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On a subclass of certain k-starlike functions with negative coefficients

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Abstract. The aim of the present paper is to show some properties of functions belonging to the class $(k, n, \alpha) - \mathcal{ST}$. The obtained results extend the results by Silverman [3].

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1 Introduction

Denote by $\mathcal{H}$ a class of functions of the form

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, by $\mathcal{S}$ the class of functions (1.1), analytic and univalent in $\mathbb{U}$, by $\mathcal{ST}(\alpha)$ subclass consisting of starlike and univalent functions of order α and by $k - \mathcal{ST} \quad (0 \leq k < \infty)$ a class of k-starlike univalent functions in $\mathbb{U}$, introduced in [2] and investigated Lecko and Wisniowska in [1].

It is known that every $f \in k - \mathcal{ST}$ has a contiounus extension to $\overline{\mathbb{U}}$, $f(\mathbb{U})$ is bounded and $f(\partial\mathbb{U})$ is a rectifiable curve [2].

Lemma 1.1. [1] *Let* $f \in \mathcal{S}$ *and* $0 \leq k < \infty$. *Then* $f \in k - \mathcal{ST}$ *iff*

$$Re\left\{\frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)}\right\} \geq 0 \tag{1.2}$$

$z \in \mathbb{U}, |\zeta| \leq k$.

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Let $\mathcal{T}$ denote the subclass consisting of functions whose nonzero coefficients, from the second on,are negative. That is, an analytic and univalent function f is in $\mathcal{T}$ if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n \in \mathbb{N}. \tag{1.3}$$

In [3], Silverman introduced the subclass of $\mathcal{T}$ denoted by $\mathcal{T}^*(\alpha)$ which consists of functions, that are starlike of order α.

Denote by $\mathcal{A}(n)$ the class of functions of the form

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m, \quad a_m \geq 0, \quad n \in \mathbb{N} \tag{1.4}$$

that are analytic in the open unit disc $\mathbb{U}$. In the present paper, a subclass $(k, n, \alpha) - \mathcal{ST}$ of starlike functions in the open unit disc $\mathbb{U}$ is introduced. A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $(k, n, \alpha) - \mathcal{ST}$ if it satisfies

$$Re\left\{\frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)}\right\} \geq \alpha \tag{1.5}$$

for some α $(0 \leq \alpha < 1)$, $z \in \mathbb{U}$, and $|\zeta| \leq k$, $k \geq 0$.

We note that $(0, 1, \alpha) - \mathcal{ST} \equiv \mathcal{T}^*(\alpha)$ and $(k, 1, 0) - \mathcal{ST} \equiv k - \mathcal{ST} \cap \mathcal{A}(n)$. Our class $(k, n, \alpha) - \mathcal{ST}$ is the generalization of the class $\mathcal{T}^*(\alpha)$ introduced by Silverman in [3].

2 Some results of the class $(k, n, \alpha) - \mathcal{ST}$

Theorem 2.1. *A function $f \in \mathcal{A}(n)$ is in the class $(k, n, \alpha) - \mathcal{ST}$ iff*

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha]\, a_m \leq 1 - \alpha. \tag{2.1}$$

Proof. Let $f \in (k, n, \alpha) - \mathcal{ST}$. Then we have from (1.5)

$$Re\left\{\frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)}\right\}$$

$$= Re\left\{\frac{z - \sum_{m=n+1}^{\infty} m a_m z^m - \zeta \sum_{m=n+1}^{\infty} a_m z^{m-1} + \zeta \sum_{m=n+1}^{\infty} m a_m z^{m-1}}{z - \sum_{m=n+1}^{\infty} a_m z^m}\right\} \geq \alpha.$$

If we choose z and ζ real and let $z \to 1^-$ and $\zeta \to -k^+$, we get,

$$\frac{1 - \sum_{m=n+1}^{\infty} m a_m + k \sum_{m=n+1}^{\infty} a_m - k \sum_{m=n+1}^{\infty} m a_m}{1 - \sum_{m=n+1}^{\infty} a_m} \geq \alpha$$

or

$$\sum_{m=n+1}^{\infty} [k(m-1) + m - \alpha]\, a_m \leq 1 - \alpha$$

which is equivalent to (2.1).

Conversely, assume that (2.1) is true. Then

$$\left\{ \frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)} \right\}$$

$$= \left\{ \frac{\zeta}{z} + \frac{(z-\zeta)\left(1 - \sum_{m=n+1}^{\infty} m a_m z^{m-1}\right)}{z - \sum_{m=n+1}^{\infty} a_m z^m} \right\}$$

$$= \left\{ \frac{1 - \sum_{m=n+1}^{\infty} m a_m z^{m-1} + \zeta \sum_{m=n+1}^{\infty} (m-1) a_m z^{m-2}}{1 - \sum_{m=n+1}^{\infty} a_m z^{m-1}} \right\}$$

for $|z| < 1$. If we choose $z \to 1^-$ and $\zeta \to -k^+$ through real values, we obtain

$$Re\left\{ \frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)} \right\} = \frac{1 - \sum_{m=n+1}^{\infty} [k(m-1) + m]\, a_m}{1 - \sum_{m=n+1}^{\infty} a_m}. \qquad (2.2)$$

If (2.1) is rewritten as

$$\sum_{m=n+1}^{\infty} [km + m - k]\, a_m \leq 1 - \alpha + \alpha \sum_{m=n+1}^{\infty} a_m,$$

and (2.2) is used, then we obtain

$$Re\left\{ \frac{\zeta}{z} + \frac{(z-\zeta)\, f'(z)}{f(z)} \right\} \geq \frac{\alpha\left(1 - \sum_{m=n+1}^{\infty} a_m\right)}{1 - \sum_{m=n+1}^{\infty} a_m} = \alpha.$$

Thus $f \in (k, n, \alpha) - \mathcal{ST}$.

Theorem 2.2. *If $f \in (k, n, \alpha) - \mathcal{ST}$, then we obtain*

$$r - \frac{1-\alpha}{(1+k)n + (1-\alpha)} r^{n+1} \leq |f(z)| \leq r + \frac{1-\alpha}{(1+k)n + (1-\alpha)} r^{n+1}$$

for $|z| = r$, with equality for

$$f(z) = z - \frac{1-\alpha}{(1+k)n + (1-\alpha)} z^{n+1}; \quad z = \mp r$$

Proof. From (2.1), we have

$$\sum_{m=n+1}^{\infty} a_m \leq \frac{1-\alpha}{(1+k)n + (1-\alpha)}. \tag{2.3}$$

Thus,

$$|f(z)| \leq r + \sum_{m=n+1}^{\infty} a_m r^m \leq r + r^{n+1} \sum_{m=n+1}^{\infty} a_m \leq r + \frac{1-\alpha}{(1+k)n + (1-\alpha)} r^{n+1}.$$

Similarly,

$$|f(z)| \geq r - \sum_{m=n+1}^{\infty} a_m r^m \geq r - r^{n+1} \sum_{m=n+1}^{\infty} a_m \geq r - \frac{1-\alpha}{(1+k)n + (1-\alpha)} r^{n+1}.$$

Theorem 2.3. *If $f \in (k, n, \alpha) - \mathcal{ST}$, then*

$$1 - \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)} r^n \leq |f'(z)| \leq 1 + \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)} r^n$$

for $|z| = r$, with equality for

$$f(z) = z - \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)} z^{n+1} \quad ; \quad z = \mp r$$

Proof. From (2.3) and Theorem 2.1, it follows that

$$\sum_{m=n+1}^{\infty} m a_m \leq \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)}.$$

Consequently, for $|z| = r < 1$, we have

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{m=n+1}^{\infty} m a_m |z|^{m-1} \leq 1 + r^n \sum_{m=n+1}^{\infty} m a_m \\ &\leq 1 + \frac{(1-\alpha)(n+1)}{(1+k)n + (1-\alpha)} r^n \end{aligned}$$

and

$$|f'(z)| \geq 1 - \sum_{m=n+1}^{\infty} ma_m|z|^{m-1} \geq 1 - r^n \sum_{m=n+1}^{\infty} ma_m$$
$$\geq 1 - \frac{(1-\alpha)(n+1)}{(1+k)n+(1-\alpha)}r^n.$$

This completes the proof of the theorem.

Theorem 2.4. *Let the functions*

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m, \quad a_m \geq 0$$

and

$$g(z) = z - \sum_{m=n+1}^{\infty} b_m z^m, \quad b_m \geq 0$$

be in the class $(k, n, \alpha) - \mathcal{ST}$*. Then for* $0 \leq \lambda \leq 1$*,*

$$h(z) = (1-\lambda) f(z) + \lambda g(z) = z - \sum_{m=n+1}^{\infty} c_m z^m, \quad c_m \geq 0$$

is in the class $(k, n, \alpha) - \mathcal{ST}$*.*

Proof. Assume that $f, g \in (k, n, \alpha) - \mathcal{ST}$. Then we have from Theorem 2.1

$$\sum_{m=n+1}^{\infty} [k(m-1)+m-\alpha]\, a_m \leq 1-\alpha$$

and

$$\sum_{m=n+1}^{\infty} [k(m-1)+m-\alpha]\, b_m \leq 1-\alpha.$$

Therefore, we can see that

$$\sum_{m=n+1}^{\infty} [k(m-1)+m-\alpha]\, c_m$$
$$= \sum_{m=n+1}^{\infty} [k(m-1)+m-\alpha]\,[(1-\lambda)a_m + \lambda b_m]$$

$$= (1-\lambda)\sum_{m=n+1}^{\infty}[k(m-1)+m-\alpha]\,a_m$$
$$+\lambda\sum_{m=n+1}^{\infty}[k(m-1)+m-\alpha]\,b_m$$
$$\leq (1-\lambda)(1-\alpha)+\lambda(1-\alpha) = (1-\alpha)$$

which completes the proof of Theorem 2.4.

Definition 2.1. *The modified Hadamard product $f * g$ of two functions*

$$f(z) = z - \sum_{m=n+1}^{\infty} a_m z^m,\ (a_m \geq 0) \quad and \quad g(z) = z - \sum_{m=n+1}^{\infty} b_m z^m,\ (b_m \geq 0)$$

is denoted by

$$(f*g)(z) = z - \sum_{m=n+1}^{\infty} a_m b_m z^m.$$

We now prove the following.

Theorem 2.5. *If $f, g \in (k,n,\alpha) - \mathcal{ST}$, then $(f*g) \in (k,n,\beta) - \mathcal{ST}$, where*

$$\beta = \frac{(1+k)n + 2(1-\alpha) - (1-\alpha)^2}{(1+k)n + 2(1-\alpha)}.$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z - \frac{(1-\alpha)}{(1+k)n + (1-\alpha)]} z^{n+1}$$

where $0 \leq \alpha < 1$ and $0 \leq k < \infty$.

Proof. From Theorem 2.1, we have

$$\sum_{m=n+1}^{\infty} \frac{[k(m-1)+m-\alpha]}{1-\alpha} a_m \leq 1 \quad \text{and}$$
$$\sum_{m=n+1}^{\infty} \frac{[k(m-1)+m-\alpha]}{1-\alpha} b_m \leq 1. \tag{2.4}$$

We have to find the largest β such that

$$\sum_{m=n+1}^{\infty} \frac{[k(m-1)+m-\beta]}{1-\beta} a_m b_m \leq 1. \tag{2.5}$$

From (2.4), we find that

$$\sum_{m=n+1}^{\infty} \frac{[k(m-1)+m-\alpha]}{1-\alpha} \sqrt{a_m b_m} \leq 1. \tag{2.6}$$

Therefore (2.5) is true if

$$\frac{[k(m-1)+m-\beta]}{1-\beta} a_m b_m \leq \frac{[k(m-1)+m-\alpha]}{1-\alpha} \sqrt{a_m b_m}$$

or

$$\sqrt{a_m b_m} \leq \frac{1-\beta}{1-\alpha} \frac{[k(m-1)+m-\alpha]}{[k(m-1)+m-\beta]}.$$

Note that from (2.6)

$$\sqrt{a_m b_m} \leq \frac{1-\alpha}{[k(m-1)+m-\alpha]}.$$

Thus if

$$\frac{1-\alpha}{[k(m-1)+m-\alpha]} \leq \frac{1-\beta}{1-\alpha} \frac{[k(m-1)+m-\alpha]}{[k(m-1)+m-\beta]}$$

or, equivalently, if

$$\beta \leq \frac{[k(m-1)+m-\alpha]^2 - (1-\alpha)^2 [k(m-1)+m]}{[k(m-1)+m-\alpha]^2 - (1-\alpha)^2}$$

then (2.5) is satisfied. Defining the function $\Theta(m)$ by

$$\Theta(m) = \frac{[k(m-1)+m-\alpha]^2 - (1-\alpha)^2 [k(m-1)+m]}{[k(m-1)+m-\alpha]^2 - (1-\alpha)^2}$$

we can see that $\Theta(m)$ is an increasing function of m. Therefore,

$$\beta \leq \Theta(n+1) = \frac{(1+k)n + 2(1-\alpha) - (1-\alpha)^2}{(1+k)n + 2(1-\alpha)}$$

which completes the assertion of theorem.

3 Extreme points for $(k, n, \alpha) - \mathcal{ST}$

Theorem 3.1. *Let* $f_n(z) = z$ *and* $f_m(z) = z - \frac{1-\alpha}{k(m-1)+m-\alpha} z^m$, $m = n+1, n+2, \cdots$. *Then* $f \in (k, n, \alpha) - \mathcal{ST}$ *iff it can be expressed in the form*

$$f(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z),$$

where $\zeta_m \geq 0$ *for* $m \geq n$ *and* $\sum_{m=n}^{\infty} \zeta_m = 1$.

Proof. Assume that

$$f(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z).$$

Then

$$\begin{aligned} f(z) &= \zeta_n f_n(z) + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\ &= \zeta_n z + \sum_{m=n+1}^{\infty} \zeta_m z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m \\ &= \left(\sum_{m=n}^{\infty} \zeta_m\right) z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m \\ &= z - \sum_{m=n+1}^{\infty} \zeta_m \frac{1-\alpha}{k(m-1)+m-\alpha} z^m. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{m=n+1}^{\infty} \zeta_m \left(\frac{1-\alpha}{k(m-1)+m-\alpha}\right)\left(\frac{k(m-1)+m-\alpha}{1-\alpha}\right) \\ &\qquad = \sum_{m=n+1}^{\infty} \zeta_m = \sum_{m=n}^{\infty} \zeta_m - \zeta_n = 1 - \zeta_n \leq 1. \end{aligned}$$

We have $f \in (k, n, \alpha) - \mathcal{ST}$.

Conversely, suppose that $f \in (k, n, \alpha) - \mathcal{ST}$. Since

$$|a_m| \leq \frac{1-\alpha}{k(m-1)+m-\alpha} \quad , \quad m = n+1, n+2, \cdots ,$$

we can set

$$\zeta_m = \frac{k(m-1)+m-\alpha}{1-\alpha} \quad , \quad m = n+1, n+2, \cdots ,$$

and

$$\zeta_n = 1 - \sum_{m=n+1}^{\infty} \zeta_m.$$

Then

$$\begin{aligned}
f(z) &= z - \sum_{m=n+1}^{\infty} a_m z^m \\
&= z - \sum_{m=n+1}^{\infty} \frac{1-\alpha}{k(m-1)+m-\alpha} \zeta_m z^m \\
&= z - \sum_{m=n+1}^{\infty} \zeta_m (z - f_m(z)) \\
&= \left(1 - \sum_{m=n+1}^{\infty} \zeta_m\right) z + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\
&= \zeta_n z + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) \\
&= \zeta_n f_n(z) + \sum_{m=n+1}^{\infty} \zeta_m f_m(z) = \sum_{m=n}^{\infty} \zeta_m f_m(z).
\end{aligned}$$

This completes the assertion of theorem.

Corollary 3.1. *The extreme points of* $(k, n, \alpha) - \mathcal{ST}$ *are given by*

$$f_n(z) = z \quad \textit{and} \quad f_m(z) = z - \frac{1-\alpha}{k(m-1)+m-\alpha} z^m, \quad m = n+1, n+2, \cdots .$$

If we take $n = 1$, $k = 0$ in Corollary 3.1, then we have the following result by Silverman [3].

Corollary 3.2. *The extreme points of* $\mathcal{T}^*(\alpha)$ *are given by*

$$f_1(z) = z \quad \textit{and} \quad f_m(z) = z - \frac{(1-\alpha)}{m-\alpha} z^m, \quad m = 2, 3, \cdots .$$

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