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Error estimates for moving least square approximations

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Abstract. In this paper we obtain error estimates for moving least square approximations for the function and its derivatives. We introduce, at every point of the domain, condition numbers of the star of nodes in the normal equation, which are practically computable and are closely related to the approximating power of the method.

Keywords: multivariate interpolation, moving least square, error estimates.

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1 Introduction

The scattered data fitting problem is encountered frequently in a wide variety of scientific disciplines and has dealt with extensively in the literature. It has been recognized also that such kinds of interpolation could be used in numerical method for PDE for a long time, but the overwhelming success of de FE method has relegated this approach. Recently, the rediscovery of meshless technology has been brought again into the spots the problem of interpolation of data in a set of computational *nodes* 'sprinkled ' through the domain [1, 3, 11].

The moving least square (MLS) as approximation method has been introduced by Shepard [12] in the lowest order case and generalized to higher degree by Lancaster and Salkauskas [7]. The use of MLS in solving PDEs was pioneered by the works of B. Nayroles, T. Belytschko and others [1, 9, 10, 11].

For this kind of applications it is fundamental to analyze the order of approximation, not only for the function itself, but also for its derivatives. In [5] M. Armentano and R. Durán have obtained error estimates in the one dimensional case. Also, D. Levin [8] has analyzed the MLS method for a particular weight

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function obtaining error estimates in the uniform norm for the approximation of a regular function in higher dimensions.

The main object of this work is to prove error estimates for the approximation of the function and the first and second order derivatives.

We use compact supported weight functions that form a Partition of Unity as is usually done in the application in solving PDEs. At every point in the domain, the MLS method use a set of nearby *nodes* (the *star* of the point) for approximating data with a polynomial in a weighted square sense. In order to obtain a good approximation, the *stars* have to satisfy certain geometrical criteria. We define in this work practically computable *condition numbers* of *stars* which are strongly related to the error estimates. Our proof is based in elementary properties of the normal equation.

The paper is organized as follows. First, in Section 2 some preliminaries are exposed. In Section 3 the MLS method is presented. In Section 4 the local normal equation appearing in the MLS method is analyzed in a slightly different way and the *conditions numbers* of *stars* are introduced. Section 5 deals with the error estimates. Finally, in Section 6 we use the error estimates to prove an error estimate in Galerkin coercive problems.

2 Preliminaries

In the n-dimensional space $\mathbb{R}^n$ let $||\,.\,||$ denote the Euclidean norm and $B_r(\mathbf{y})$ denote the open ball $\{\,\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{y}|| < r\,\}$ with center $\mathbf{y}$ and radius r. We use standard multi-index notation. In particular, given any multi-index $\nu = (\nu_1, ..., \nu_n) \in \mathbb{N}^n$, $|\nu|$ denote the sum $\nu_1 + ... + \nu_n$, and, if f is a sufficiently smooth function, $D^\nu f$ denote the partial derivative

$$\frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} ... \partial x_n^{\nu_n}} f.$$

On low order derivatives, we shall also write

$$D^i f = \frac{\partial f}{\partial x_i}, \qquad D^{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \text{etc.}$$

Let Ω be an open bounded domain in $\mathbb{R}^n$ and Q_N denote an arbitrarily chosen set of N points $x_\alpha \in \overline{\Omega}$ referred to as *nodes*:

$$Q_N = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}, \quad \mathbf{x}_\alpha \in \overline{\Omega}$$

Let $\mathcal{I}_N := \{\omega_\alpha\}_{\alpha=1}^N$ denote a finite open covering of $\overline{\Omega}$ consisting of N *clouds* ω_α such that $\mathbf{x}_\alpha \in \omega_\alpha$ and ω_α is "centered" around $\mathbf{x}_\alpha$ in some way, and

$$\overline{\Omega} \subset \bigcup_{\alpha=1}^{N} \omega_\alpha, \tag{1}$$

We define the *radius* d_α of ω_α as $\max_{\mathbf{x}\in\partial\omega_\alpha}\{||\mathbf{x}-\mathbf{x}_\alpha||\}$.

A class of functions $S_N := \{\mathcal{W}_\alpha\}_{\alpha=1}^N$ is called a partition of unity subordinated to the open covering $\mathcal{I}_N$ if it possesses the following properties:

- $\mathcal{W}_\alpha \in C_0^s(\mathbb{R}^n), \quad s \geq 0$ or $s = +\infty$
- $supp(\mathcal{W}_\alpha) \subseteq \overline{\omega}_\alpha$
- $\mathcal{W}_\alpha(\mathbf{x}) > 0, \quad x \in \omega_\alpha$
- $\sum_{\alpha=1}^N \mathcal{W}_\alpha(\mathbf{x}) = 1$, for every $\mathbf{x} \in \overline{\Omega}$.

There is no unique way to build a partition of unity as defined above. A widely used approach in practice is the following:

For each $\alpha = 1, ..., N$, ω_α is an open ball $B_{d_\alpha}(\mathbf{x}_\alpha)$ such that 1 is verified. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ a C^s-function such that $\varphi(\mathbf{x}) > 0$ if $\mathbf{x} \in B_1(\mathbf{0})$ and $supp(\varphi) = \overline{B_1(\mathbf{0})}$. For $\alpha = 1, ..., N$, let define a function ϕ_α by formula

$$\phi_\alpha(\mathbf{x}) = \varphi\left(\frac{\mathbf{x}-\mathbf{x}_\alpha}{d_\alpha}\right)$$

and $\mathcal{W}_\alpha$ by

$$\mathcal{W}_\alpha(\mathbf{x}) = \frac{\phi_\alpha(\mathbf{x})}{\sum \phi_\beta(\mathbf{x})}$$

From now on we shall assume that the compact domain $\overline{\Omega}$ satisfies the condition of regularity:

(R1) There exists a number $\gamma \geq 1$ such that any two points $\mathbf{x}, \mathbf{y}$ in $\overline{\Omega}$ can be joined by a rectifiable curve Γ in $\overline{\Omega}$ with length $|\Gamma| \leq \gamma||\mathbf{x}-\mathbf{y}||$

Following H. Whitney [13], a function $f : \overline{\Omega} \to \mathbb{R}$ is said to be of class C^q in $\overline{\Omega}$ if and only if functions $D^k f(\mathbf{x})$ and $R_k(\mathbf{x};\mathbf{y})$ ($|k| \leq q$) exist in $\overline{\Omega}$ such that Taylor's formula holds:

$$D^k f(\mathbf{x}) = \sum_{|s|\leq q-|k|} \frac{1}{s!} D^{k+s} f(\mathbf{y})(\mathbf{x}-\mathbf{y})^s + R_k(\mathbf{x};\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in \overline{\Omega} \tag{2}$$

The remainder terms $R_k(\mathbf{x};\mathbf{y})$ shall have the following property: given any point $\mathbf{z}\in\overline{\Omega}$ and $\varepsilon>0$, there is a $\delta>0$ such that

$$|R_k(\mathbf{x};\mathbf{y})|\leq\varepsilon||\mathbf{x}-\mathbf{y}||^{q-|k|},\qquad \text{for every}\quad \mathbf{x},\mathbf{y}\in\overline{\Omega},\ ||\mathbf{x}-\mathbf{z}||<\delta,\ ||\mathbf{y}-\mathbf{z}||<\delta \tag{3}$$

(2) and (3) imply that f is continuous in $\overline{\Omega}$ and f has continuous partial derivatives up to order q in Ω satisfying

$$D^k f(\mathbf{x})=\frac{\partial^{|k|}}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}}f(\mathbf{x}),\qquad |k|\leq q,\mathbf{x}\in\Omega$$

A function f is said of class $C^{q,1}$ in $\overline{\Omega}$ if and only if f is of class C^q in $\overline{\Omega}$ and the partial derivatives $D^k f$ of f of order q ($|k|=q$) are Lipschitz continuous in $\overline{\Omega}$. The semi-norm $|\,.\,|_{q,1}$ is defined as

$$|\,f\,|_{q,1}=\sup\left\{\frac{|D^k f(\mathbf{x}_1)-D^k f(\mathbf{x}_2)|}{||\mathbf{x}_1-\mathbf{x}_2||}:\mathbf{x}_1,\mathbf{x}_2\in\overline{\Omega},\ \mathbf{x}_1\neq\mathbf{x}_2,\ |k|=q\right\}$$

The following important estimates of the remainder terms R_k are obtained in [13]:

Lemma 1. *Let $\overline{\Omega}$ satisfy (R1) and let f be of class $C^{q,1}$ in $\overline{\Omega}$. Then, for every $\mathbf{x},\mathbf{y}\in\overline{\Omega}$*

$$|R_k(\mathbf{x};\mathbf{y})|\leq c_{q-|k|}\gamma^{q-|k|}||\mathbf{x}-\mathbf{y}||^{q-|k|+1}|\,f\,|_{q,1} \tag{4}$$

where $c_s=\frac{n^s}{(s-1)!}$ $(s>0)$ or $c_0=1$.

3 The moving least square method

Given data values $\mathbf{f}=(f_\alpha)_{\alpha=1}^N$ at *nodes* x_α, the MLS method produces a function $\widehat{f}\in C^s(\mathbb{R}^n)$ that interpolate data $\mathbf{f}$ in a weighted square sense. Let $\mathcal{P}_q$ the space of polynomial of degree q, $q\ll N$ and $q\leq s$, and let $\mathcal{B}_q=\{p_0,p_1,\ldots,p_m\}$ be any basis of $\mathcal{P}_q$. For each $\mathbf{z}\in\overline{\Omega}$ (fixed) we consider

$$P^*(\mathbf{z},\mathbf{x})=\sum_{0\leq j\leq m}a_j(\mathbf{z})\ p_j(\mathbf{x})$$

where $\left\{a_j(\mathbf{z})\right\}_{0\leq j\leq m}$ are chosen such that

$$J_{\mathcal{B}_q,z}(a)=\sum_{\alpha=1}^N\mathcal{W}_\alpha(\mathbf{z})\left(\sum_{0\leq j\leq m}a_j\ p_j(\mathbf{x}_\alpha)-f_\alpha\right)^2 \tag{5}$$

is minimized. Then, we define the approximation $\widehat{f}$ in z by

$$\widehat{f}(\mathbf{z}) = P^*(\mathbf{z}, \mathbf{z})$$

Observe that the polynomial $P^*(\mathbf{z}, \mathbf{x})$ can be obtained by solving the normal equations for the minimization problem. In fact, if we denote

$$F(\mathcal{B}_q) = \begin{pmatrix} p_0(\mathbf{x}_1) & p_0(\mathbf{x}_2) & \cdots & p_0(\mathbf{x}_N) \\ p_1(\mathbf{x}_1) & p_1(\mathbf{x}_2) & \cdots & p_1(\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_1) & p_m(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_N) \end{pmatrix},$$

$$W(\mathbf{z}) = \begin{pmatrix} \mathcal{W}_1(\mathbf{z}) & 0 & \cdots & 0 \\ 0 & \mathcal{W}_2(\mathbf{z}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_N(\mathbf{z}) \end{pmatrix}$$

then, $\mathbf{a} = (a_0(\mathbf{z}), ..., a_m(\mathbf{z}))$ is the solution of the following system:

$$F(\mathcal{B}_q)W(\mathbf{z})F^T(\mathcal{B}_q)\,\mathbf{a} = F(\mathcal{B}_q)W(\mathbf{z})\,\mathbf{f} \tag{6}$$

In order to have the moving least square approximation well defined we need that the minimization problem has a unique solution at every $\mathbf{z} \in \overline{\Omega}$ and this is equivalent to the non-singularity of matrix $F(\mathcal{B}_q)W(\mathbf{z})F^T(\mathcal{B}_q)$. Our error estimates are obtained with the following assumption about the system of *nodes* and weight functions $\{\mathcal{Q}_N,\ \mathcal{S}_N\}$:

Property $\mathbf{R}_q$: for any $\mathbf{z} \in \overline{\Omega}$, the normal matrix $F(\mathcal{B}_q)W(\mathbf{z})F^T(\mathcal{B}_q)$ is non singular.

Definition 2. *If $\#\mathcal{P}_q = m$, a set of nodes $\{\mathbf{x}_j \in \mathbb{R}^n : j = 1, ..m\}$ is called $\mathcal{P}_q-$unisolvent if the Vandermondian*

$$F(\mathcal{B}_q) = \begin{pmatrix} p_0(\mathbf{x}_1) & p_0(\mathbf{x}_2) & \cdots & p_0(\mathbf{x}_m) \\ p_1(\mathbf{x}_1) & p_1(\mathbf{x}_2) & \cdots & p_1(\mathbf{x}_m) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_1) & p_m(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_m) \end{pmatrix}$$

is non singular.

It is clear that this property does not depends on the basis $\mathcal{B}_q$.

If $\{\mathbf{x}_j\}_{j=1,...m}$ is $\mathcal{P}_q-$*unisolvent*, we can choose a basis $\mathcal{B}_q = \{q_0, q_1, ..., q_m\}$ of $\mathcal{P}_q$ such that $q_i(\mathbf{x}_j) = \delta_{i,j}$.

Definition 3. *Given* $\mathbf{z} \in \overline{\Omega}$, *the set* $\mathcal{ST}(\mathbf{z}) := \{\alpha | \mathcal{W}_\alpha(\mathbf{z}) \neq 0\}$ *will be called the star at* $\mathbf{z}$.

The next theorem, proved in [14], give us necessary and sufficient conditions for the satisfaction of property $\mathbf{R}_q$ in a stable way.

Theorem 4. *A necessary condition for the satisfaction of* ***Property*** $\boldsymbol{R}_q$ *is that for any* $\mathbf{z} \in \overline{\Omega}$

$$\#\mathcal{ST}(\mathbf{z}) \geq \#\mathcal{P}_q = m$$

If in addition to the above condition the set

$$\{\mathbf{x}_{\alpha_k} \mid \alpha_k \in \mathcal{ST}(\mathbf{z})\}$$

contains a $\mathcal{P}_q$*−unisolvent subset , then the above condition is also sufficient.*

The case $q = 1$ and $n = 2$ was proved also in [4]. It should be remarked that a related issue was considered by W. Han and X. Meng [6] in the context of approximations based also on partition of unity.

Let $\mathcal{F} := \mathbb{R}^N$ be the set of possible values $\mathbf{f} = (f_\alpha)_{\alpha=1}^N$ of functions at the *nodes* x_α. With the assumption above, the MLS method provides an operator $\mathcal{A} : \mathcal{F} \to C^s(\overline{\Omega})$ defined by

$$\mathcal{A}(\mathbf{f})(\mathbf{z}) = \widehat{f}(\mathbf{z}), \qquad \mathbf{f} \in \mathcal{F}, \quad \mathbf{z} \in \overline{\Omega}$$

This is not an interpolation operator in the sense that, in general, $\mathcal{A}(\mathbf{f})(\mathbf{x}_\alpha) \neq f_\alpha$.

Given a function $f \in C^p(\overline{\Omega})$, the associated vector in $\mathcal{F}$ is $\mathbf{f} = (f(\mathbf{x}_\alpha))_{\alpha=1}^N$ and we will write $\mathcal{A}(f)$ for $\mathcal{A}(\mathbf{f})$. The operator $\mathcal{A}$ is linear and $q - reproductive$, that is, $\mathcal{A}(P) = P$ when P is a polynomial of degree q.

Property $\mathbf{R}_q$ does not depend on the basis of $\mathcal{P}_q$ and this property will play a fundamental role in our work. In fact, if $\mathcal{A}_q = \{q_0, q_1, ..., q_m\}$ is another basis of $\mathcal{P}_q$ such that $\mathcal{B}_q = G\mathcal{A}_q$, G a non-singular matrix, then

$$J_{\mathcal{A}_q,z} = J_{\mathcal{B}_q,z} \circ G, \qquad \mathbf{z} \in \overline{\Omega}$$

and $J_{\mathcal{A}_q,z}$ has a unique minimum if and only if $J_{\mathcal{B}_q,z}$ does.

Therefore, in analyzing the normal equation in a neighborhood of a given point $\mathbf{z} \in \overline{\Omega}$, we can choose a convenient basis at this point. For our work, this basis will be the Taylor monomial centered at $\mathbf{z}$:

$$\mathcal{T}_{\mathbf{z}}^q = \{(\mathbf{x} - \mathbf{z})^\eta\}_{0 \leq |\eta| \leq q}$$

The following results follows easily from the considerations above.

Proposition 5. *The system $\{\mathcal{Q}_N,\ S_N\}$ satisfies* ***Property*** $\mathbf{R}_q$ *if and only if the matrix $F(\mathcal{T}_{\mathbf{z}}^q)W(\mathbf{z})F^T(\mathcal{T}_{\mathbf{z}}^q)$ is non singular at each $\mathbf{z}\in\overline{\Omega}$.*

Remark 6. If $1 \leq p < q$, the satisfaction of **Property** $\mathbf{R}_q$ implies the satisfaction of **Property** $\mathbf{R}_p$. This easily proved fact will be useful later.

Assumption. In what follows, we shall assume that $q \leq 2$. These cases are the most used in practice.

3.1 The derivatives of $\mathcal{A}(\mathbf{f})$

Given $\mathbf{f} \in \mathcal{F}$, for each $\mathbf{c} \in\overline{\Omega}$ (fixed) we want to make explicit, following [7], the formulas of the derivatives of $\mathcal{A}(\mathbf{f})$ at $\mathbf{c}$ that will be useful in future calculations.

In all what follow we will use the basis $\mathcal{T}_{\mathbf{c}}^q = \{(\mathbf{x}-\mathbf{c})^\eta\}_{0\leq|\eta|\leq q}$ of $\mathcal{P}_q$, $q = 1, 2$, and, in order to simplify notation, we will drop any reference to this basis in the normal equation. Therefore, we have

$$\mathcal{A}(\mathbf{f})(\mathbf{x}) = \sum_{0\leq|\eta|\leq q} a_\eta(\mathbf{x})(\mathbf{x}-\mathbf{c})^\eta \tag{7}$$

where $\mathbf{a} = (a_\eta(\mathbf{x}))_{0\leq|\eta|\leq q}$ is the solution of:

$$FW(\mathbf{x})F^T\,\mathbf{a} = FW(\mathbf{x})\,\mathbf{f} \tag{8}$$

In order to calculate the values $D^\eta\mathcal{A}(\mathbf{f})(\mathbf{c})$, $0 \leq |\eta| \leq q$, it is useful to use the following notation:

- $\mathbf{0} \in\mathbb{R}^N$ is the multi-index $(0, 0, ..., 0)$.
- For $i = 1, ..., n$, $\mathbf{e}_i$ is the multi-index with $|\mathbf{e}_i| = 1$, $\mathbf{e}_i = (0, ..., 1, ..., 0)$, with 1 in the ith place.
- For $i = 1, ..., n$, $\mathbf{e}_{ii}$ is the multi-index with $|\mathbf{e}_{ii}| = 2$, $\mathbf{e}_{ii} = (0, ..., 2, ..., 0)$, with 2 in the ith place, and for $j = i+1, ..., n$, $\mathbf{e}_{ij}$ is the multi-index with $|\mathbf{e}_{ij}| = 2$, $\mathbf{e}_{ij} = (0, ..., 1, ..., 1, ..., 0)$, with 1 in the ith and jth places.

First at all,

$$\mathcal{A}(\mathbf{f})(\mathbf{c}) = a_0(\mathbf{c}) \tag{9}$$

Then, for $i = 1, ..., n$, we have

$$D^{\mathbf{e}_i}\mathcal{A}(\mathbf{f})(\mathbf{c}) = D^{\mathbf{e}_i} a_0(\mathbf{c}) + a_{\mathbf{e}_i}(\mathbf{c}) \tag{10}$$

while we can get $D^{\mathbf{e}_i} a_{\mathbf{0}}(\mathbf{c})$ from the solution $\mathbf{a}_i = (D^{\mathbf{e}_i} a_\eta(\mathbf{c}))_{0\le|\eta|\le q}$ of

$$FW(\mathbf{c})F^T(\mathbf{a}_i) = F(D^{\mathbf{e}_i} W(\mathbf{c}))\,(\mathbf{f} - F^T\mathbf{a}) \tag{11}$$

Our next goal is to calculate the second order derivatives.
For $i = 1, ..., n$ and $j = i+1, .., n$, we have

$$\begin{aligned} D^{\mathbf{e}_{ii}}\mathcal{A}(\mathbf{f})(\mathbf{c}) &= D^{\mathbf{e}_{ii}} a_{\mathbf{0}}(\mathbf{c}) + 2D^{\mathbf{e}_i} a_{\mathbf{e}_i}(\mathbf{c}) + a_{\mathbf{e}_{ii}}(\mathbf{c}) \\ D^{\mathbf{e}_{ij}}\mathcal{A}(\mathbf{f})(\mathbf{c}) &= D^{\mathbf{e}_{ij}} a_{\mathbf{0}}(\mathbf{c}) + D^{\mathbf{e}_i} a_{\mathbf{e}_j}(\mathbf{c}) + D^{\mathbf{e}_j} a_{\mathbf{e}_i}(\mathbf{c}) + a_{\mathbf{e}_{ij}}(\mathbf{c}) \end{aligned} \tag{12}$$

So, it remains to show how to get $D^{\mathbf{e}_{ij}} a_0(\mathbf{c})$, $i = 1, ...n$ and $j = i, ..., n$. Applying operator $D^{\mathbf{e}_j}$ to both side of (11) one can easily get:

$$\begin{aligned} FW(\mathbf{c})F^T(D^{\mathbf{e}_{ij}}\,\mathbf{a}) &= F(D^{\mathbf{e}_{ij}} W(\mathbf{c}))\,(\mathbf{f} - F^T\mathbf{a}) \\ &\quad - F(D^{\mathbf{e}_i}\, W(\mathbf{c}))F^T(D^{\mathbf{e}_j}\,\mathbf{a}) \\ &\quad - F(D^{\mathbf{e}_j}\, W(\mathbf{c}))F^T(D^{\mathbf{e}_i}\,\mathbf{a}) \end{aligned} \tag{13}$$

We have now all machinery needed to calculate the derivatives up to order 2 of $\mathcal{A}(\mathbf{f})$ at $\mathbf{c}$.

3.2 The Star of nodes at a point $\mathbf{c} \in \overline{\Omega}$

As it is well know, in working with the normal equation (8) and all related equations, one can consider only those *nodes* $\mathbf{x}_\alpha$ or indexes α such that $\mathcal{W}_\alpha(\mathbf{c}) \neq 0$, that is, the *star* $\mathcal{ST}(\mathbf{c})$. If $\mathcal{ST}(\mathbf{c}) = \{\alpha_1, ..., \alpha_k\}$, matrices F, W and $\mathbf{f}$ can be considered as

$$F = \begin{pmatrix} p_0(\mathbf{x}_{\alpha_1}) & p_0(\mathbf{x}_{\alpha_2}) & \cdots & p_0(\mathbf{x}_{\alpha_k}) \\ p_1(\mathbf{x}_{\alpha_1}) & p_1(\mathbf{x}_{\alpha_2}) & \cdots & p_1(\mathbf{x}_{\alpha_k}) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\mathbf{x}_{\alpha_1}) & p_m(\mathbf{x}_{\alpha_2}) & \cdots & p_m(\mathbf{x}_{\alpha_k}) \end{pmatrix}$$

$$W = \begin{pmatrix} \mathcal{W}_{\alpha_1}(\mathbf{c}) & 0 & \cdots & 0 \\ 0 & \mathcal{W}_{\alpha_2}(\mathbf{c}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{W}_{\alpha_k}(\mathbf{c}) \end{pmatrix}$$

$$\mathbf{f} = (f_{\alpha_1}, f_{\alpha_2}, ..., f_{\alpha_k})$$

and so on. Also in calculating derivatives up to order 2 of $\mathcal{A}(\mathbf{f})$ at $\mathbf{c}$ we can use this setting, as we have shown in above.

The *mesh size* of the *star* $\mathcal{ST}(\mathbf{c})$ is defined by the number $h(\mathcal{ST}(\mathbf{c})) := \max\{d_{\alpha_1}, ..., d_{\alpha_k}\}$.

4 The condition numbers of the star $\mathcal{ST}(\mathbf{c})$

In all of this section $\mathbf{c} \in \overline{\Omega}$ is a fixed point and $\mathcal{ST}(\mathbf{c})$ is the *star* at c. In order to gain clearness, we shall drop subscript α from the weight functions and *nodes* in the *star*. Then, for $i = 1, ..., k$, $\mathcal{W}_i$ means $\mathcal{W}_{\alpha_i}$, $\mathbf{x}_i$ means $\mathbf{x}_{\alpha_i}$, $x_{i,j}$ is the jth coordinate of $\mathbf{x}_{\alpha_i}$, etc. It will be also useful to introduce a linear change of coordinates by the formula $\mathbf{y} = \mathbf{x} - \mathbf{c}$. We set also $h = h(\mathcal{ST}(\mathbf{c}))$.

We now record the fundamental result which lead directly to the error estimates described in the next Section.

Theorem 7. *There exist numbers $CN_s(\mathcal{ST}(\mathbf{c}))$, $s = 1, ...q$, which are computable measures of the quality of the star $\mathcal{ST}(\mathbf{c})$, and constants C_s, $s = 1, ...q$, $C_s = C(n, k, CN_1(\mathcal{ST}(\mathbf{c})), ..., CN_q(\mathcal{ST}(\mathbf{c})))$, such that*

$$|a_\eta| \leq C_q\, h^{-|\eta|}\, ||V||, \quad 0 \leq |\eta| \leq q \tag{14}$$

where $V \in \mathbb{R}^k$, and $\mathbf{a} = (a_\eta)_{1\leq|\eta|\leq q}$ is the solution of:

$$FWF^T\mathbf{a} = FV \tag{15}$$

Proof. We will consider only the case $q = 2$.

We shall arrive at the proof of the theorem and the practical meaning of $CN_s(\mathcal{ST}(\mathbf{c}))$ in several steps:

Step I

Let introduce the spherical coordinate system :

$$\begin{aligned}
y_1 &= r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1\\
y_2 &= r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_2\cos\theta_1\\
y_3 &= r\sin\theta_{n-1}\sin\theta_{n-2}\cdots\sin\theta_3\cos\theta_2\\
&\vdots\\
y_{n-1} &= r\sin\theta_{n-1}\cos\theta_{n-2}\\
y_n &= r\cos\theta_{n-1}
\end{aligned}$$

In this coordinate system, we write for the point $\mathbf{y} = \mathbf{x} - \mathbf{c}$ the symbols (r, θ). We introduce also numbers ρ_l, $l = 1, ..., k$ such that $r_l = ||\mathbf{y}_l|| = \rho_l \cdot h$. Using these symbols, it is clear that every element A_{ij} of matrix $A = FWF^T$ can be rewritten

$$A_{ij} = h^{p_{ij}} \cdot a_{ij}(\rho, \theta)$$

for some integer p_{ij} that we will call the *order* of A_{ij}. We do the same for elements of matrix F

$$F_{ij} = h^{p_{ij}} \cdot f_{ij}(\rho, \theta)$$

Taking care of the *order* of elements, it is clear that we can partitioning A as follows

$$A = \begin{pmatrix} 1 & h\,\widehat{A}_{12}(\rho, \theta) & h^2 \widehat{A}_{13}(\rho, \theta) \\ h\,\widehat{A}_{21}(\rho, \theta) & h^2\,\widehat{A}_{22}(\rho, \theta) & h^3\,\widehat{A}_{23}(\rho, \theta) \\ h^2\,\widehat{A}_{31}(\rho, \theta) & h^3\,\widehat{A}_{32}(\rho, \theta) & h^4\,\widehat{A}_{33}(\rho, \theta) \end{pmatrix} \tag{16}$$

where $\widehat{A}_{12}$ is a $1 \times n-$ matrix , $\widehat{A}_{13}$ is a $1 \times \left(\frac{n(n+1)}{2}\right) -$matrix and so on.

For matrix F , we get:

$$F = \begin{pmatrix} \widehat{F}_0 \\ h\,\widehat{F}_1(\rho, \theta) \\ h^2\,\widehat{F}_2(\rho, \theta) \end{pmatrix} \tag{17}$$

where $\widehat{F}_0$ is the $1 \times k-$ matrix $(1, 1, ..., 1)$, $\widehat{F}_1$ is an $n \times k-$matrix, and $\widehat{F}_2$ is an $\left(\frac{n(n+1)}{2}\right) \times k-$matrix.

Step II

Our next goal is to proceed to the first steps of Gaussian elimination with the standard pivot A_{11} in order to make the subdiagonal elements of the first column equals to 0. We left multiply A by the matrix

$$G_1 = \begin{pmatrix} 1 & 0 & & 0 \\ -A_{21} & 1 & & \\ \vdots & & \ddots & 0 \\ -A_{u1} & 0 & & 1 \end{pmatrix}$$

where we have denoted $u = 1 + n + \left(\frac{n(n+1)}{2}\right)$. Of course, since we are dealing with equation (15), we must make also the multiplication $G_1 F$. Then, system (15) is transformed in

$$\overline{A}\mathbf{a} = \overline{F}V \tag{18}$$

where $\overline{A} = G_1 F W F^T$ and $\overline{F} = G_1 F$.

The central point to be remarked about this multiplication and subsequent matrix multiplications is that this process does not change *orders* of elements.

After the multiplication above, system (15) can be written

$$\begin{pmatrix} 1 & h\,\overline{A}_{12} & h^2\,\overline{A}_{13} \\ 0 & h^2\,\overline{A}_{22} & h^3\,\overline{A}_{23} \\ 0 & h^3\,\overline{A}_{32} & h^4\,\overline{A}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \overline{F}_0 \\ h\,\overline{F}_1 \\ h^2\,\overline{F}_2 \end{pmatrix} V \tag{19}$$

where, for the sake of simplicity, we have eliminated in notations the dependence on (ρ, θ).

Since matrix $\overline{A}_{22}$ must be non singular, we can left multiplies the last equality by matrix

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{A}_{22}^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$$

getting

$$\begin{pmatrix} 1 & h\,\overline{A}_{12} & h^2\,\overline{A}_{13} \\ 0 & h^2\,I & h^3(\overline{A}_{22}^{-1}\,\overline{A}_{23}) \\ 0 & h^3\,\overline{A}_{32} & h^4\,\overline{A}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \overline{F}_0 \\ h\,(\overline{A}_{22}^{-1}\overline{F}_1) \\ h^2\,\overline{F}_2 \end{pmatrix} V \tag{20}$$

Our next goal is to make the matrix $h^3\overline{A}_{32}$ equal to the zero matrix by a Gaussian reduction process, that is, we left multiply both sides of equation by the matrix

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & -h\overline{A}_{32} & I \end{pmatrix}$$

and the system can be wrtten now

$$\begin{pmatrix} 1 & h\,\overline{A}_{12} & h^2\,\overline{A}_{13} \\ 0 & h^2\,I & h^3(\overline{A}_{22}^{-1}\,\overline{A}_{23}) \\ 0 & 0 & h^4\,\widetilde{A}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix} = \begin{pmatrix} \overline{F}_0 \\ h\,(\overline{A}_{22}^{-1}\overline{F}_1) \\ h^2\,(\overline{F}_2 - \overline{A}_{32}\overline{A}_{22}^{-1}\overline{F}_1) \end{pmatrix} V \tag{21}$$

where

$$\widetilde{A}_{33} = \overline{A}_{33} - \overline{A}_{32}\overline{A}_{22}^{-1}\overline{A}_{23}$$

and $\widetilde{A}_{33}$ is non singular.

Solving (21), we get

$$\begin{aligned} \mathbf{a}_2 &= h^{-2}\,\widetilde{A}_{33}^{-1}(\overline{F}_2 - \overline{A}_{32}\overline{A}_{22}^{-1}\overline{F}_1)V = h^{-2}\widetilde{F}_2 V \\ \mathbf{a}_1 &= h^{-1}\left(\overline{A}_{22}^{-1}\overline{F}_1 - (\overline{A}_{22}^{-1}\overline{A}_{23})\widetilde{F}_2\right) = h^{-1}\widetilde{F}_1 V \\ \mathbf{a}_0 &= \left(\overline{F}_0 - \overline{A}_{12}\widetilde{F}_1 - \overline{A}_{13}\widetilde{F}_2\right) V = \widetilde{F}_0 V \end{aligned} \tag{22}$$

Moreover, using the fact that members of the original matrix have all the form

$$\sum_{i=1}^{k} W_i g_{st}(\rho,\theta)$$

with $|g_{st}(\rho,\theta)| \leq 1$, one can arrive soon to the conclusion that there exist a constant $K = K(n,k)$ such that

$$\max\{||\overline{A}_{12}||, ||\overline{A}_{13}||, ||\overline{A}_{23}||, ||\overline{F}_0||, ||\overline{F}_1||, ||\overline{F}_2||\} \leq K$$

We can understand immediately the relevance of the condition numbers of the non singular matrices $\overline{A}_{22}(\rho,\theta)$ and $\widetilde{A}_{33}(\rho,\theta)$.

If we define the numbers $CN_s(\mathcal{ST}(\mathbf{c}))$, $s = 1, \ldots q$, by formulas:

$$CN_1(\mathcal{ST}(\mathbf{c})) := ||(\overline{A}_{22}(\rho,\theta)^{-1}||$$
$$CN_2(\mathcal{ST}(\mathbf{c})) := ||(\widetilde{A}_{33}(\rho,\theta)^{-1}||$$

The proof of the theorem follows by applying all our estimates in (22). □

Several interesting question related to $CN_s(\mathcal{ST}(\mathbf{c}))$ are:

1. In practical calculation, we did not need to introduce spherical coordinates as in Step I. For a matrix element A_{ij} of *order* p, we have

   $$A_{ij} = h^p \cdot a_{ij}(\rho,\theta)$$

   but this term is clearly equal to

   $$h^p \cdot \left(\frac{A_{ij}(\mathbf{y})}{h^p}\right)$$

   i.e., $a_{ij}(\rho,\theta) = \left(\frac{A_{ij}(\mathbf{y})}{h^p}\right)$. Similarly for matrix F. Having doing the appropriate division on matrices FWF^T and F, we can process directly at Step II.

2. $CN_s(\mathcal{ST}(\mathbf{c}))$ depend only on (ρ, θ). Then, they are invariant by transformations $\mathbf{x} \to \lambda(\mathbf{x} - \mathbf{c}) + \mathbf{c}$, $\lambda \in \mathbb{R}$. That is, $CN_s(\mathcal{ST}(\mathbf{c}))$ are really measures of the geometrical quality of the *star*, independently of his width. We will pursue this interesting point in a forthcoming paper.

3. As it is naturally expected, in general $(\overline{A}_{22})^{-1}$ has a more well behavior than $(\widetilde{A}_{33})^{-1}$. In very unstructured or unsymmetrical *star*, the condition number of $\widetilde{A}_{33}$ can be very large. For example, in dimension two and triangular structured meshes like that used in Finite Element, $CN_s(\mathcal{ST}(\mathbf{c}))$ are very stable at interior points. I have obtain a mean value of

$$cond(\overline{A}_{22}) = 1.28$$
$$cond(\widetilde{A}_{33}) = 4.32$$

with uniform *meshsize* equal to 0.2, 0.1, 0.05. But it is at boundary points, as expected, where $cond(\widetilde{A}_{33})$ can be very high, even when matrix A is non-singular and additional precautions must be taken. We shall pursue this geometrical question in a work in progress. However, in meshless methods using moving least square, $q = 1$ is used and in this case the behaviour is stable.

5 Error estimates for MLS

Our next goal is to apply theorem 7, joined to the results of Subsection 3.1, in order to obtain estimates of $D^\eta \mathcal{A}(\mathbf{f})(\mathbf{c})$, $0 \le |\eta| \le q$ and $\mathbf{f} \in \mathcal{F}$. As we have remarked before, only data $(f_1, ..., f_k)$ at *nodes* of the *star* participate in this calculations.

Proposition 8. *Let $G_q > 0$ a constant such that*

$$|D^\eta \mathcal{W}_i(\mathbf{c})| \le \frac{G_q}{h^{|\eta|}}, \; i = 1, ..., k \text{ and } 1 \le |\eta| \le q$$

and let data $\mathbf{f} = (f_1, ..., f_k)$ be written as

$$\mathbf{f} = h^{q+1}\mathbf{g}.$$

Then, there exist constants $\widetilde{C}_q$, $q = 1$ or 2,

$$\widetilde{C}_1 = \widetilde{C}_1(n, k, C_q, CN_1(\mathcal{ST}(\mathbf{c}))),$$
$$\widetilde{C}_2 = \widetilde{C}_2(n, k, C_q, C_H, CN_1(\mathcal{ST}(\mathbf{c})), CN_2(\mathcal{ST}(\mathbf{c})))$$

such that

$$|D^\eta \mathcal{A}(\mathbf{f})(\mathbf{c})| \le \widetilde{C}_q \, h^{q+1-|\eta|} \, ||\mathbf{g}||, \qquad 0 \le |\eta| \le q$$

Proof. The proof of this result consists in a iterative application of theorem 7 over formulas (9), (10), (11), (12), (13).

Case $|\eta| = 0$ is easy. In fact, $\mathcal{A}(\mathbf{f})(\mathbf{c}) = a_0$ where a_0 is the first coordinate of the solution of the system $FWF^T\mathbf{a} = FW(h^{q+1}\mathbf{g})$. Written $V = W(h^{q+1}\mathbf{g})$ and applying theorem 3, we get

$$|a_\eta| \leq C_q h^{q+1-|\eta|} ||\,\mathbf{g}|| \tag{23}$$

We shall prove now the case $|\eta| = 1$.

Calculations in Subsection 3.1 show that, for $i = 1, ..., n$, we have

$$D^{\mathbf{e}_i}\mathcal{A}(\mathbf{f})(\mathbf{c}) = D^{\mathbf{e}_i} a_0(\mathbf{c}) + a_{\mathbf{e}_i}(\mathbf{c})$$

By (23), it follows that

$$|a_{\mathbf{e}_i}(\mathbf{c})| \leq C_q h^q \,||\,\mathbf{g}||$$

while we can get $D^{\mathbf{e}_i} a_{\mathbf{0}}(\mathbf{c})$ from the first coordinate of the solution $\mathbf{a}_i = (D^{\mathbf{e}_i} a_\eta(\mathbf{c}))_{0\leq|\eta|\leq q}$ of system

$$FW(\mathbf{c})F^T(\mathbf{a}_i) = F(D^{\mathbf{e}_i} W(\mathbf{c}))\,(\mathbf{f} - F^T\mathbf{a})$$

Written now $V = (D^{\mathbf{e}_i} W(\mathbf{c}))\,(\mathbf{f} - F^T\mathbf{a})$, theorem 7 implies that we only need an estimate of the form

$$||V|| \leq D_q\, h^q \,||\,\mathbf{g}\,||$$

The coordinates of vector $F^T\mathbf{a}$ are:

$$(F^T\mathbf{a})_i = \sum_{0\leq|\eta|\leq q} a_\eta(\mathbf{c})(\mathbf{x}_i - \mathbf{c})^\eta$$

so that, by 23, we have

$$|(F^T\mathbf{a})_i| \leq \sum_{0\leq|\eta|\leq q} \left(C_q h^{q+1-|\eta|}||\,\mathbf{g}||\right)\left(h^{|\eta|}\right) \leq D(n, C_q)\, h^{q+1}\,||\,\mathbf{g}||$$

Now, we obtain

$$||V|| \leq ||(D^{\mathbf{e}_i} W(\mathbf{c}))|| \cdot ||\,(\mathbf{f} - F^T\mathbf{a})|| \leq ||(D^{\mathbf{e}_i} W(\mathbf{c}))||\,\widetilde{D}(n, C_q)\, h^{q+1}\,||\,\mathbf{g}||$$

On the other hand, the norm of the diagonal matrix $(D^{\mathbf{e}_i} W(\mathbf{c}))$ can be estimate by the maximum of the absolute values of the diagonal, that is

$$||(D^{\mathbf{e}_i} W(\mathbf{c}))|| \leq K \max_{s=1,\ldots k}\{|D^{\mathbf{e}_i} W_s(\mathbf{c})|\} \leq K G_q\, h^{-1}$$

Finally, we get

$$||V|| \leq K G_q \, h^{-1} \widetilde{D}(n, C_q) \, h^{q+1} \, ||\mathbf{g}|| = D_q h^q || \mathbf{g} ||$$

Case $|\eta| = 2$ requires more lengthy calculations and considerable stamina, but follows easily along the similar lines. □

Now, we are ready to establish our main local error estimate:
Given $f \in C^{q,1}(\overline{\Omega})$ and $\mathbf{c} \in \overline{\Omega}$, we will concern with the problem of estimating

$$|D^\eta f(\mathbf{c}) - D^\eta \mathcal{A}(f)(\mathbf{c})|, \qquad 0 \leq |\eta| \leq q \tag{24}$$

Writing

$$f(\mathbf{x}) = P_{\mathbf{c}}^q(\mathbf{x}) + R_0(\mathbf{c}; \mathbf{x})$$

where $P_{\mathbf{c}}^q$ is the q-order Taylor polynomial of f at $\mathbf{c}$. By lemma 1 we have

$$|R_0(\mathbf{c}; \mathbf{x})| \leq c_q \gamma^q ||\mathbf{x} - \mathbf{c}||^{q+1} | \, f \, |_{q,1} \tag{25}$$

Since MLS procedure is q-reproductive and it is a linear operator of data, we can write

$$\mathcal{A}(f) = P_{\mathbf{c}}^q + \mathcal{A}(R_0)$$

The $\eta-$derivative of $\mathcal{A}(f)$ at $\mathbf{c}$ is then

$$D^\eta \mathcal{A}(f)(\mathbf{c}) = D^\eta P_{\mathbf{c}}^q(\mathbf{c}) + D^\eta \mathcal{A}(R_0)(\mathbf{c})$$

and, because $D^\eta P_{\mathbf{c}}^q(\mathbf{c}) = D^\eta f(\mathbf{c})$, we have

$$D^\eta \mathcal{A}(f)(\mathbf{c}) - D^\eta f(\mathbf{c}) = D^\eta \mathcal{A}(R_0)(\mathbf{c}) \qquad 0 \leq |\eta| \leq q$$

On the other hand, estimate (25) implies that, we have the estimate

$$|(\mathbf{R}_0)_i| = |R_0(\mathbf{c}; \mathbf{x}_i)| \leq c_q \gamma^q ||\mathbf{x} - \mathbf{c}||^{q+1} | \, f \, |_{q,1} \leq c_q \gamma^q \, [h(\mathcal{ST}(\mathbf{c}))]^{q+1} | \, f \, |_{q,1}$$

over the *star* $\mathcal{ST}(\mathbf{c}) = \{\mathbf{x}_1, ..., \mathbf{x}_k\}$. Then, we can write

$$\mathbf{R}_0 = [h(\mathcal{ST}(\mathbf{c}))]^{q+1} \cdot \mathbf{f}$$

with $||\mathbf{f}|| \leq c_q \gamma^q | \, f \, |_{q,1}$. Our local fundamental results as an immediate corollary of Proposition 8.

Theorem 9. *Given* $\mathbf{c} \in \overline{\Omega}$ *and a constant* $G_q > 0$ *such that*

$$|D^\eta \mathcal{W}_i(\mathbf{c})| \leq \frac{G_q}{h^{|\eta|}}, \ i = 1, ..., k \text{ and } 1 \leq |\eta| \leq q$$

there exist constants C_q, $q = 1$ *or* 2,

$$C_1 = C_1(c_q, \gamma, n, k, C_q, CN_1(\mathcal{ST}(\mathbf{c}))),$$
$$C_2 = C_2(c_q, \gamma, n, k, C_q, C_H, CN_1(\mathcal{ST}(\mathbf{c})), CN_2(\mathcal{ST}(\mathbf{c})))$$

such that, for each $f \in C^{q,1}(\overline{\Omega})$

$$|D^\eta f(\mathbf{c}) - D^\eta \mathcal{A}(f)(\mathbf{c})| \leq C_q\,[h(\mathcal{ST}(\mathbf{c}))]^{q+1-|\eta|}\,|\,f\,|_{q,1}, \qquad 0 \leq |\eta| \leq q$$

This theorem make emphasis in the local character of the MLS approximation. Of course, if we make global assumptions about parameters, one can easily get global error estimates along the same lines of the local theorem.

Assumption G. We impose the following conditions:
There exist

(H0) An upper bound of the overlap of *clouds:*

$$M = \sup_{\mathbf{c} \in \overline{\Omega}} \#\{\mathcal{ST}(\mathbf{c})\}$$

(H1) Upper bounds of the condition number:

$$CB_q = \sup_{\mathbf{c} \in \overline{\Omega}} \{CN_q(\mathcal{ST}(\mathbf{c}))\}, \qquad q = 1, 2$$

(H2) An upper bound of the *meshsize* of *stars*:

$$d = \sup_{\mathbf{c} \in \overline{\Omega}} \{h(\mathcal{ST}(\mathbf{c}))\}$$

(H3) An uniform bound of the derivatives of $\{\mathcal{W}_\alpha\}$. That is, a constant $G_q > 0$, $q = 1,2$, such that

$$||D^\eta \mathcal{W}_\alpha||_{L^\infty} \leq \frac{G_q}{h^{|\eta|}}, \qquad 1 \leq |\eta| \leq q$$

Assuming all these conditions, we can prove:

Theorem 10. *There exist constants* C_q, $q = 1$ *or* 2,

$$C_1 = C_1(c_q, \gamma, n, M, G_1, CB_1),$$
$$C_2 = C_2(c_q, \gamma, n, M, C_2, CB_1, CB_2)$$

such that, for each $f \in C^{q,1}(\overline{\Omega})$

$$||D^\eta f - D^\eta \mathcal{A}(f)||_{L^\infty(\Omega)} \leq C_q\, d^{q+1-|\eta|}\, |\, f\,|_{q,1}, \qquad 0 \leq |\eta| \leq q$$

This situation arises in considering uniform system $\{\mathcal{Q}_N,\ S_N\}$. However, as we have remarked before, the number $CN_2(\mathcal{ST}(\mathbf{c}))$ can be very high near the boundary points. This drawback can degrade appreciatively the global error estimate when $q = 2$.

6 Error estimates in Galerkin approximations

Our goal in this Section is to present a simple example of obtaining error estimates in Galerkin approximations using MLS. We shall follows [5].

Let $\{\mathcal{Q}_N,\ S_N\}$ be a system where **Property** $\mathbf{R}_1$ holds. Then, the MLS approximation is a linear operator

$$\mathcal{A} : \mathcal{F} \to C^s(\overline{\Omega})$$

Let $\{\mathcal{E}_\alpha\}_{\alpha=1}^N$ the natural basis of $\mathcal{F}$, that is, $\mathcal{E}_\alpha := (0, ..., 1., 0, ..., 0)$ with 1 in the α*-node.* Defining functions $\varphi_\alpha := \mathcal{A}(\mathcal{E}_\alpha)$, $\alpha = 1, ..., N$, it is not difficult to see that $\{\varphi_\alpha\}_{\alpha=1}^N$ it is also a partition of unity and

$$\mathcal{A}(\mathbf{f}) = \sum_{\alpha=1}^{N} f_\alpha\, \varphi_\alpha,\ \ \forall\, \mathbf{f} \in \mathcal{F}$$

Given the following variational problem: find $u \in V \subset H^1(\Omega)$ such that

$$B(u, v) = L(v) \qquad \forall v \in V,$$

where B is a bilinear, continuous and coercive on V and L is a linear continuous operator, we can use the MLS method to define Galerkin approximation in the following way [1, 11]:

Assuming that $\varphi_\alpha \in V$, $\alpha = 1, ..., N$, let $V_d = span\{\varphi_1, ..., \varphi_N\}$. Therefore we can define the Galerkin approximation $\widehat{u} \in V_d$ of the real solution u as

$$\widehat{u}(\mathbf{x}) = \sum_{\alpha=1}^{N} u_\alpha \varphi_\alpha(\mathbf{x})$$

where $u_1, ..., u_N$ is the solution of the following system

$$\sum_{\beta=1}^{N} B(\varphi_\alpha, \varphi_\beta)\, u_\beta = L(\varphi_\alpha), \qquad 1 \leq \alpha \leq N$$

If $u \in C^{q,1}(\overline{\Omega})$ and assumption G above holds, then from Céa's lemma [2] and Theorem 10 we have the following error estimate:

$$||u - \widehat{u}||_V \leq \frac{K}{\lambda} \min_{v \in V_d} ||u - v||_V \leq \frac{K}{\lambda} ||u - \mathcal{A}(u)|| \leq C\, d^q \mid u \mid_{q,1}$$

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