{-2}k^2) \leq k^{-2\delta} \log\log k^2,$$

one can get the last inequality in (3.2). Choosing an appropriate δ such that $2\beta(1-\delta) > 0$, one gets that (3.2) goes to 0.

To prove $I_2 < \infty$, by $EX_1 = 0$ and $n > M(\epsilon)$, firstly notice the following fact

$$\begin{aligned} \frac{n|EX_{n1}|}{\epsilon\sqrt{2n\log\log n}} &= \frac{n|EX_1 I(|X_1| > \epsilon\sqrt{2n\log\log n})|}{\epsilon\sqrt{2n\log\log n}} \\ &\leq \frac{nEX_1^2 I(|X_1| > \epsilon\sqrt{2n\log\log n})}{2\epsilon^2 n\log\log n} \\ &\leq \frac{EX_1^2 I(|X_1| > \epsilon\sqrt{2n\log\log n})}{\eta^{\frac{2}{\beta+1}}} \to 0, \quad (\eta\to\infty). \end{aligned} \tag{3.3}$$

We now consider I_2, take $p > 2(\beta+1)$, by Lemma 3.1 and chebyshev's inequality, one can show that

$$\begin{aligned} I_2 &\leq \sum_{n>[M(\epsilon)]} \frac{(\log\log n)^{\beta}}{n\log n} P(|\sum_{i=1}^{n} X_{ni}| \geq H(n)/2) \\ &\leq C \sum_{n>[M(\epsilon)]} \epsilon^{-p} n^{-p/2-1} (\log n)^{-1} (\log\log n)^{\beta-p/2} E|\sum_{i=1}^{n} X_{ni}|^p \\ &\leq C \sum_{n>[M(\epsilon)]} \epsilon^{-p} n^{-1} (\log n)^{-1} (\log\log n)^{\beta-p/2} (E|X_{n1}|^2)^{p/2} \\ &\quad + C \sum_{n>[M(\epsilon)]} \epsilon^{-p} n^{-p/2} (\log n)^{-1} (\log\log n)^{\beta-p/2} E|X_{n1}|^p \\ &\triangleq I_3 + I_4. \end{aligned}$$

To estimate I_2, we study I_3 and I_4, respectively. Note that $EX_1^2 < \infty$, and this yields

$$\begin{aligned}\epsilon^{2(\beta+1)} I_3 &\leq C\epsilon^{2(\beta+1)-p} \sum_{n\geq[M(\epsilon)]} n^{-1}(\log n)^{-1}(\log\log n)^{\beta-p/2} \\ &\leq C\epsilon^{2(\beta+1)-p} \int_{[M(\epsilon)]}^{\infty} d(\log\log x)^{\beta-p/2+1} \qquad (3.4)\\ &\leq C\epsilon^{2(\beta+1)-p}\eta^{\frac{2(\beta+1-\frac{p}{2})}{\beta+1}}\epsilon^{p-2(\beta+1)} \to 0, \quad (\eta\to\infty).\end{aligned}$$

Turn to I_4, applying Fubini´s theorem, it follows that

$$\begin{aligned}\epsilon^{2(\beta+1)} I_4 \ &\leq \ C\epsilon^{2(\beta+1)-p} \sum_{n\geq[M(\epsilon)]} n^{-p/2}(\log n)^{-1}(\log\log n)^{\beta-p/2} \\ &\quad\times \sum_{k=1}^{n} E|X_1|^p I(H(k-1)\leq |X_1| < H(k)) \\ &\leq \ \epsilon^{2(\beta+1)-p} \sum_{k>[M(\epsilon)]} E|X_1|^p I(H(k-1)\leq |X_1| < H(k)) \qquad (3.5)\\ &\quad\times \sum_{n\geq k} n^{-p/2}(\log n)^{-1}(\log\log n)^{\beta-p/2} \\ &\triangleq \ I_5.\end{aligned}$$

Let $k\geq 3$, by integration by parts, for (3.5) we have

$$\begin{aligned}&\sum_{n\geq k} n^{-p/2}(\log n)^{-1}(\log\log n)^{\beta-p/2} \\ &\leq C\int_k^{\infty} x^{-p/2+1}(\log\log x)^{\frac{\beta+1-p}{2}} d(\log\log x)^{\frac{\beta+1}{2}} \qquad (3.6)\\ &\leq Ck^{-p/2+1}(\log\log k)^{\beta+1-p/2} = C\epsilon^{p-2}(H(k))^{-p+2}(\log\log k)^{\beta},\end{aligned}$$

from (3.2) and (3.6), note that the moment condition $EX_1^2(\log\log|X_1|)^{\beta} < \infty$, for I_5 we can get

$$\begin{aligned}I_5 \leq \ & C\epsilon^{2(\beta+1)-2} \sum_{k>[M(\epsilon)]} E|X_1|^p I(H(k-1)\leq |X_1| < H(k)) \\ &\times (H(k))^{-p+2}(\log\log k)^{\beta} \\ &\leq C\,\epsilon^{2(\beta+1)-2\delta\beta} \sum_{k>[M(\epsilon)]} EX_1^2 I(H(k-1)\leq |X_1| < H(k))(\log\log k)^{\beta} \\ &\leq C\,\epsilon^{2\beta(1-\delta)} EX_1^2(\log\log X_1^2)^{\beta} I(|X_1|\geq H([M(\epsilon)])) \to 0, \ \ \epsilon\downarrow 0,\end{aligned}$$

where δ is similar to (3.2).

Proof of Theorem 1.2. The proof follows from Propositions 3.1 $\sim$ 3.3.

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Yuexu Zhao
Dept. of Infor. and Math. Sci.
Hangzhou Dianzi University
Hangzhou, 310018
CHINA

E-mail: yxzhao@hdu.edu.cn