A Note on Convergence in Probability*

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We give short proofs of generalizations of the main results contained in [1] and [2]. Let S be a metric space with metric d and balls $B(x, \delta) = \{y : d(y, x) < \delta\}$ and $\mathscr{S}$ the Borel σ-field of S. If X is a random variable defined on a probability space $(\Omega, \mathscr{A}, P)$ with values in S we denote with $\mathscr{L}(X)$ the distribution of X, and sometimes with $\mathscr{L}_P(X)$ if more than one probability measure is involved. The symbol $\xrightarrow{\omega}$ (resp. $\xrightarrow{P}$) indicates weak converge (resp. convergence in probability). If $A \in \mathscr{A}$ is such that $P(A) > 0$, $P(\cdot \mid A)$ will indicate the conditional probability given A.

If P and Q are probabilities on $\mathscr{A}$ which are mutually absolutely continuous we will say that they are equivalent and write $P \equiv Q$.

The symbol Δ indicates symmetric difference; ∂C denotes the boundary of the set $C \subseteq S$; A^c indicates the complement of A. For notation and basic properties of weak convergence the reader is refered to [3]. The main results are contained in Theorems 1 and 2. The proof of Theorem 1 is based on the following Lemma.

LEMMA 1 – *Let $(\Omega, \mathscr{A}, \mu)$ be a non-atomic finite measure space, S a polish space, ν a Borel measure on $\mathscr{S}$ such that $\mu(\Omega) = \nu(S)$. Then there exists a mearusable transformation $X : \Omega \to S$, such that $\forall B \in \mathscr{S}$*

$$\mu[X \in B] = \nu(B).$$

PROOF: It is sufficient to consider the case $\mu(\Omega) = \nu(S) = 1$. In this case construct a random variable Y on Ω such that its distribution is uniform on $[0, 1]$. Then construct a random variable with values in S on the probability space $([0, 1], \mathscr{B}, \lambda)$, (where $\mathscr{B}$ is the Borel σ-field of $[0, 1]$ and λ is the Lebesgue measure) and such that $\mathscr{L}(Z) = \nu$. The composition $X = Z \circ Y$ gives the desired transformation.

In the next Theorem S is a polish space.

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THEOREM 1 – *Let* $(\Omega, \mathscr{A}, P)$ *be a non-atomic probability space,* X *a random variable with values in* S*. Then* $X_n \xrightarrow{P} X$ *for all sequences* X_n *such that* $\mathscr{L}(X_n) \xrightarrow{\omega} \mathscr{L}(X)$*, if and only if* X *is degenerate.*

PROOF: We only prove the "only if" part; the other is well known. If X is not degenerate there exists a Borel set C such that $P[X \in C] > 0$ and $P[X \notin C] > 0$. Let's assume that $P[X \in C] \geq 1/2$. Let $A \subseteq [X \in C]$ be such that $P(A) = P(X \notin C]$. Let X_1 be a measurable transformation on $(A, \mathscr{A} \cap A)$ such that $\forall B \in \mathscr{S}$

$$P[X_1 \in B] = P([X \in B] \cap [X \notin C]).$$

Let X_2 a measurable transformation on $([X \notin C], \mathscr{A} \cap [X \notin C])$ such that $\forall\, B \in \mathscr{S}$.

$$P[X_2 \in B] = P([X \in B] \cap A).$$

Define

$$Y(\omega) = \begin{cases} X(\omega) & \text{if} \quad \omega \in [X \in C] - A \\ X_1(\omega) & \text{if} \quad \omega \in A \\ X_2(\omega) & \text{if} \quad \omega \in [X \notin C] \end{cases}$$

It is easy to check that $\mathscr{L}(Y) = \mathscr{L}(X)$ but $P[Y \neq X] > 0$. If we define $X_n = Y, \forall n$, we clearly have $\mathscr{L}(X_n) \xrightarrow{\omega} \mathscr{L}(X)$ but $X_n \not\xrightarrow{P} X$.

On atomic spaces this result is false as the following easy example shows.

$$\Omega = \{0, 1\} \qquad P(\{0\} = \tfrac{1}{3} \qquad P(\{1\}) = \tfrac{2}{3} \qquad X(0) = 0, \qquad X(1) = 1.$$

X is not degenerate and if $\mathscr{L}(X_n) \xrightarrow{\omega} \mathscr{L}(X)$, is easy to verify that $X_n \to X$ at all points.

In the next Theorem S is a separable space.

THEOREM 2 – $X_n \xrightarrow{P} X$ *if and only if* $\mathscr{L}_Q(X) \xrightarrow{\omega} \mathscr{L}_Q(X)$, $\forall Q \equiv P$.

PROOF: The "only if" part follows immediatly. The other implication is a consequence of the following two lemmas.

LEMMA 2 – $\mathscr{L}_Q(X_n) \xrightarrow{\omega} \mathscr{L}_Q(X)$, $\forall Q \equiv P$ *implies* $P([X_n \in C] \,\Delta\, [X \in C]) \to 0$ $\forall C$ *such that* $P[X \in \partial C] = 0$.

PROOF: Let C be such that $P[X \in \partial C] = 0$. If $P[X \in C] > 0$ define

$$Q = \frac{P(\cdot\,|[X \in C] + P}{2}.$$

$Q \equiv P$ and then by hypothesis $Q[X_n \in Q] \to Q[x \in C]$.

This last equation is equivalent to

$$P[X_n \in C,\ X \in C] \to P[X \in C].$$

We also have

$$P[X_n \in C] \to P[X \in C].$$

If $P[X \in C] = 0$ this two equations are trivially satisfyied. It is easyly seen that they imply the result.

LEMMA 3 – *If* $\forall C$ *such that* $P[X \in \partial C] = 0$, $P([X_n \in C] \,\Delta\, [X \in C]) \to 0$, *then* $X_n \xrightarrow{P} X$.

PROOF: Let $\varepsilon > 0, 0 < \delta < 1$, and $A = \{(x, y) : (x, y) \in S \times S$ and $d(x, y) > \varepsilon\}$. Let $\{x_i\}_{i=1,2,\ldots}$ be a countable dense subset of S. Select $0 < \gamma < \varepsilon/2$ such that $P[d(X, x_i) = \gamma,\ i = 1, 2, \ldots] = 0$. Select now N such that

$$P[X \in \bigcup_{i=1}^{N} B(x_i\,, \gamma)] > 1 - \delta$$

and check that if $K = \bigcup_{i=1}^{N} B(x_i\,, \gamma)$

$$A \cap (K \times S) \subseteq \bigcup_{i=1}^{N} [B(x_i\,, \gamma) \times B(x_i\,, \gamma)^c].$$

Therefore

$$\begin{aligned} P[d(X, X_n) > \varepsilon] &\leq \delta + P[X \in K,\ d(X, X_n) > \varepsilon] \\ &= \delta + P[(X, X_n) \in A \cap (K \times S)] \\ &\leq \delta + \sum_{i=1}^{N} P[(X, X_n) \in B(x_i\,, y) \times B(x_i\,, \gamma)^c] \\ &= \delta + \sum_{i=1}^{N} P[X \in B(x_i\,, \gamma),\ X_n \notin B(x_i\,, \gamma)] \\ &\leq \delta + \sum_{i=1}^{N} P([X \in B(x_i\,, \gamma)] \,\Delta\, [X_n \in B(x_i\,, \gamma)]). \end{aligned}$$

Therefore

$$\lim_n \sup P[d(X, X_n) > \varepsilon] \leq \delta \quad \forall \delta > 0$$

which is the desired result.

REFERENCES

[1] A. R. PADMANABHAN – (1970) – *Some conditions for convergence in probability and convergence in law to coincide.* Mathematica Japonicae, Vol. 15, n.º 2, 105-109.

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