Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol.32, No. 2, 107-144

Bifurcation of the essential dynamics of Lorenz maps and applications to Lorenz-like flows: contributions to the study of the expanding case*

Rafael Labarca and Carlos Moreira

Abstract. In this article we provide, by using kneadings sequences, the combinatorial bifurcation diagramme associated to a typical two parameter of Lorenz maps on the real line. We apply these results to two parameter families of geometric Lorenz-like flows.

Keywords: Expanding Lorenz maps; combinatorial bifurcation diagramme; geometric Lorenz-like flows.

1 Introduction

In a remarkable contribution, E.N. Lorenz [17], showed numerical evidence of the existence of a strange attractor for a quadratic system of ordinary differential equations in three variables. Some time later J. Guckenheimer, [9], produced a work where he introduced symbolic dynamics in order to understand the topologically equivalence classes for nearly similar attractors. At that time R.F. Williams, [19], introduced a geometrical model in order to understand the dynamics of these Lorenz attractors. Using this geometrical model the dynamical behavior of the three dimensional vector field can be reduce to the dynamical behavior of a one-dimensional map with one discontinuity and Guckenheimer and Williams, [11], used this fact to shown uncountable many classes of non-equivalent geometric Lorenz attractors. The evidence of the non-equivalence where the kneading sequences associated to these one-dimensional maps. Later

Received 14 February 2001.
*Partially supported by Fondecyt grants #1970720, #1990903, DICYT-USACH – Chile and PRONEX on Dynamical Systems. Brazil.

Afrajmovich, Bykov and Shilnikov, [1],studied a two parameter family of three dimensional vector fields that unfolds a codimension two bifurcation which appears in a vector field defined in a neighborhood of the origin $0 \in \mathbb{R}^3$. This vector field has an hyperbolic singularity, at $0 \in \mathbb{R}^3$, which has a two dimensional stable manifold and a one dimensional unstable manifold. For the connected components Λ_1 and Λ_2 of the set $W_0^u - \{0\}$, the vector field X satisfies $\Lambda_1 \subset W_0^s$ and $\Lambda_2 \subset W_0^s$. In that work they described part of the bifurcation theory that appears in a generic two parameter unfolding of the vector field X.

So, its seem to be natural to try to obtain a bifurcation theory for a generic unfolding of X using symbolic dynamics. In this direction de Melo and Martens announced, in an unpublished paper [7], the existence of parameterized families of contacting Lorenz-like flows that are topologically universal in the sense that given any geometric Lorenz flow then its dynamics is ''essentially'' the same as the dynamics of some element of the family. In [16] we describe, the way in which such a contracting family realize all the allowed dynamics for Lorenz-like maps. The same universality is not true when we work with families of expanding Lorenz-like, flows as de Melo and Martens knew and as we will show in the present paper.

Also we have to mention the work by Hubbard and Sparrow [12], were they defined a set of pair of sequences which model all the topological dynamics exhibed by expanding Lorenz maps. Recently, in [14], we extend the Hubbard-Sparrow model for expanding maps to an Universal Model for the ''essential dynamics'' of Lorenz maps and we called it the *Lexicographical World* (in the sequel denoted by LW). Now, it is clear that any parameterized family of Lorenz maps has a bifurcation theory induced by the lexicographical world (an extremely interesting problem focused for several authors in this an another contexts, see for instance [6], [5], [10], [2], [4]). So, it seems to be natural to develop a programme to obtain the bifurcation theories associated to some typical parameterized families of Lorenz maps. We developed such a programme. In [15] and [16] we describe the bifurcation theory associated to a two parameter family of linear maps and to a two parameter family of contracting maps, respectively. In certain form, both bifurcation theories are similar. In the present paper we describe the bifurcation theory associated to a typical two parameter family of expanding Lorenz maps and the associated bifurcation theory is dramatically different from the previous one's.

Of course, there are other points of view which may be used to described the bifurcation theory associated to parameterized families of Lorenz maps. In fact, starting with the work of Arnold ([2]), a lot of work have been produced with respect to the bifurcation theory of the canonical family (see for instance [4] and the references there in). The approaches,in this case, was done by using the rotation number. This approach seems to be useful when applied to homeomorphisms of the circle but looks very complicated when we deal with non injective maps and, in fact, a bifurcation diagram for the dynamics (of the canonical family) is unknown (at least for us). The better approximation to the bifurcation theory of the canonical map seems to be the work by Boyland ([4]). We have to mention that the lexicographical world includes the dynamics of all the interpolated maps used by Boyland ([4] page 359) in its construction and, then, it is also a model for these maps.

To be more specific, in the present paper we study the two parameter family of quadratic expanding Lorenz maps:

$$G_{\mu,\nu}(x) = \begin{cases} -\mu + \sqrt{x}\,, & x > 0 \\ \nu - \sqrt{-x}\,, & x < 0 \end{cases}$$

and we give the corresponding bifurcation theory in the parameter space.

This paper is organized as follows: In section 2 we state our results, in section 3 we describe the lexicographical world, in Section 4 we prove our results, in section 5 we present a generalization of them and, finally, in section 6 we will relate them to three dimensional geometric vector fields.

We observe that a number of authors have studied paremeterized families of Lorenz-like maps from several points of view, namely combinatorial, differentiable, topological or geometrical (see the references).

2 Statement of our results

2.1 The set DM_0

In the sequel DM_0 will denote the set of maps $f\colon (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ such that:

(1) The restriction maps $f|_{(-\infty,0)}\colon (-\infty, 0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)}\colon (0, \infty) \to \mathbb{R}$ are continuous and non-decreasing maps.

(2)

$$f(0^+) = \lim_{x \downarrow 0} f(x) \in (-\infty, 0]$$

and

$$f(0^-) = \lim_{x \uparrow 0} f(x) \in [0, \infty[$$

An element in DM_0 will be called *injective* if its restriction to the interval $]f(0^+), f(0^-)[$ is an injective map.

We will say that $f \in DM_0$ is *increasing* if the restriction maps $f|_{(-\infty,0)}: (-\infty, 0) \mapsto \mathbb{R}$ and $f|_{(0,\infty)}: (0, \infty) \to \mathbb{R}$ are increasing.

We call the elements in DM_0 *Lorenz maps.*

2.2 The lexicographical order

Let Σ_2 denote the set of sequences $\theta: \mathbb{N} \to \{0, 1\}$ endowed with the topology given by the metric

$$d(\alpha, \beta) = \sum_{i=0}^{\infty} \frac{\overline{d}(\alpha_i, \beta_i)}{2^i},$$

where

$$\overline{d}(\alpha_i, \beta_i) = \begin{cases} 0, \alpha_i = \beta_i \\ 1, \alpha_i \neq \beta_i \end{cases}.$$

Let $\sigma: \Sigma_2 \to \Sigma_2$ be the shift map $\sigma(\theta_0, \theta_1, \theta_2, \dots) = (\theta_1, \theta_2, \dots)$. Let Σ_0 and Σ_1 denote the sets $\{\theta \in \Sigma_2; \theta_0 = 0\}$ and $\{\theta \in \Sigma_2; \theta_0 = 1\}$ respectively. It is clear that $\Sigma_2 = \Sigma_0 \cup \Sigma_1$.

In Σ_2 we consider the *lexicographical order*: $\theta < \alpha$ for any $\theta \in \Sigma_0$ and $\alpha \in \Sigma_1$ or $\theta < \alpha$ if there is $n \in \mathbb{N}$ such that $\theta_i = \alpha_i$ for $i = 0, 1, 2, \dots, n-1$ and $\theta_n = 0$ and $\alpha_n = 1$.

For $a \leq b$ in Σ_2 let $[a, b]$ denote the interval $\{\theta \in \Sigma_2 | a \leq \theta \leq b\}$. $\Sigma_{a,b}$ will denote the set $\bigcap_{n=0}^{\infty} \sigma^{-n}([a, b])$.

2.3 The set Σ_{a_f, b_f}

For $f \in DM_0$ let $\Gamma_f = (\mathbb{R} \setminus \bigcup_{j=0}^{\infty} f^{-j}(0))$ denote the set of "continuity" of the map f.

For $x \in \Gamma_f$ we define $I_f(x) \in \Sigma_2$ by

$$I_f(x)(i) = 0 \quad \text{if } f^i(x) < 0 \quad \text{and} \quad I_f(x)(i) = 1 \quad \text{if } f^i(x) > 1.$$

For $x = 0$ we define:

$$I_f(0^+) = \lim_{x \downarrow 0, x \in \Gamma_f} I_f(x) \quad \text{and} \quad I_f(0^-) = \lim_{x \uparrow 0, x \in \Gamma_f} I_f(x).$$

In the same way to any $x \in \bigcup_{j=0}^{\infty} f^{-j}(0)$ such that $f^i(x) \neq 0$, $0 \leq i < n$; $f^n(x) = 0$ we associate the sequences:

$$I_f(x^+) = (I_f(x)(0), \dots, I_f(x)(n-1), I_f(0^+))$$

and

$$I_f(x^-) = (I_f(x)(0), \ldots, I_f(x)(n-1),\ I_f(0^-)).$$

For $x \in \Gamma_f$ we define $I_f(x^+) = I_f(x^-) = I_f(x)$.

Let $I_f = \{I_f(x^+);\ x \in [f(0^+), f(0^-)[\} \cup \{I_f(x^-);\ x \in]f(0^+), f(0^-)]\}$.

Clearly $\sigma(I_f) \subset I_f$. Let us denote by $a_f = I_f((f(0^+))^+)$ and $b_f = I_f((f(0^-))^-)$ the kneading sequences associated to the map f.

Lemma 1. $I_f = \bigcap_{n=0}^{\infty} \sigma^{-n}([a_f, b_f]) = \Sigma_{a_f,b_f}$.

We observe that associated to any $f \in DM_0$ we can define a continuous map

$$h\colon [f(0^+)\,,\, f(0^-)] \cap \Gamma_f \to \Sigma_{a_f,b_f} \subset \Sigma_2,$$

such that $h \circ f = \sigma \circ h$. The map h is given by $h(x) = I_f(x)$and collapses intervals into points. This map cannot be extended, continuously, to the set $\cup_{i=0}^{\infty} f^{-i}(0)$. There are two kinds of intervals that the map h can collapses: The wandering intervals and the intervals that are contained in the stable manifold of periodic sinks. An interval $I \subset [f(0^+)\,,\, f(0^-)]$ is called a *wandering interval*, for the map f, if for any $x \in I$ we have that x is a wandering point. We will call a point x a *nonwandering* point if for any neighborhood U_x of x and any positive integer N we can find $n \geq N$ such that $f^n(U_x) \cap U_x \neq \emptyset$. The set of nonwandering points of the map f is denoted by Ω_f. A point $x \notin \Omega_f$ is called a *wandering point.* Given any interval, I, *the orbit* of this interval is the sequence of iterations $(f^n(I)\,, n \in \mathbb{N})$. Concerning the existence of wandering intervals we have the following:

Lemma 2 ([14]). *Let $\{\varphi_\lambda\,, \lambda \in \mathbb{R}\} \subset DM_0$ be a one parameter family of C^2 increasing maps such that for each λ there are sequences $\lambda_n \to \lambda$ and $\mu_n \to \lambda$ with $\varphi_{\lambda_n}(x) > \varphi_\lambda(x)$ and $\varphi_{\mu_n}(x) < \varphi_\lambda(x)\,, \forall x$ then there is a residual set of parameters λ for which φ_λ has no wandering intervals.*

We observe that our two parameter family of expanding maps satisfy this property.

Definition 1. Given $f, g \in DM_0$. We will say that f has *essentially* the same dynamics as g if $I_f = I_g$.

We note that in this situation, up to the existence of some intervals where the itineraries of the points are the same, the dynamics of the maps f and g are topologically equivalent (see [8]).

2.4 The lexicographical world

Let $Min_2 = \{a \in \Sigma_0\,;\ \sigma^k(a) \geq a\,, k \in \mathbb{N}\}$ and $Max_2 = \{b \in \Sigma_1\,;\ \sigma^k(b) \leq b\,, k \in \mathbb{N}\}$.

Definition 2. The $LW = \{(a\,, b) \in Min_2 \times Max_2\,;\ \{a\,, b\} \subset \Sigma_{a,b}\}$ will be called the *lexicographical world*.

For $a \in Min_2$ its LW-fiber is the set $LW_0(a) = \{b \in Max_2\,;\ (a\,, b) \in LW\}$. For $b \in Max_2$ its LW-fiber is the set $LW_1(b) = \{a \in Min_2\,;\ (a\,, b) \in LW\}$.

Remark 1. It is clear that given $(a\,, b) \in LW$ then $\Sigma_{a,b} \neq \emptyset$.

Let us now consider $(a, b) \in LW$.

Lemma 3 ([15]). *There is $f \in DM_0$ such that $I_f = \Sigma_{a,b}$.*

We will call this result the *realization lemma*. We observe that the maps, in this lemma, may be obtained as increasing maps.

Therefore, we have a surjective map $I\colon DM_0 \to LW$, $I(f) = (a_f, b_f)$ and $DM_0 = \bigcup_{(a,b)\in LW} I^{-1}(\{(a, b)\})$.

In this context the next definition is natural.

Definition 3. Let $\alpha\colon U \subset \mathbb{R}^k \to DM_0$ be a map.

(1) We will say that α is an *a-universal family* if $\forall a \in Min_2$ there is a nonempty set, $A(a) \subset U$, such that $a_f = a\,, \forall f \in \alpha(A(a))$.

(2) We will say that α is an *b-universal family* if $\forall b \in Max_2$ there is a nonempty set, $B(b) \subset U$, such that $b_f = b\,, \forall f \in \alpha(B(b))$.

(3) We will say that α is an *LW-universal family* if $\forall\,(a, b) \in LW$ there exists a nonempty set $A(a, b) \subset U$ such that $I \circ \alpha(A(a, b)) = (a, b)$.

It is clear that associated to any map α, as above, we have an a $(b\,, LW)$-decomposition of its domain. We will call this a (resp. $b\,, LW$)-decomposition the a (resp. $b\,, LW$) bifurcation theory defined by α.

Open problem. There are LW-universal families?

Certainly, this is a very hard problem and we believe that there is not a finite $k \in \mathbb{N}$ with this property. In the present paper we prove the following

Theorem 1. *The given two parameter family of expanding Lorenz maps is a-universal, is b-universal but it is not LW-universal. Moreover, the respective a , b , LW-bifurcation theories are given.*

3 Symbolic dynamics and the lexicographical world

Here we introduce some results and notations that are necessary for our results. The results are proved in [14] or in [16].

3.1 Dynamical properties for sequences in LW

Let $a_1 \leq a_2$ be two periodic sequences in Σ_0. The sequence $m(a_1 , a_2) = \underline{a_1\, a_2}$ will be called the *average* of the sequences a_1 and a_2.

Example. For $a_1 = \underline{01}$, $a_2 = \underline{011}$ we have $m(a_1 , a_2) = \underline{01011}$.

Let $A_0 = \{\underline{0_n 1} , \underline{01_m} \,;\, n, m \in \mathbb{N}\backslash\{0\}\}$ and $A_{n+1} = A_n \cup \{m(a_1 , a_2);\ a_1 , a_2 \in A_n$ are consecutive sequences$\}$. Set $A_\infty = \bigcup_{n=0}^{n=\infty} A_n$. The elements in A_∞ will be called *primary sequences*. As we will see in the next section, primary sequences are associated with primary bifurcations.

The elements in $\overline{A_\infty}$ are characterized by the following property:

Lemma 4. *$a \in \overline{A_\infty}$ if and only if*

$$(*) \qquad a \in Min_2 \text{ and } \sigma(a) \geq \sigma(b),$$

for $b = \sup\{\sigma^k(a) \,;\, k \in \mathbb{N}\}$.

In the proof of this lemma (see [16]) we define the renormalization map $R_{a,b}\colon \Sigma_2 \to \Sigma(a, b)$, by $R_{a,b}(a_0, a_1, \cdots) = (\tilde{a}_0, \tilde{a}_1, \cdots)$ where $\tilde{a}_i = a$ if $a_i = 0$ and $\tilde{a}_i = b$ if $a_i = 1$. Here $\Sigma(a, b) = \{\theta\colon \mathbb{N} \to \{a, b\}\}$. We have

Proposition 1. *Assume $a \in \Sigma_0$ satisfy $(*)$ then $R_{0,01}(a)$ satisfy $(*)$ in $\Sigma_{0,01}$.*

Let us denote $A^0_\infty = A_\infty$ and define, for any $a \in A^0_\infty$ the set:

$$A^1_1(a) = \{c \in \Sigma_0 \,;\, c = \underline{a\_b_+a^n} \quad \text{or} \quad c = \underline{a\_b^m b_+} \,, \text{for } n , m \in \mathbb{N}\}.$$

Here for $b = \underline{b_0 b_1 \dots b_k 0}$ we have $b_+ = b_0 b_1 \dots b_k 1$.

Observe that if $c_n = \underline{a\_b_+a^n}$ then $c_n \to a\_b_+\underline{a}$ and if $d_m = \underline{a\_b^m b_+}$ then $d_m \to a\_\underline{b}$.

Set

$$A_2^1(a) = \{m(a_1, a_2);\ a_1, a_2 \in A_1^1(a) \text{ are consecutive sequences}\} \cup A_1^1(a)$$

and in general

$$A_{n+1}^1(a) = \{m(a_1, a_2);\ a_1, a_2 \in A_n^1(a) \text{ are consecutive sequences}\} \cup A_n^1(a),\ n \geq 2.$$

Let $A_\infty^1(a) = \bigcup_{n=1}^{n=\infty} A_n^1(a)$ and $A_\infty^1 = A_\infty^0 \cup \bigcup_{a \in A_\infty^0} A_\infty^1(a)$.

At this stage we have to point out the following: Assume $a \in A_\infty^0$. Let $\tilde{a} = \sup\{\sigma^k(a); k \in \mathbb{N} \text{ and } \sigma^k(a) \in \Sigma_0\}$ and $\tilde{b} = \inf\{\sigma^k(a); k \in \mathbb{N} \text{ and } \sigma^k(a) \in \Sigma_1\}$. Consider $\Sigma_{\tilde{a},\tilde{b}} = \{\theta : \mathbb{N} \to \{\tilde{a}, \tilde{b}\}\}$ be the set of sequences of the two symbols $\tilde{a}$ and $\tilde{b}$. Replace $0 = \tilde{a}$ and $\tilde{b} = 1$ and define $A_0(a) = \{\underline{0_n 1}, \underline{01_m}; n, m \in \mathbb{N} \setminus \{0\}\}$ and $A_{n+1}(a) = A_n(a) \cup \{m(a_1, a_2); a_1, a_2 \in A_n(a) \text{ are consecutive sequences}\}$. Set $A_\infty(a) = \bigcup_{n=0}^{\infty} A_n(a)$ and $A_\infty^\star(a) = \{\inf\{\sigma^k(\alpha); k \in \mathbb{N}\}; \alpha = R_{\tilde{a},\tilde{b}}(\eta);\ \eta \in A_\infty(a)\}$.

Lemma 2. $\bar{A}_\infty^\star(a) = \bar{A}_\infty^1(a)$.

As before, let $\sigma_{\tilde{a},\tilde{b}} : \Sigma_{\tilde{a},\tilde{b}} \to \Sigma_{\tilde{a},\tilde{b}}$ be the shift map. We have

Lemma 3. *$\alpha \in \bar{A}_\infty^\star(a)$ if and only if*

($\star$) $\alpha \in Min_2(\tilde{a}, \tilde{b})$ and $\sigma_{\tilde{a},\tilde{b}}(\alpha) \geq \sigma_{\tilde{a},\tilde{b}}(\beta)$ for $\beta = \sup\{\sigma_{\tilde{a},\tilde{b}}^k(\alpha); k \in \mathbb{N}\}$

Inductively, for any $a \in A_\infty^n(a)$, let

$$A_1^{n+1}(a) = \{c \in \Sigma_0;\ c = \underline{a_- b_+ a^j} \quad \text{or} \quad c = \underline{a_- b^k b_+} \quad \text{for} \quad j, k \in \mathbb{N}\}$$

Now, we define

$$A_2^{n+1}(a) = \{m(a_1, a_2);\ a_1, a_2 \in A_1^{n+1}(a) \text{ are consecutive sequences}\} \bigcup A_1^{n+1}(a)$$

and

$$A_{m+1}^{n+1}(a) = \{m(a_1, a_2);\ a_1, a_2 \in A_m^{n+1}(a) \text{ are consecutive sequences}\} \bigcup A_m^{n+1}(a),\ m \geq 2.$$

As before, define $A^{n+1}_\infty(a) = \bigcup_{m=1}^\infty A^{n+1}_m(a)$; $A^{n+1}_\infty = \bigcup_{a\in A^n_\infty} A^{n+1}_\infty(a) \cup A^n_\infty$ and finally, $A^\infty_\infty = \bigcup_{j=0}^\infty A^j_\infty$.

Note. A similar construction; as we did in Lemma 2 and Lemma 3, for $a \in A^\circ_\infty$, we can do for any $a \in A^\infty_\infty$.

The elements in $(A^\infty_\infty \setminus A^0_\infty)$ will be called secondary sequences. As we will see in the next section secondary sequences are associated with secondary bifurcations.

Let denote by B^∞_∞ the set $\{\sup\{\sigma^k(a)\,, k \in \mathbb{N}\}\,, a \in A^\infty_\infty\}$. We will denote by $b(a)$ the sequence $\sup\{\sigma^k(a)\,, k \in \mathbb{N}\}$ for $a \in \Sigma_0$ and by $a(b)$, the sequence $\inf\{\sigma^k(b)\,, k \in \mathbb{N}\}$ for $b \in \Sigma_1$. Clearly, $b(a) \in Max_2$ and $a(b) \in Min_2$.

It is clear that $\Sigma_{a,\underline{1}} \neq \phi$ for any $a \in \Sigma_0$. Hence we can define maps $\varphi\,, \psi\,, \chi : \Sigma_0 \to \Sigma_1$ by:

$$\varphi(a) = \inf\{b \in \Sigma_1 |\, \Sigma_{a,b} \neq \phi\},$$
$$\psi(a) = \inf\{b \in \Sigma_1\,;\ \Sigma_{a,b} \text{ contains } \infty\text{-elements}\}$$

and

$$\chi(a) = \inf\{b \in \Sigma_1\,;\ \Sigma_{a,b} \text{ is uncountable}\}.$$

Clearly, $a_1 \le a_2$ imply $\varphi(a_1) \le \varphi(a_2)$, $\psi(a_1) \le \psi(a_2)$ and $\chi(a_1) \le \chi(a_2)$ and for all $c \in \Sigma_1$ such that $c < \varphi(a)$ we have $\Sigma_{a,c} = \Phi$.

Examples. For any $0\underline{01} \le a \le \underline{01}$ we have $\varphi(a) = \underline{10}$, $\psi(\underline{01}) = \chi(\underline{01}) = 1\underline{10}$. Also $\varphi(\underline{0}) = \psi(\underline{0}) = \chi(\underline{0}) = 1\underline{0}$; $\varphi(0\underline{1}) = \psi(0\underline{1}) = \underline{1}$; $\varphi(\underline{0_n 1}) = \underline{10_n}$, $\psi(\underline{0_n 1}) = \chi(\underline{0_n 1}) = 1\underline{10_n}$ and $\varphi(\underline{01_m}) = \underline{1_m 0}$, $\psi(\underline{01_m}) = \chi(\underline{01_m}) = \underline{11_m 0}$.

3.2 The Morse-Smale and the Entropy Zero cases

Definition 4.

a) We will call a map $f \in DM_0$ *Morse-Smale* if $a_f \in A^0_\infty$ and $b_f = \varphi(a_f)$.

b) We will call a map $f \in DM_0$ *essentially Morse-Smale* if $a_f \in A^\infty_\infty$ and $b_f = \varphi(a_f)$.

We will denote by $MS_0 \subset DM_0$ the set of Morse-Smale and essentially Morse-Smale maps. We call these maps Morse-Smale because its dynamics essentially reduces to a periodic orbit.

Lemma 5. *Given $f \in DM_0$ be a "Morse-Smale" map we have that $[f(0^+), f(0^-)] = \bigcup_{i=0}^{per(a_f)-1} \overline{I_i}$, where $x \in I_0$ implies $I_f(x) = a_f$ and $I_f|_{I_j}$ is constant and equal to $a_j = \sigma^j(a_f)$ for $0 < j \leq per(a_f) - 1$.*

Lemma 6. *Let $a \in A^0_\infty$. There is an injective $f \in DM_0$ such that $a_f = a$ and $b_f = \varphi(a)$.*

The next result follows immediately from this lemma:

Corollary 1. *Let $a \in \overline{A^0_\infty} \setminus A^0_\infty$. There is an injective map $f \in DM_0$ such that $a_f = a$ and $b_f = \varphi(a)$.* □

We note that these maps can be considered as bijective maps on the circle.

Lemma 7. *Let $a \in A^\infty_\infty$. There is $f \in DM_0$ an increasing map, such that $a_f = a$ and $\varphi(a) = b_f$.*

In a similar way we obtain

Lemma 8. *Associated to any $a \in Min_2$ there is an increasing map $f \in DM_0$ such that $a_f = a$ and $b_f = \varphi(a_f)$.*

In general we get

Lemma 9. *For $a \in A^\infty_\infty$ and $d \in B^\infty_\infty \cap LW_0(a)$. There is an increasing map $f \in DM_0$ such that $a_f = a$, $b_f = d$.*

As a consequence of these lemmas we have

Proposition 2. *Given $(a, b) \in LW$ there is an increasing map $f \in DM_0$ such that $I(f) = (a_f, b_f) = (a, b)$.*

This result is a generalization of a similar result obtained for expansive maps in [12].

Definition 5. We will call a map $f \in DM_0$ an *entropy zero* map if $b_f \leq \chi(a_f)$.

One of the most interesting problems related with the bifurcation theory associated to a parameterized family of dynamical systems $\{f_\lambda ; \lambda \in U \subset \mathbb{R}^k\}$, is to describe the set $\{\lambda \in U ; f_\lambda$ is an entropy zero map$\}$ (see for instance [18], [3] and the references there in). For our family of expanding Lorenz maps we will prove some results, in this direction, in section 4.6.

We observe that any Morse-Smale or essentially Morse-Smale map is an entropy zero map.

4 Expanding maps

4.1 The map G

Let us define $G_{(\mu,\nu)}\colon (\mathbb{R}\setminus\{0\}) \to \mathbb{R}$, $(\mu,\nu) \in \mathbb{R}^2$, the two parameter family of maps in DM_0:

$$G_{(\mu,\nu)}(x) = \begin{cases} -\mu + x^{1/2}, & x > 0 \\ \nu - (-x)^{1/2}, & x < 0 \end{cases}$$

In this section we will provide the bifurcation theory associated to this family of elements in DM_0.

4.2 The injective maps

Let us define the set

$$IM = \{(\mu,\nu);\ G_{(\mu,\nu)}|_{[-\mu,\nu]}\colon [-\mu,\nu] \to [-\mu,\nu] \text{ is an injective map}\}.$$

It is nor hard to see that $IM = \{(\mu,\nu);\ \mu \geq 0\,, \nu \geq 0 \text{ and } (\sqrt{\nu} - 1/2)^2 + (\sqrt{\mu} - 1/2)^2 \geq 1/2\}$.

4.3 Fixed points

The fixed points of the map $G_{(\mu,\nu)}$ are given by:

a) $x_{\pm}(\mu) = \left[\dfrac{1 \pm \sqrt{1-4\mu}}{2}\right]^2$ defined for $\mu \leq 1/4$.

b) $y_{\pm}(\nu) = -\left[\dfrac{1 \pm \sqrt{1-4\nu}}{2}\right]^2$ defined for $\nu \leq 1/4$.

We have $y_+(\nu) \leq y_-(\nu) \leq 0 \leq x_-(\mu) \leq x_+(\mu)$.

Note. The maps $x_-(\mu)$ and $y_-(\nu)$ are defined for $0 \leq \mu \leq \frac{1}{4}$; $0 \leq \nu \leq \frac{1}{4}$.

4.4 Preimages of 0

a) $\nu - (-x)^{1/2} = 0$ imply $x = -\nu^2 = y_1(\nu)$, $\nu \geq 0$;
$\nu - (-x)^{1/2} = y_1(\nu)$ imply $x = -(\nu - y_1(\nu))^2 = -[\nu + \nu^2]^2 = y_2(\nu)$;
$\nu - (-x)^{1/2} = y_n(\nu)$ imply $x = -[\nu - y_n(\nu)]^2 = y_{n+1}(\nu)$, $\nu \geq 0$.

b) $-\mu + x^{1/2} = 0$ imply $x = \mu^2 = x_1(\mu)$, $\mu \geq 0$;
$-\mu + x^{1/2} = x_1(\mu)$ imply $x = (\mu + x_1(\mu))^2 = (\mu + \mu^2)^2 = x_2(\mu)$;
$-\mu + x^{1/2} = x_n(\mu)$ imply $x = (\mu + x_n(\mu))^2 = x_{n+1}(\mu)$, $\mu \geq 0$.

4.5 Primary bifurcations

a) $G(0^+)$ = fixed point of the left hand-side

$$-\mu = -\left[\frac{1 \pm \sqrt{1-4\nu}}{2}\right]^2$$

that is:

$$\mu = \left[\frac{1 \pm \sqrt{1-4\nu}}{2}\right]^2.$$

b) $G(0^-)$ = fixed point of the right hand-side

$$\nu = \left[\frac{1 \pm \sqrt{1-4\mu}}{2}\right]^2$$

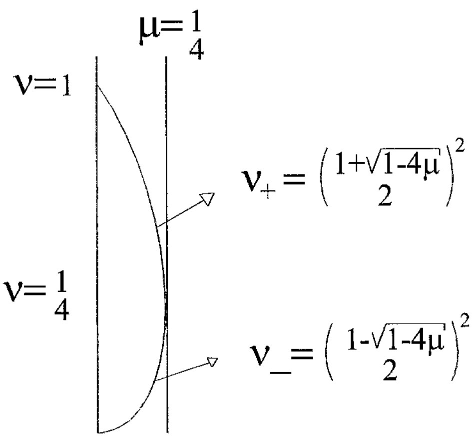

Figure 1: (b)

Let $L_{\underline{0}} = \{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+;\quad a(G_{(\mu,\nu)}) = \underline{0}\}$ and $R_{\underline{1}} = \{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+;\quad b(G_{(\mu,\nu)}) = \underline{1}\}$.

We observe that

$$L_{\underline{0}} = \{(\mu, \nu);\ 0 \le \nu \le \nu_-(\mu),\ 0 \le \mu \le 1/4 \\ \text{or } 0 \le \nu \le 1/4 \quad \text{for } \mu \ge 1/4\}$$

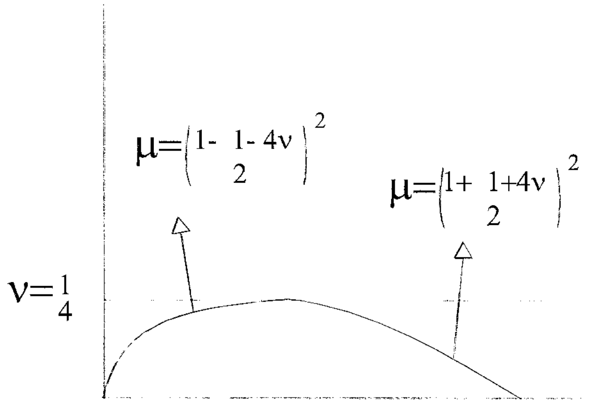

Figure 2: (a)

and

$$R_{\underline{1}} = \{(\mu,\nu);\ 0 \le \mu \le \mu_-(\nu),\ 0 \le \nu \le 1/4 \text{ or } 0 \le \mu \le 1/4 \quad \text{for } \nu \ge 1/4\}.$$

Here $\nu_-(\mu)$ is given by $\mu = \left(\frac{1-\sqrt{1-4\nu_-(\mu)}}{2}\right)^2$ and $\mu_-(\nu)$ is given by $\nu = \left(\frac{1-\sqrt{1-4\mu_-(\nu)}}{2}\right)^2$.

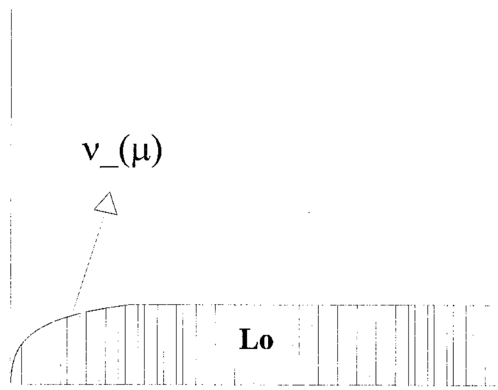

Figure 3:

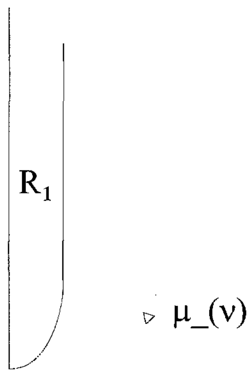

Figure 4:

c) $G(0^+) = n$th preimage of zero $= y_n(\nu)$.

These equations define the curves $\mu_n(\nu) = -y_n(\nu)$ that satisfies: $(\mu, \nu) \in \text{Graph}(\mu_n)$ imply $a(G_{(\mu,\nu)}) = \underline{0_n 1}$. Moreover, the inverses $\hat{\nu}_n(\mu) = \mu_n^{-1}(\nu)$ converges (uniformly in the C^1 topology on compact intervals) to the curve $s(\mu)$ given by:

$$s(\mu) = \begin{cases} \dfrac{1-(2\mu^{1/2}-1)^2}{4}, & 0 \le \mu \le 1/4 \\ \dfrac{1}{4}, & \mu \ge 1/4 \end{cases}$$

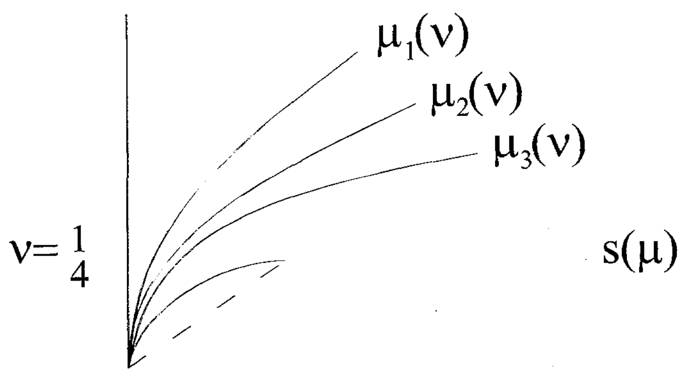

Figure 5:

d) $G(0^-) = n$th preimage of zero $= x_n(\mu)$.

These equations define the curves $\nu_n(\mu) = x_n(\mu)$ that satisfies: $(\mu, \nu) \in$ Graph(ν_n) imply $b(G_{(\mu,\nu)}) = \underline{1_n 0}$. Moreover, the inverses $\hat{\mu}_n(\nu) = \nu_n^{-1}(\mu)$ converges (uniformly in the C^1 topology in compact intervals) to the curve $s(\nu)$ given by

$$s(\nu) = \begin{cases} \dfrac{1 - (2\nu^{1/2} - 1)^2}{4}, & 0 \le \nu \le 1/4 \\ \dfrac{1}{4}, & \nu \ge 1/4 \end{cases}$$

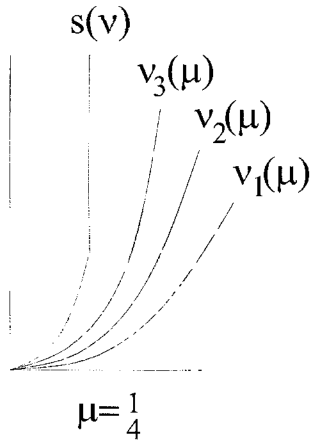

Figure 6:

Proposition 3. *The graph of the curve $\mu_n(\nu)$ is transversal to graph of the curve $\nu_m(\mu)$ all $m, n \in \mathbb{N}$.*

Proof. We have $\mu_1(\nu) = \nu^2$ and $\nu_1(\mu) = \mu^2$. So, the result is true for $n = 1$ and $m = 1$.

For $\nu_2(\mu)$ we have $\nu_2(\mu) = (\mu + \nu_1(\mu))^2$, then $\nu_2'(\mu) = 2(\mu + \nu_1(\mu))(1 + \nu_1'(\mu)) \ge 2(1/4 + 1/2)(1 + \nu_1'(\mu)) = \frac{3}{2}(1 + \nu_1'(\mu)) = \frac{3}{2}(1 + 2\mu) \ge \frac{3}{2}(1 + 1/2) \ge (3/2)^2 \ge 2$. Hence, we conclude the result for $n = 1$, $m = 2$.

Assume the result is true for $n = 1$ and $m = p$ with $\nu_p'(\mu) > 1$. For $\nu_{p+1}(\mu)$ we have $\nu_{p+1}(\mu) = (\mu + \nu_p(\mu))^2$ and $\nu_{p+1}'(\mu) = 2(\mu + \nu_p(\mu))(1 + \nu_p'(\mu)) \ge 2(1/4 + 1/2)(1 + 1) > 4(3/4) = 3$. Therefore, we get the result for $n = 1$ and any $m \in \mathbb{N}$.

Assume that the result is true for $n = k$ and any $m \in \mathbb{N}$ with $\mu_k'(\nu) > 1$ and $\nu_m'(\mu) > 1$. For $\mu_{k+1}(\nu) = (\nu + \mu_k(\nu))^2$ we have $\mu_{k+1}'(\nu) = 2(\nu + \mu_k(\nu))(1 + \mu_k'(\nu)) > 2(1/2 + 1/4)(1 + 1) > 3$. So, we conclude the result for $n = k + 1$ and any $m \in \mathbb{N}$. □

e) Let us now assume that $G(0^+) < y_1(\nu)$. In this situation there is a point, $\overline{x_1}(\mu, \nu) \geq 0$, such that $-\mu + \overline{x}_1^{1/2} = y_1(\nu)$, that is $\overline{x_1}(\mu, \nu) = (\mu + y_1(\nu))^2$.

If we look for the condition $G(0^-) = \overline{x_1}(\mu, \nu)$ we get the curve $\overline{\mu_1}(\nu) = \nu^{1/2} - y_1(\nu)$. This curve is tangent to the curve $C_{\underline{10}}$ at $\nu = 0$; transversally intersects the curves $\mu_n(\nu)$, $n \geq 2$ and the curve $s(\mu)$. We note that $(\mu, \nu) \in \text{Graph}(\overline{\mu_1})$ imply $b(G_{(\mu,\nu)}) = \underline{100}$.

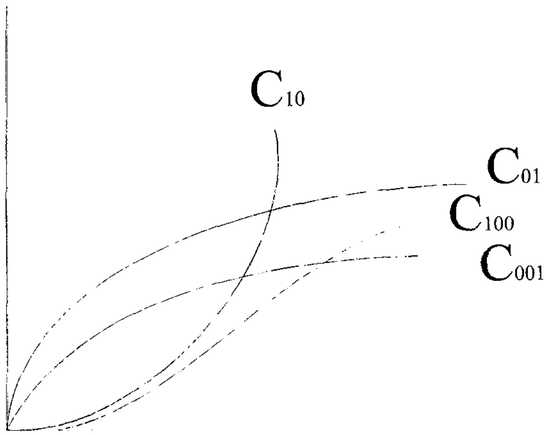

Figure 7:

In the same way, if we assume that $G(0^+) < y_n(\nu)$, $n \geq 2$, we will find a point $\overline{x_n}(\mu, \nu) \leq \overline{x_{n-1}}(\mu, \nu)$ such that $-\mu + \overline{x_n}^{1/2} = y_n(\nu)$, that is

$$\overline{x_n}(\mu, \nu) = (\mu + y_n(\nu))^2.$$

If we look for the condition $G(0^-) = \overline{x_n}(\mu, \nu)$ we obtain a curve $\overline{\mu_n}(\nu) = \nu^{1/2} - y_n(\nu)$, that is: $\overline{\mu_n}(\nu) = \nu^{1/2} + \mu_n(\nu)$. This curve is tangent to $\overline{\mu_{n-1}}(\nu)$ at $\nu = 0$; transversally intersects the curves $\mu_m(\nu)$, $m \geq n + 1$ and the curve $s(\mu)$. We observe that $(\mu, \nu) \in \text{Graph}(\overline{\mu_n})$ imply $b(G_{(\mu,\nu)}) = \underline{10_{n+1}}$.

f) Let us now assume that $G(0^-) > x_1(\mu)$. In this situation we can find a point $\overline{y_1}(\mu, \nu)$ such that $\nu - (-\overline{y_1})^{1/2} = x_1(\mu)$, that is $\overline{y_1}(\mu, \nu) = -[\nu - x_1(\mu)]^2$.

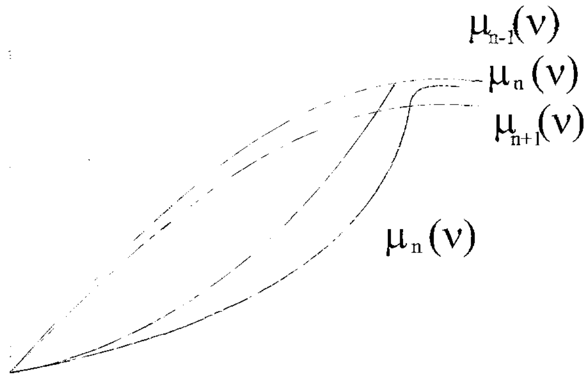

Figure 8:

If we look for the condition $G(0^+) = \overline{y_1}(\mu, \nu)$ we obtain a curve $\overline{\nu_1}(\mu) = \mu^{1/2} + x_1(\mu)$. This curve is tangent to the curve $C_{\underline{01}}$ at $\mu = 0$; transversally intersects the curves $\nu_n(\mu)$, $n \geq 2$ and the curve $s(\nu)$.

Also we have $(\mu, \nu) \in \text{Graph}(\overline{\nu_1})$ imply $a(G_{(\mu,\nu)}) = \underline{011}$.

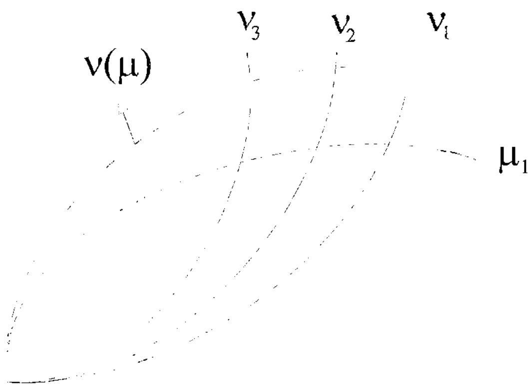

Figure 9:

Similarly: if we assume that $G(0^-) > x_n(\mu)$ we can find a point $\overline{y_n}(\mu, \nu)$ such that $\nu - (-\overline{y_n})^{1/2} = x_n(\mu)$ that is $\bar{y}_n(\mu, \nu) = -[\nu - x_n(\mu)]^2$.

If we look for the condition $G(0^-) = \overline{y_n}(\mu, \nu)$ we obtain a curve $\overline{\nu_n}(\mu) = \mu^{1/2} + x_n(\mu) = \mu^{1/2} + \nu_n(\mu)$. This curve is tangent to $\overline{\nu_{n-1}}(\mu)$ at $\mu = 0$; transversally intersects the curves $\nu_n(\mu)$; $m \geq n+1$ and the curve $s(\nu)$. We note that $(\mu, \nu) \in \text{Graph}(\overline{\nu_n})$ imply $a(G_{(\mu,\nu)}) = \underline{01_{n+1}}$.

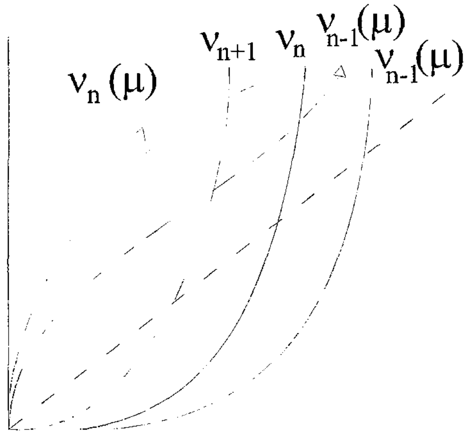

Figure 10:

Remark.

a) Associated to any $a \in A_0 = \{\underline{0_n 1}; \quad \underline{01_n}; \quad n \in \mathbb{N}\}$ we have constructed two curves C_a, C_b, $b = b(a)$, such that $(\mu, \nu) \in C_a$ imply $a(G_{(\mu,\lambda)}) = a$ and $(\mu, \nu) \in C_b$ imply $b(G_{(\mu,\lambda)}) = b$. These two curves transversally intersects at a point $\{P(a, b)\} = C_a \cap C_b$;

b) For $a_1, a_2 \in A_0$ such that $C_{a_1} \cap C_{b_2} \neq \emptyset$, where $b_2 = b(a_2)$, it is not hard to prove that the intersection is transversal and contains a unique point.

g) Let us now ask for the sets $L_{a_-\underline{b}}$ and $R_{b_+\underline{a}}$ for $a \in A_0$, $b = b(a)$.

At this point we establishes one of the main differences between contracting and expanding families.

Let

$$a \in A_0, \; R_{b_+\underline{a}} = \{(\mu, \nu);\; b(G_{(\mu,\nu)}) = b_+\underline{a}\} \quad \text{and}$$
$$L_{a_-\underline{b}} = \{(\mu, \nu);\; a(G_{(\mu,\lambda)}) = a_-\underline{b}\}.$$

Lemma 10. *$R_{b_+\underline{a}}$ is formed by*

a) a noncompact set, $R^1_{b_+\underline{a}}$, whose boundary is formed by two curves, $\gamma_1(b_+\underline{a})$ and $\gamma_2(b_+\underline{a})$ such that:

(i) $\gamma_2(b_+\underline{a}) \subset C_b$; $P(a, b) \in \gamma_2(b_+\underline{a})$

(ii) $P(a, b) \in \overline{\gamma_1(b_{+}\underline{a})}$ *and* $\gamma_1(b_{+}\underline{a})$ *is tangent to* C_b *at* $P(a, b)$ *and*

b) *a curve,* $C_{b_{+}\underline{a}}$, *which is tangent to* $\gamma_1(b_{+}\underline{a})$ *at a point* $Q(b_{+}\underline{a}) \neq P(a, b)$ *that satisfy* $C_{b_{+}\underline{a}}$ *is tangent to* C_b *at* $(0, 0) \in \overline{C_{b_{+}\underline{a}}}$.

The figure (11) displays the geometry of this Lemma.

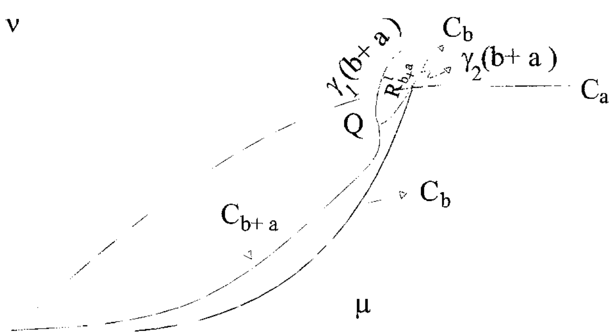

Figure 11:

Lemma 11. $L_{a_{-}\underline{b}}$ *is formed by*

a) *a noncompact set,* $L^1_{a_{-}\underline{b}}$, *whose boundary is formed by two curves,* $\gamma_1(a_{-}\underline{b})$ *and* $\gamma_2(a_{-}\underline{b})$ *such that:*

(i) $\gamma_2(a_{-}\underline{b}) \subset C_a$; $P(a, b) \in \gamma_2(a_{-}\underline{b})$

(ii) $P(a, b) \in \overline{\gamma_1(a_{-}\underline{b})}$ *and* $\gamma_1(a_{-}\underline{b})$ *is tangent to* C_a *at* $P(a, b)$ *and*

b) *a curve,* $C_{a_{-}\underline{b}}$, *which is tangent to* $\gamma_1(a_{-}\underline{b})$ *at a point* $\underline{Q(a_{-}\underline{b})} = Q(b_{+}\underline{a})$, *that satisfy* $C_{a_{-}\underline{b}}$ *is tangent to* C_a *at* $(0, 0) \in \overline{C_{a_{-}\underline{b}}}$.

The following figure (12) displays the geometry of this Lemma.

Proof. The proof of these two lemmas can be carried out in the following steps.

(i) assume $a = \underline{0_n 1}$. In this case consider the point $P = P(\underline{0_n 1}, \underline{10_n}) \in \mathbb{R}^2$.

(ii) let $U(P)$ be a small neighborhood of the point P.

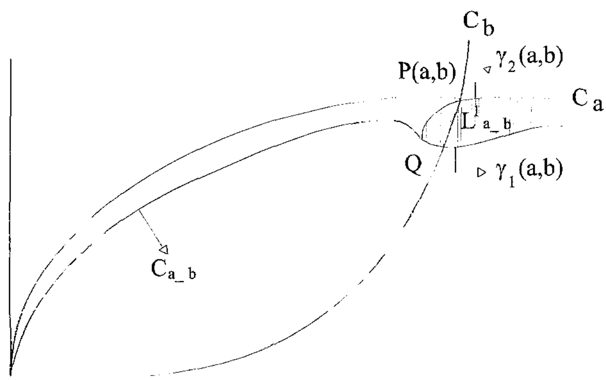

Figure 12:

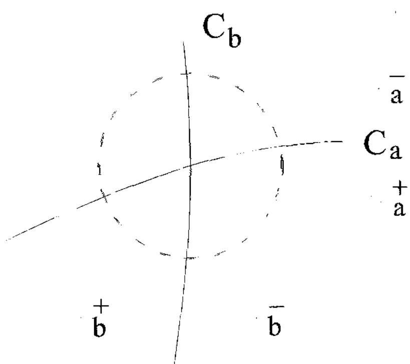

Figure 13:

(iii) the intersection $C_a \cap U(P)$ and $C_b \cap U(P)$ divides $U(p)$ into two disjoint regions U_a^+, U_a^-; U_b^+ and U_b^-, respectively (see figure (13)).

(iv) verify that there is a point $SN(b) \in C_a \cap U_b^+$ where we can find a saddle node for $F^{\#\mathrm{per}(b)}$, near $x = \nu$. Moreover, for $(\mu, \nu) \in]SN(b), P(a,b)[\subset C_a \cap \overline{U_b^+}$ we have $b(F_{(\mu,\nu)}) = b_+\underline{a}$.

(v) verify that there is a point $SN(a) \in C_b \cap U_a^+$ where we can find a saddle-node for $F^{\#\mathrm{per(a)}}$ near $x = 0^+$. Moreover, for $(\mu, \nu) \in]SN(a), P(a,b)[\subset C_b \cap U_a^+$ we have $a(F_{(\mu,\nu)}) = a_-\underline{b}$.

(vi) there is a point $Q(a,b) \in U_b^+ \cap U_a^+$ such that

a) $Q(a,b) \in \overline{SN(a)} \cap \overline{SN(b)}$;

b) the map $F^{\#\mathrm{per(a)}}_{Q(a,b)}$ has a fixed point with derivative 1 which is expanding.

(vii) there is a curve, $C_{b_+\underline{a}}$, tangent to C_b at $P(a,b)$; $P(a,b) \in (\overline{C_{b_+\underline{a}}} \setminus C_{b_+\underline{a}})$; such that $(\mu,\nu) \in C_{b_+\underline{a}}$ imply $b(F_{(\mu,\nu)}) = b_+\underline{a}$. This curve is unbounded and coincides with the saddle-node curve, $SN(b)$, up to $Q(a,b)$.

(viii) there is a curve, $C_{a_-\underline{b}}$, tangent to C_a at $P(a,b)$; $P(a,b) \in (\overline{C_{a_-\underline{b}}} \setminus C_{a_-\underline{b}})$; such that $(\mu,\nu) \in C_{a_-\underline{b}}$ imply $a(F_{(\mu,\nu)}) = a_-\underline{b}$. This curve is unbounded and coincides with the saddle-node curve, $SN(a)$, up to $Q(a,b)$.

(ix) the existence of the other component of the curves, $C_{b_+\underline{a}}$ and $C_{a_-\underline{b}}$, can be now easily computed.

Corollary 2. *(The geometry of sets L_a and R_b, $a \in A_0$, $b = b(a)$).*

a) $L_a = \{(\mu,\nu); a(F_{(\mu,\nu)}) = a\}$ *is unbounded and* $(\overline{L_a} \setminus L_a) \subset C_a \cup \gamma_1(b_+\underline{a})$ *(see figure (14)).*

b) $R_b = \{(\mu,\nu); b(F_{(\mu,\nu)}) = b\}$ *is unbounded and* $(\overline{R_b} \setminus R_b) \subset C_b \cup \gamma_1(a_-\underline{b})$ *(see figure (15)).*

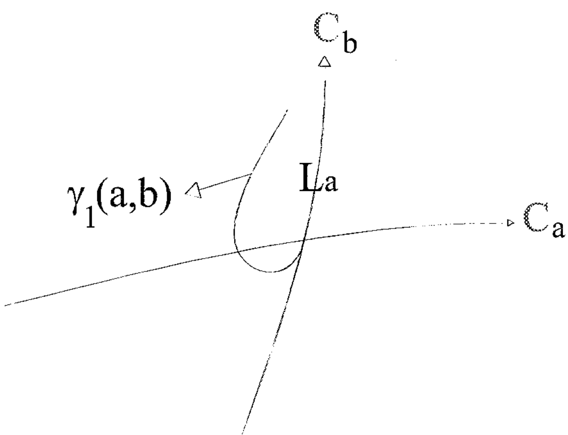

Figure 14:

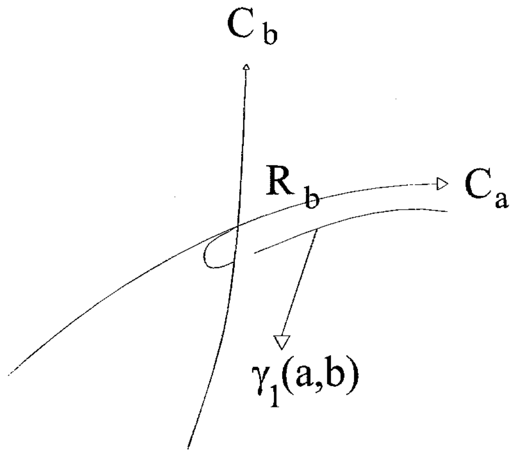

Figure 15:

4.6 The average

Let $a_1, a_2 \in A_0$ be two consecutive sequences, $a_1 < a_2$. Let $b_1 = b(a_1) < b_2 = b(a_2)$. Denote $a = m(a_1, a_2)$ and $b = m(b_1, b_2) = b(a)$.

Lemma 12. *There are curves C_a and C_b such that:*

(i) $(0,0) \in \overline{C_a}$, $(0,0) \in \overline{C_b}$.

(ii) *C_a is tangent to C_{a_i} at $(0,0)$, $i = 1,2$ and C_b is tangent to C_{b_i} at $(0,0)$, $i = 1,2$.*

(iii) *$C_a \cap C_b = \{P(a,b)\}$ and C_a transversally intersects C_b at $P(a,b)$.*

The figure (16) displays the geometry of the curves C_a and C_b.

Note.

(i) As in the previous section we obtain a similar result for $L_{a-\underline{b}}$, $R_{b+\underline{a}}$, L_a and R_b.

(ii) In this way we have obtained the geometry of the sets $L_{a-\underline{b}}$, $R_{b+\underline{a}}$, L_a and R_b for $a \in A_1 = A_0 \cup \{m(a_1, a_2);\quad a_1, a_2 \in A_0$ are consecutive sequences$\}$.

Inductively, we can obtain the geometry of the sets $L_{a-\underline{b}}$, $R_{b+\underline{a}}$, L_a and R_b for $a \in A_{n+1} = A_n \cup \{m(a_1, a_2);\quad a_1, a_2 \in A_n$ are consecutive sequences$\}$.

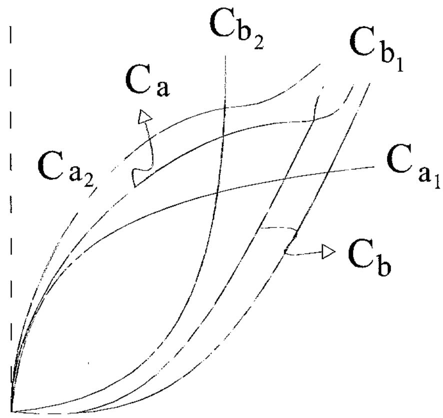

Figure 16:

4.7 The closure of the primary bifurcations

Let $A_\infty = \bigcup_{n=0}^{\infty} A_n$. Assume $a \in \overline{A_\infty}$ is a sequence which is not periodic nor eventually periodic. Let $b = \sup\{\sigma^k(a); \quad k \in \mathbb{N}\}$ and $a_n \in A_\infty$, $a_n \to a$. Set $b_n = b(a_n)$, we have $b_n \to b$.

In this situation we can find values of the parameter (μ_a, ν_a) and curves C_a, C_b, such that $C_a = \{(\mu, \nu); \quad \nu = \varphi_a(\mu)\}$; $C_b = \{(\mu, \nu); \quad \nu = \psi_b(\mu)\}$, where $\varphi_a\colon (0, \infty) \to \mathbb{R}$ and $\psi_b\colon (0, \infty) \to \mathbb{R}$ satisfies

$$\varphi_a(\mu) = \psi_b(\mu) \quad \text{for} \quad \mu \geq \mu_a;$$

The maps $\varphi_a|_{(0,\mu_a)}$ and $\psi_b|_{(0,\mu_a)}$ are tangent at μ_a. The curves C_{a_n} converges to C_a and the curves C_{b_n} converges to C_b.

The intersections $\{P(a_n, b_n)\} = C_{a_n} \cap C_{b_n}$ converges to $\{P(a, b)\} = \{(\mu_a, \varphi_a(\mu_a))\}$. Moreover, $(\mu, \nu) \in C_a$ imply $a(F_{(\mu,\nu)}) = a$ and $(\mu, \nu) \in C_b$ imply $b(F_{(\mu,\nu)}) = b$.

In fact, the curve C_a is the limit of the sequence of curves C_{a_n} and the curve C_b is the limit of the sequence of curves C_{b_n}.

4.8 Secondary bifurcations

Let $a \in A_\infty$. Associated with a define $A(a) = \{c \in \Sigma_0; \quad c = \underline{a\_b_+a^m}$ or $c = \underline{a\_b^m b_+}\,; m \in \mathbb{N}\}$. Denote $b = b(a)$.

Lemma 13. *For any $c \in A(a)$ we can find a curve, C_c, such that*

(i) $\{(0,0), P(a,b)\} \subset (\overline{C_c} \setminus C_c)$;

(ii) C_c *is located in the bounded region limited by* C_a *and* $C_{a\_\underline{b}}$ *and*

(iii) $(\mu, \nu) \in C_c$ *imply* $a(F_{(\mu,\lambda)}) = c$.

The following figures describe these properties of the curves C_c.

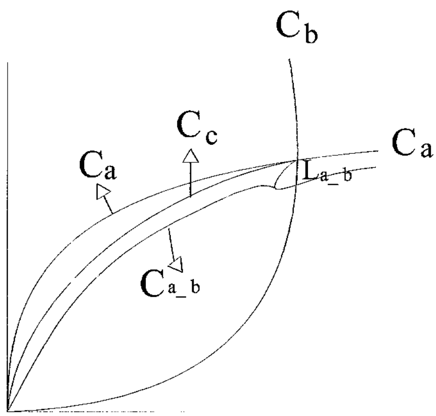

Figure 17:

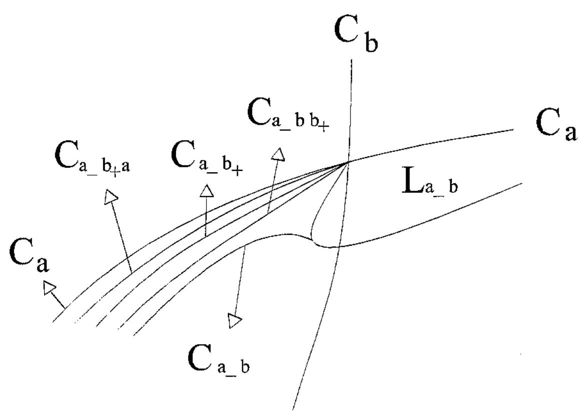

Figure 18:

Now define $B(b) = \{d \in \Sigma_1; \quad d = \underline{b_+ a^m a_-}$ or $d = \underline{b_+ a_- b^m}, m \in \mathbb{N}\}$.

Lemma 14. *For any $d \in B(b)$ we can find a curve, C_d, such that*

(i) $\{(0,0), P(a,b)\} \subset (\overline{C_d} \setminus C_d)$;

(ii) *C_d is located in the bounded region limited by C_b and $C_{b_{+}\underline{a}}$ and*

(iii) *$(\mu, \nu) \in C_d$ imply $b(F_{(\mu,\lambda)}) = d$.*

The figures (19) and (20) describe these properties of the curves C_d.

Remark 2.

1) We observe that: given $c \in A(a)$ and $d = \sup \sigma^k(c); k \in \mathbb{N}$ then $C_c \cap C_d = \emptyset$.

2) For any $c \in A(a)$ define $A(c) = \{\gamma \in \Sigma_0; \gamma = \underline{c_- d_+ c^m}$ or $\gamma = \underline{c_- d^m d_+}, m \in \mathbb{N}\}$, where $d = a(c)$. The same result is true for any $\gamma \in A(c)$ (that is, C_γ is a curve located in a region bounded by C_c and $C_{c_- \underline{d}}$; $\{(0,0), P(a,b)\} \subset (\overline{C_\gamma} \setminus C_\gamma)$).

3) For elements $c \in (\overline{A(a)} \setminus A(a))$ we obtain the corresponding curve C_c, as the limit of the curves C_{c_n} $c_n \in A(a)$ and $c_n \to c$.

4) Given $a \in A_\infty$, $c \in A(a)$ and $\gamma \in A(c)$ we can continue as in 2) above.

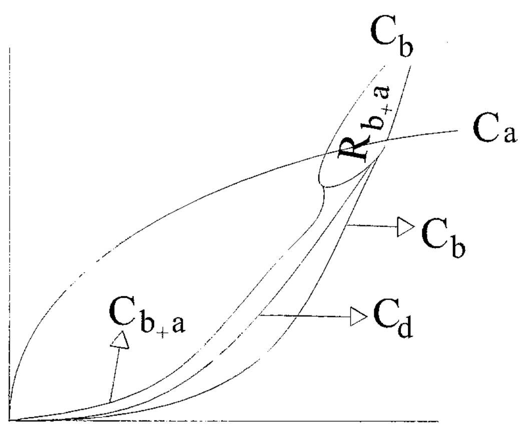

Figure 19:

4.9 Tongues

Consider $a \in A_\infty$ and $c \in A(a)$ or $c = a$.

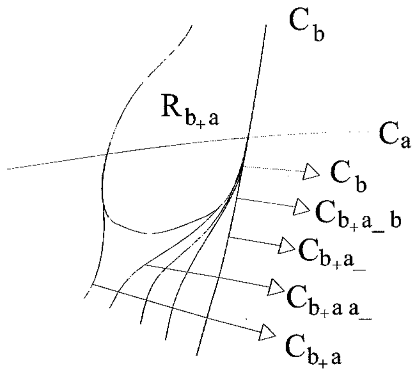

Figure 20:

Lemma 15. *Associated with any of these sequences there is a tongue* $L_{c\_b_+\underline{a}} = \{(\mu,\nu);\quad a(F_{(\mu,\nu)}) = c\_b_+\underline{a}\}$ *contained in* $R_{b_+\underline{a}}$. *Also, there are: a point* $Q(c) \in C_c$, *a curve* $C_{c\_b_+\underline{a}}$ *and a point* $\tilde{Q}(c\_b_+\underline{a}) \in C_{c\_b_+\underline{a}}$ *such that*

(i) $\{P(a,b),(0,0)\} \subset \overline{C_{c\_b_+\underline{a}}}$;

(ii) $[Q(c),P(a,b)] \subset C_c$, $[Q(c),\tilde{Q}(c\_b_+\underline{a})] \subset R_{b_+\underline{a}}$, $[\tilde{Q}(c\_b_+\underline{a}),P(a,b)] \subset C_{c\_b_+\underline{a}}$ *and* $\partial L_{c\_b_+\underline{a}} = [Q(c),P(a,b)] \cup [Q(c),\tilde{Q}(c\_b_+\underline{a})] \cup [\tilde{Q}(c\_b_+\underline{a}),P(a,b)]$.

The figure (21) displays these facts.

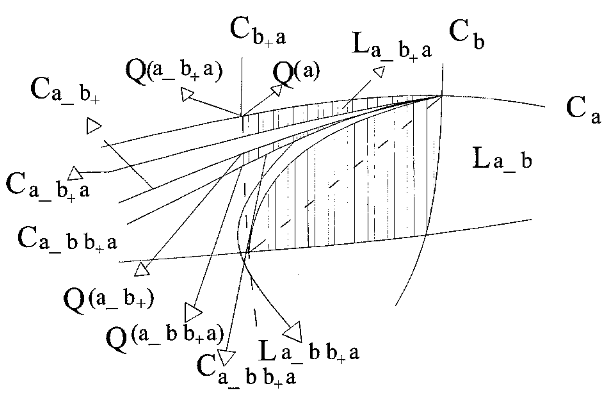

Figure 21:

Now, consider $d \in B(b)$ or $d = b$, $b = b(a)$.

Lemma 16. *Associated with any of these sequences there is a tongue* $R_{d_+a_-\underline{b}} = \{(\mu, \nu);\ b(F_{(\mu,\nu)}) = d_+a_-\underline{b}\}$ *contained in* $L_{a_-\underline{b}}$. *Also, there are: a point* $Q(d) \in C_d$, *a curve* $C_{d_+a_-\underline{b}}$ *and a point* $\tilde{Q}(d_+a_-\underline{b}) \in C_{d_+a_-\underline{b}}$ *such that*

(i) $\{P(a, b), (0, 0)\} \subset C_{d_+a_-\underline{b}}$

(ii) $[Q(d), P(a, b)] \subset C_d$, $[Q(d), \tilde{Q}(d_+a_-\underline{b})] \subset C_{a_-\underline{b}}$, $[\tilde{Q}(c_-b_+\underline{a}), P(a, b)] \subset C_{c_-b_+\underline{a}}$ *and* $\partial R_{d_+a_-\underline{b}} = [Q(d), P(a, b)] \cup [Q(d), \tilde{Q}(d_+a_-\underline{b})] \cup [\tilde{Q}(d_+a_-\underline{b}), P(a, b)]$.

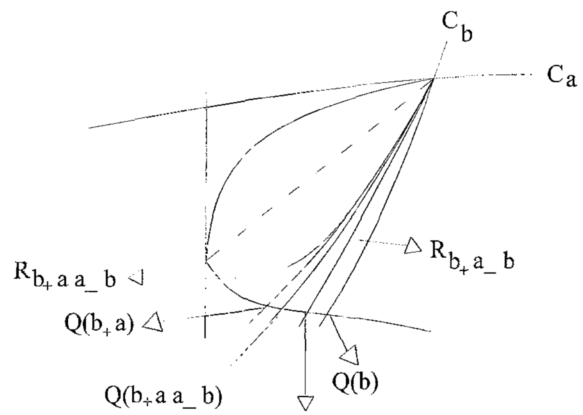

Figure 22:

4.10 Completing the secondary bifurcations

Let $a \in A_\infty$, $A_1(a) = A(a)$ and define $A_2(a) = A_1(a) \cup \{m(a_1, a_2);\ a_1, a_2 \in A_1(a)$ are consecutive sequences$\}$.

Inductively, $A_{n+1}(a) = A_n(a) \cup \{m(a_1, a_2); a_1, a_2 \in A_1(a)$ are consecutive sequences$\}$ and $A_\infty(a) = \bigcup_{n=1}^{\infty} A_n(a)$.

For any $c \in A_\infty(a)$ we have a curve, C_c, as in 4.8, also a tongue, $L_{c_-b_+\underline{a}}$, as in 4.9.

Take

$$a \in A_\infty\,,\ b = b(a)\,,\ B_1(b) = B(b) \quad \text{and}$$
$$B_2(b) = B_1(b) \cup \{m(d_1, d_2)\, d_1, d_2 \in B_1(b) \text{ are consecutive sequences}\}\,.$$

Inductively,

$$B_{n+1}(b) = B_n(b) \cup \{m(d_1, d_2);\ d_1, d_2 \in B_n(b)$$

$$\text{are consecutive sequences}\} \quad \text{and} \quad B_\infty(b) = \bigcup_{n=1}^{\infty} B_n(b)\,.$$

For any $d \in B_\infty(b)$ we have a curve, C_d, as in 4.8 and a tongue, $L_{d_+ a_- \underline{b}}$, as in 4.9.

4.11 The doubling period sequences

Let $a \in A_\infty$; $c \in A_\infty(a)$. Define

$$a_0 = c, \quad b_0 = b(c); \quad a_1 = \underline{(a_0)_-(b_0)_+}\,,$$
$$b_1 = \underline{(b_0)_+(a_0)_-} \quad \text{and} \quad a_{n+1} = \underline{(a_n)_-(b_n)_+}\,,$$
$$b_{n+1} = \underline{(b_n)_+(a_n)_-}\,, \quad n \geq 1.$$

For any a_n, we have a curve, C_{a_n}, as in 4.8 and a tongue, $L_{(a_n)_-(a(a_n))_+\underline{b_n}}$, as in 4.9.

For any b_n, we have a curve, C_{b_n}, as in 4.8 and a tongue, $R_{(b_n)_+(a(b_n))_-\underline{b_n}}$, as in 4.9.

For $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$ we have a curve as in 4.8.

4.12 $\underline{0}$ and $\underline{1}$ tongues

Let $a \in Min_2$ be any periodic sequence. Associated with the sequence $\tilde{a} = a_-\underline{1}$ there is a tongue $L_{\tilde{a}} \subset R_{\underline{1}}$ a curve $C_{\tilde{a}} \subset R_{\underline{1}}$ and an interval $I_{\tilde{a}} \subset \{(1/4, \nu);\ \nu \in [1/4, 3/4]\}$ such that:

- $\partial\overline{L_{\tilde{a}}} = C_a \cap L_{\tilde{a}} \cup C_{\tilde{a}} \cup I_{\tilde{a}}$.
- $(\mu, \nu) \in L_{\tilde{a}}$ imply $a(\mu, \nu) = \tilde{a}$.
- $(\mu, \nu) \in I_{\tilde{a}} \cup C_{\tilde{a}}$ imply $a(\mu, \nu) = \tilde{a}$.
- $(0, 0) \in \overline{C_{\tilde{a}}}$ and $\overline{C_a}$ is tangent to $\overline{C_{\tilde{a}}}$ at $(0, 0)$.

The next figure (23) represent these facts.

Let $b \in Max_2$ be any periodic sequence. Associated with the sequence $\tilde{b} = b_+\underline{0}$ there is a tongue $R_{\tilde{b}} \subset L_{\underline{0}}$ a curve $C_{\tilde{b}} \subset L_{\underline{0}}$ and an interval $I_{\tilde{b}} \subset \{(\mu, 1/4);\ \mu \in [1/4, 3/4]\}$ such that:

– $\partial \overline{R_{\tilde{b}}} = C_b \cap R_{\tilde{b}} \cup C_{\tilde{b}} \cup I_{\tilde{b}}$.

– $(\mu\,,\ \nu) \in R_{\tilde{b}}$ imply $b(\mu, \nu) = \tilde{b}$.

– $(\mu\,,\ \nu) \in I_{\tilde{b}} \cup C_{\tilde{b}}$ imply $b(\mu, \nu) = \tilde{b}$.

– $(0, 0) \in \overline{C_{\tilde{b}}}$ and $\overline{C_b}$ is tangent to $\overline{C_{\tilde{b}}}$ at $(0, 0)$.

The next figure (24) represent these facts.

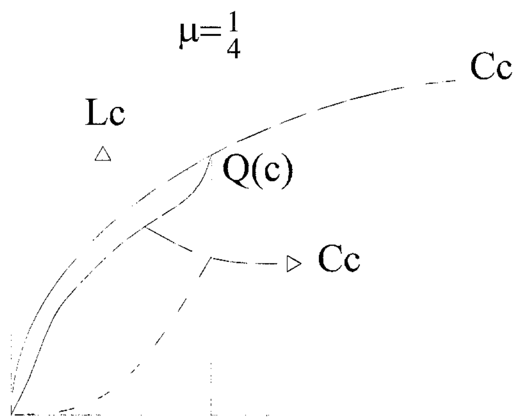

Figure 23:

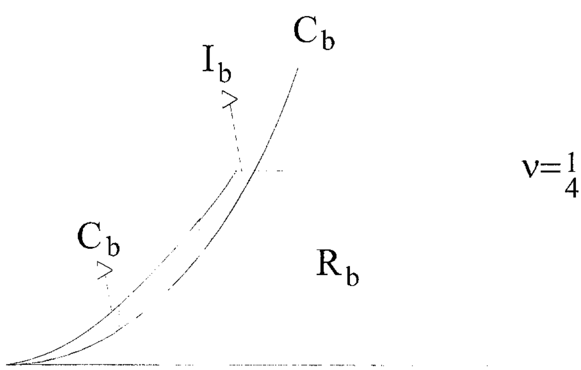

Figure 24:

4.13 Completing the bifurcation diagram

Assume $a \in Min_2, b \in Max_2$, $b \neq b(a)$ and $C_a \cap C_b \neq \emptyset$.

(i) Let us assume that a and b are periodic sequences. In this situation we have:

a) there are two curves, $C_{a_-\underline{b}}$ and $C_{\underline{a_-b_+}}$, tangents to C_a at $P(a,b)((0,0))$ such that $P(a,b) \in \overline{C_{a_-\underline{b}}}$ (or $C_{\underline{a_-b_+}}$) and $(0,0) \in \overline{C_{a_-\underline{b}}}$ (or $C_{\underline{a_-b_+}}$).

b) there are two curves, $C_{b_+\underline{a}}$ and $C_{\underline{b_+a_-}}$, tangents to C_b at $P(a,b)((0,0))$ such that $P(a,b) \in \overline{C_{b_+\underline{a}}}$ (or $P(a,b) \in \overline{C_{\underline{b_+a_-}}}$) and $(0,0) \in \overline{C_{b_+\underline{a}}}$ (or $\overline{C_{\underline{b_+a}}}$).

c) there is a curve, $C_{a_-b_+\underline{a}}$, tangent to C_a at $P(a,b)$ and $(0,0)$ such that $P(a,b) \in \overline{C_{a_-b_+\underline{a}}}$ and $(0,0) \in \overline{C_{a_-b_+\underline{a}}}$.

d) there is a curve, $C_{b_+a_-\underline{b}}$, tangent to C_b at $P(a,b)$ and $(0,0)$ such that $P(a,b) \in \overline{C_{b_+a_-\underline{b}}}$ and $(0,0) \in \overline{C_{b_+a_-\underline{b}}}$.

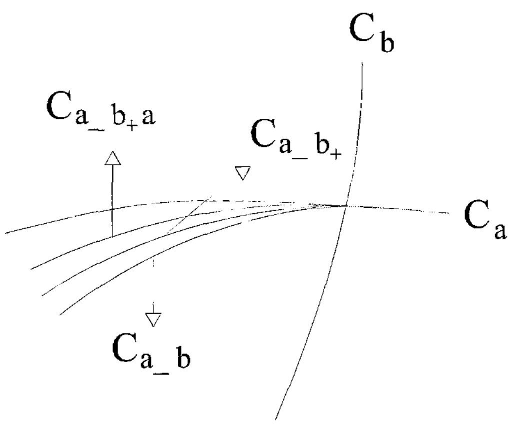

Figure 25: (a) and (c)

(ii) Let us now assume that b is a periodic sequence and a it is not a periodic sequence. In this situation there is a curve, C_{b_+a}, tangent to C_b at $P(a,b)$ and $(0,0)$. The figure (27) represent these facts.

(iii) Let us assume that a is a periodic sequence and b is not a periodic sequence. In this situation there is a curve, $C_{a_-\underline{b}}$, tangent to C_a at $P(a,b)$ and $(0,0)$. The figure (28) represent these facts.

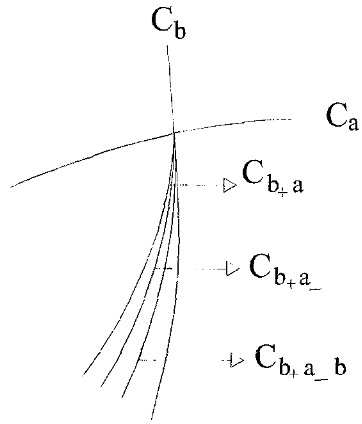

Figure 26: (b) and (d)

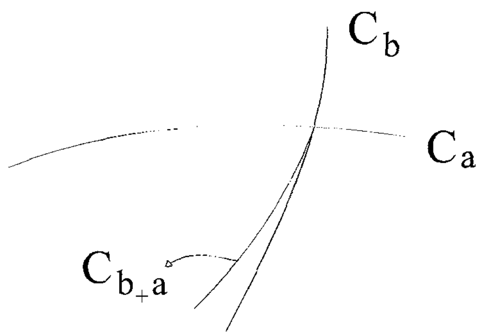

Figure 27: (ii)

4.14 Measure of the bifurcation set

Set $\pi = \{(\mu, \nu)\,;\, \mu \geq 0, \nu \geq 0\}$: $AA = \{(\mu, \nu) \in \pi\,;\, G_{(\mu,\nu)}$ satisfy the Axiom $A\}$ and $B = \pi \setminus AA$.

For $a \in A_\infty$ let $\Delta_a = L_a \cup L_{a_-\underline{b}} \cup R_{b_+\underline{a}}$, where $b = b(a)$ and $\Delta_\infty = \bigcup_{a \in A_\infty} \Delta_a \cup L_{\underline{0}} \cup R_{\underline{1}}$.

Lemma 17.

a) $m(\Delta_\infty \setminus AA) = 0$

b) *B contains open sets in π.*

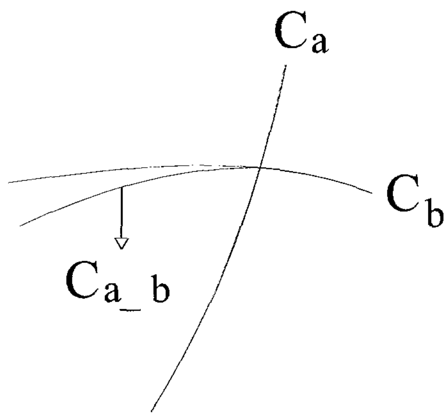

Figure 28: (iii)

This result follows as in theorem 2 in [13].

4.15 Entropy zero

For $a \in A_\infty$ let $\Gamma_a = L_a \cup R_b \cup L_{a\_b+\underline{a}} \cup R_{b+a\_\underline{b}}$, where $b = b(a)$ and $\Gamma_\infty = \bigcup_{a \in A_\infty} \Gamma_a \cup L_{0\underline{1}} \cup R_{1\underline{0}}$.

Let $\pi^+ = \{(\mu, \nu) \in \pi;\, \mu \neq 0 \text{ and } \nu \neq 0\}$.

Lemma 18. *$EZ \cap \pi^+ = \Gamma_\infty$, here EZ denotes the set of parameter values (μ, ν) where the respective map $G_{(\mu,\nu)}$ has entropy zero.*

This result follows from theorem 1 in [14].

4.16 The global picture

Taking together the results in this section the global picture of the bifurcation diagramme is represented in figure 29.

5 The general case

Let us consider the five parameter family of maps $G_{(\alpha,c_1,c_2,\mu,\nu)} : (R \setminus \{0\}) \to \mathbb{R}$, $\mu \geq 0$, $\nu \geq 0$, $\alpha < 1$, $c_1 > 0$, $c_2 > 0$ in DM_0 given by:

$$G_{((\alpha,c_1,c_2,\mu,\nu)}(x) = \begin{cases} -\mu + c_1 x^\alpha, & x > 0 \\ \nu - c_2(-x)^\alpha, & x < 0 \end{cases}$$

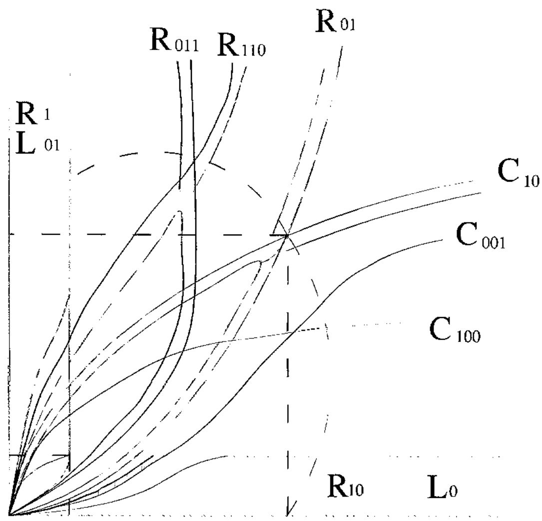

Figure 29: The global picture.

We observe that all the previous results, for the expanding family, are (still) true when we fix the parameters α , c_1 and c_2. That is, we can obtain the a and b-decomposition, associated to any two parameter family $F_{(\mu,\nu)} = F_{(\alpha,c_1,c_2,\mu,\nu)}$ in the same way as for the previous family. Nevertheless, the situation for topological equivalence is dramatically different as we will show in the next subsection.

5.1 Topologically equivalence for families

Assume $F = \{F_{(\mu,\nu)};\ (\mu,\nu) \in \pi\}$ and $G = \{G_{(\mu,\nu)};\ (\mu,\nu) \in \pi\}$ are two-parameter families of maps in DM_0.

Definition 6. We will say that F and G are *equivalent* if there is a bijection $\phi\colon \pi \to \pi$ such that $\forall(\mu,\nu) \in \pi$ the map $F_{(\mu,\nu)}$ is topologically equivalent to $G_{\phi(\mu,\nu)}$.

For the five parameter family of maps $G_{(\alpha,c_1,c_2,\mu,\nu)}$ we have

Lemma 19. *There are parameter values* (α, c_1, c_2) $(\tilde{\alpha}, \tilde{c_1}, \tilde{c_2})$ *such that the two parameter families* $F_{(\mu,\nu)} = G_{(\alpha,c_1,c_2,\mu,\nu)}$ *and* $H_{(\mu,\nu)} = G_{(\tilde{\alpha},\tilde{c_1},\tilde{c_2},\mu,\nu)}$ *are not equivalent.*

Proof.

(i) We have $G'(x) = 1$ for $x > 0$ if and only if $x(\alpha, c_1) = (\alpha c_1)^{\frac{1}{1-\alpha}}$. We have $G'(x) = 1$ for $x < 0$ if and only if $y(\alpha, c_2) = -(\alpha c_2)^{\frac{1}{1-\alpha}}$.

(ii) The equation for a fixed point of the saddle node type, for $x < 0$, are $\nu - c_2(-x)^\alpha = x$ and $x = -(\alpha c_2)^{\frac{1}{1-\alpha}}$, that is; $\nu = \nu_{sn} = c_2(\alpha c_2)^{\frac{\alpha}{1-\alpha}}(1-\alpha)$, is the parameter value where the map $G_{(\mu,\nu)}$ presents a saddle-node fixed point at $x(\alpha, c_2) = -(\alpha c_2)^{\frac{1}{1-\alpha}}$.

(iii) Now we ask for the curve $G(0^+) = -\mu =$ fixed point in $x < 0$. We have the equations $-\mu = y_2(v)$ and $\nu - c_2(-y_2(v))^\alpha = y_2(v)$, that is, $\nu - c_2\mu^\alpha = -\mu$, therefore; $\nu(\mu) = c_2\mu^\alpha - \mu$.

This curve is tangent to the curve $\nu = \nu_{sn}$ at the value $\mu = \mu_{max} = (\alpha c_2)^{\frac{1}{1-\alpha}}$.

(iv) Figure 30 displays the region of the parameters (μ, ν) such that $a(\mu, \nu) = \underline{0}$. This region is given by $\{(\mu, \nu); 0 \leq \nu \leq \nu(\mu); 0 \leq \mu \leq \mu_{max}\}$ or $0 \leq \nu \leq \nu_{sn}$ for $\mu \geq \mu_{max}$.

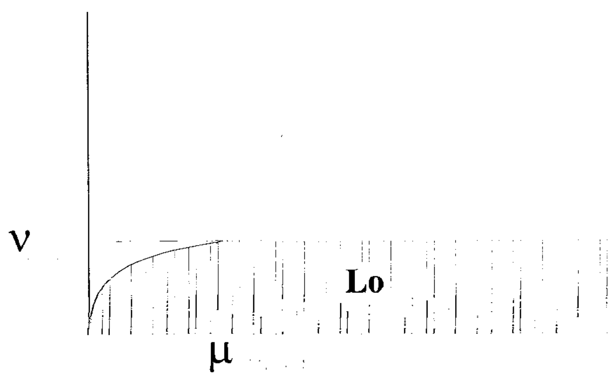

Figure 30:

(v) The set $\{(\mu, \nu), b(\mu, \nu) = \underline{10}\}$ contains the curve $\nu_{\underline{10}} = \left(\frac{\mu}{c_1}\right)^{1/\alpha}$. See Figure 31 for this picture.

(vi) We have $\nu_{\underline{10}}(\mu) = \nu_{sn}$ if and only of

$$\mu = \bar{\mu} = c_1 c_2^\alpha (1-\alpha)^\alpha \cdot (\alpha c_2)^{\frac{\alpha^2}{1-\alpha}}.$$

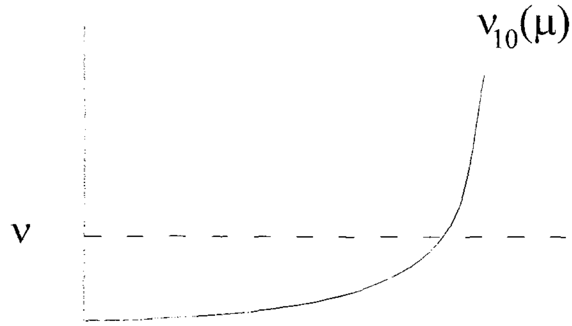

Figure 31:

(vii) In this situation

$$\frac{\bar{\mu}}{\mu_{max}} = \frac{c_1 c_2^{\alpha}(1-\alpha)^{\alpha}(\alpha c_2)^{\frac{\alpha^2}{1-\alpha}}}{(\alpha c_2)^{\frac{1}{1-\alpha}}} = \frac{c_1}{c_2}\frac{1-\alpha}{\alpha^{1+\alpha}} = S(c_1, c_2, \alpha).$$

It is clear that there are values (c_1, c_2, α) and $(\tilde{C}_1, \tilde{C}_2, \tilde{\alpha})$ such that

$$S(c_1, c_2, \alpha) < 1 < S(\tilde{c}_1, \tilde{c}_2, \tilde{\alpha}).$$

In the first case the curve $\nu_{10}(\mu)$ is like in figure 32.

In the second case the curve $\nu_{10}(\mu)$ is like in figure 33.

Clearly, each case define a two parameter families as announced in the lemma.

6 Application to geometric vector fields

Assume $X_0 \colon U \subset \mathbb{R}^3 \to \mathbb{R}^3$ is a three dimensional vector field that satisfies:

a) $0 \in U$ is an hyperbolic singularity of the vector field X, whose eigenvalues satisfies $-\lambda_1 < -\lambda_2 < 0 < \lambda_3$;

b) The components, γ_1 and γ_2, of the set $(W_0^u \setminus \{0\})$ satisfies $\gamma_1 \subset W_0^s$, $\gamma_2 \subset W_0^s$;

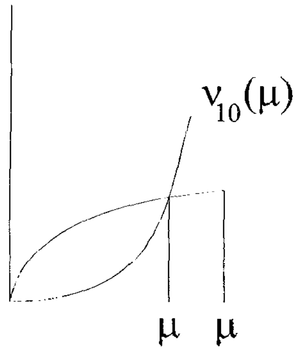

Figure 32:

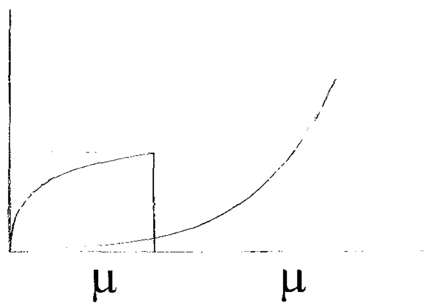

Figure 33:

c) There is a transversal section, $\Sigma \subset U$, such that: Σ is transversal to W_0^s and $\Sigma \setminus W_0^s = \Sigma_1 \cup \Sigma_2$ are two disjoint sets in Σ that satisfies:

$$\pi_1(\Sigma_1) \subset \Sigma_1 \quad \text{and} \quad \pi_2(\Sigma_2) \subset \Sigma_2$$

where $\pi_i : \Sigma_i \to \Sigma, i = 1, 2$ is the first return map associated to the respective cross section.

Let $\mathcal{U}_0 \subset \mathfrak{X}^r(U, \mathbb{R}^3)$ be a neighborhood of the vector field X_0 in $\mathfrak{X}^r(U, \mathbb{R}^3) = \{X : U \subset \mathbb{R}^3 \to \mathbb{R}^3;\ X \text{ is } C^r\}$ with the usual C^r-topology.

In the above conditions there are codimension one submanifold, $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{U}_0$; such that:

a) $Y \in \mathcal{N}_i$ if and only if $\gamma_i(Y) \subset W^s_{\sigma(Y)}$ here $\sigma(Y)$ is the hyperbolic singularity, near 0, that correspond to the vector field Y. Also, $(W_{\sigma(Y)}\hat{\text{U}}\backslash\{\sigma(Y)\}) = \gamma_1(Y) \cup \gamma_2(Y)$, where $\gamma_i(Y)$ is the natural extension of the component γ_i;

b) $(\mathcal{U}_0 \setminus \mathcal{N}_i) = \mathcal{U}_1^i \cup \mathcal{U}_2^i$ where $Y \in \mathcal{U}_1^1$ if and only if $\pi_1(\Sigma_1) \subset \Sigma_1$ and $Y \in \mathcal{U}_2^2$ if and only if $\pi_2(\Sigma_2) \subset \Sigma_2$ and

c) $\mathcal{N}_1$ is transversal to $\mathcal{N}_2$.

Finally we will assume that Σ can be foliated by one-dimensional submanifolds $\{\mathcal{F}_x; \quad x \in \Sigma\}$ such that

(i) $x \in W^s_Y \cap \Sigma$ imply $\mathcal{F}_x = W^s_Y \cap \Sigma$ and

(ii) $\pi_i(\mathcal{F}_x) \subset \mathcal{F}_{\pi_i(x)}, x \in \Sigma_i$.

In this situation the interesting part of the bifurcation theory, for elements in $\mathcal{U}_0$, is located in $\mathcal{U}_1^2 \cap \mathcal{U}_2^1$. In this set we can apply the results in Sections 3,4 and 5 to obtain, bis a bis, similar results for generic two parameter families of vector fields in $\mathcal{U}_0$.

Acknowledgments. Part of this paper is an outgrowth of research during a visit of the authors to ICTP (Trieste – Italy), a visit of RL to IMPA-Brazil and a visit of the second author to the Universidad de Santiago de Chile. We thanks to these institutions for their support and hospitality while preparing this paper.

References

[1] Afraimovich V.S., Bykov V.V. and Shil'nikov L.P., *On structurally stable attracting limit sets of Lorenz attractor type*. Trans. Mosc.Math. Soc., **44** (1983), 153-216.

[2] Arnold V.I., *Small Denominators. I. On the mapping of the circumference onto itself.* Amer. Math. Soc. Transl. 2^{nd} Ser., **46** (1965), 213-284.

[3] Babrilea F. and Jimênez Lôpez V., *A Characterization of chaotic functions with antropy zero via their scrambled sets*. Math. Bohen., **120**(2) (1998), 293-296.

[4] Boyland P., *Bifurcations of Circle Maps: Arnold's Tongues, Bistability and Rotation Intervals*. Comm. Math. Phys., **106** (1986), 353-381.

[5] Brucks K.M., Misiurewicz M. and Tresser Ch., *Monotonicity properties of the family of trapezoidals maps*. Comm. Mat. Phys., **137** (1991), 1-12.

[6] Campbell D.K., Galeeva R., Tresser CH. and Uherka D.J., *Piecewise linear models for the quasiperiodic transition to chaos*. Chaos, **6**(2) (1996), 121-154.

[7] De Melo W. and Martens M., *Universal Models for Lorenz maps*. Preprint IMPA. (1996).

[8] De Melo W. and Van Strien S., *Lectures on One Dimensional Dynamics*. Springer Verlag. (1993).

[9] Guckenheimer J., *A Strange, Strange Attractor*. In: *Hopf Bifurcations and its applications*. J.E. Marsden and M. McCracken Eds. Springer Verlag. Berlin. (1976), 368-381.

[10] Galeeva R., Mar[illegible]s M. and Tresser Ch., *Inducing, slopes and conjugacy classes*. Israel Journal of Math., **99** (1997), 123-147.

[11] Guckenheimer J. and Williams R.F., *Structural Stability of Lorenz Attractors*. Publ. Math. IHES, **50** (1979), 59-72.

[12] Hubbard J.H. and Sparrow C.T., *The classification of topologically expansive Lorenz maps*. Comm. on Pure and App. Math., **XLIII** (1990), 431-443.

[13] Labarca R., *Bifurcation of Contracting Singular Cycles*. Ann. Scient. Ec. Norm. Sup. $4^{\grave{e}me}$ série, **t.28** (1995), 705-745.

[14] Labarca R. and Moreira C., *Essential Dynamics for Lorenz maps on the real line and the Lexicographical World*. Preprint 2000 and submitted.

[15] Labarca R. and Moreira C., *Bifurcations of the Essential Dynamics of Lorenz Maps on the real line and the bifurcation scenario for the linear family*. Preprint 2000 and accepted for publication in Scientia.

[16] Labarca R. and Moreira C., *Bifurcation of the Essential Dynamics of Lorenz Maps on the real line and the bifurcation scenario for Lorenz Like Flows: The Contracting Case*. Preprint 2000.

[17] Lorenz E.N., *Deterministic non-periodic flow*. J. Atmos. Sci., **20** (1963), 130-141.

[18] Mañé, R., *Ergodic Theory and Differentiable Dynamics*. Springer Verlag, 1987.

[19] Williams R.F., *The structure of Lorenz Attractors*. In *Turbulence Seminar Berkeley 1976/1977*. P. Bernard, T. Ratiu (Eds.) Springer-Verlag. New York, Heidelberg, Berlin, 94-112.

R. Labarca
Departamento de Matematica y CC
Universidad de Santiago de Chile
Casilla 307 Correo 2 – Santiago
Chile

C. Moreira
Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110
22460-320 – Jardim Botânico
Rio de Janeiro, RJ.
Brazil