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Singular cycles of vector fields on regular parts of the boundary of Morse-Smale systems

Rodrigo Bamón[1]

Abstract. At the boundary of the class of Morse-Smale vector fields there are vector fields whose unique degenerate phenomena is a singular cycle. We first characterize and classify all singular cycles which contains only one degeneracy (the *simple singular cycles: ssc*). Each of these cycles defines a codimension one submanifold of vector fields. For some ssc its codimension one submanifold is a regular part of the boundary of the Morse-Smale systems. We characterize those ssc that defines this type of submanifold. Our ambient space is n dimensional, $n \geq 2$.

Keywords: Vector fields, Singular cycles, Morse-Smale.

1. Introduction.

We consider C^r, r big enough, differentiable vector fields defined on n dimensional manifolds, $n \geq 2$. A *cycle* of a vector field is a chain recurrent set formed by a finite collection of critical elements (singularities and periodic orbits) together with a finite collection of regular orbits. A cycle is *singular* if at least one of its critical elements is a singularity.

As we shall see, vector fields with singular cycles are not structurally stable. However, they may appear persistently in one-parameter families of vector fields. In fact, there are codimension-one submanifolds in the space of vector fields defined by systems with a singular cycle as unique degenerate phenomenon. The degenerate phenomenon appearing on these singular cycles is either a non-hyperbolic critical element or a non-transversal intersection between invariant manifolds.

One of the aims of this work is to characterize and classify all these singular cycles. We call them *simple* singular cycles.

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Vector fields with simple singular cycles may belong to the boundary of the Morse-Smale systems. This part of the boundary is regular, that is, a codimension one submanifold. Generic one-parameter families of vector fields leaves the Morse-Smale systems through systems with simple singular cycles.

A second aim in this work is to characterize and classify those simple singular cycles of vector fields on the boundary of the Morse-Smale systems.

The motivation for this study, beyond the intrinsic mathematical interest of classifying systems, comes from bifurcation theory: simple singular cycle may separate Morse-Smale systems from systems with non-trivial recurrent sets: an Ω- explosion phenomena. Although this kind of Ω - explosion is studied by many authors for particular singular cycles ([**BLMP**], [**L**], [**PR**], [**Ri**], [**Ro**], [**S**]), there is no global view of all the singular cycles to be considered for this respect. We intend to cover this gap.

In Section 2 we give some general definitions and we state our results. In Section 3 we prove Lemma 1, which essentially characterize the singular cycles that contains only one degeneracy: the simple singular cycles. Further, in this Section we also classify these simple singular cycles, and, we impose generic conditions to them. In Section 4 we prove our results. In Section 5 we prove various simple technical lemmas which complete the proofs of Section 4. In Section 6 we speculate about general statements for the unfolding of simple singular cycles.

2. Statement of results.

A *cycle* Γ of a vector field X is a set

$$\Gamma = \left(\bigcup_{i=0}^{k-1} \sigma_i\right) \bigcup \left(\bigcup_{i=0}^{k-1} \gamma_i\right)$$

where $\{\sigma_i\}_{i=0}^{k-1}$ are critical elements and $\{\gamma_i\}_{i=0}^{k-1}$ are regular orbits such that $\alpha(\gamma_i) = \sigma_i$ and $\omega(\gamma_i) = \sigma_{i+1}$, $\forall i = 0, \cdots, k-1 \pmod{k}$. Here, $\alpha(\gamma)$ and $\omega(\gamma)$ denotes the α and ω limit sets of the orbit γ. For a critical element σ, $W^s(\sigma) = \{x/\omega(x) = \sigma\}$ and $W^u(\sigma) = \{x/\alpha(x) = \sigma\}$ denotes, respectively, the stable and unstable sets of σ. For a cycle Γ, $\gamma_i \subseteq W^u(\sigma_i) \cap W^s(\sigma_{i+1})$, $\forall i = 0, \cdots k-1$. A cycle is *singular* if at least one of the critical elements of Γ is a singularity.

A cycle is *hyperbolic* if all of its critical elements are hyperbolic; otherwise it is called *non-hyperbolic.* A cycle is *homoclinic* if contains only one critical element. If not, it is *heteroclinic.*

A singular cycle Γ is *simple* if either:

a) It is hyperbolic, it contains a unique singularity and every regular orbit γ_i in Γ, with the exception of exactly one (say γ_j), is a transversal intersection between $W^s(\sigma_{i+1})$ and $W^u(\sigma_i)$. Further,

$$\dim(W^s(\sigma_{j+1})) + \dim(W^u(\sigma_j)) = n$$

and $W^s(\sigma_{j+1}), W^u(\sigma_j)$ are in general position (that is, $T_pW^s(\sigma_{j+1}) + T_pW^u(\sigma_j)$ is a hyperspace in $T_pM \ \ \forall p \in \gamma_j$).

b) Only one of the critical elements of Γ is non-hyperbolic and of saddle-node type. There is a unique singularity in Γ and every regular orbit γ_i in Γ is a transversal intersection between $W^s(\sigma_{i+1})$ and $W^u(\sigma_i)$.

In case a) the orbit γ_j is called the *non-transversal orbit* (*non-transversal intersection*) of Γ.

Every simple singular cycle (ssc) contains a unique degeneracy. The next Lemma states this and its inverse: simplicity gives a characterization for those singular cycles with just one degeneracy.

Lemma 1. (a) *Vector fields with simple singular cycles lie in codimension one submanifolds in the function space of vector fields. They involve only one degeneracy.* (b) *Vector fields with non simple singular cycles lie in submanifolds of higher codimension. They involve more than one degeneracy.*

Part (a) is nowadays an standard fact in bifurcation theory. Part (b) will be proved.

From now on, given a vector field X with a ssc Γ, we denote by $\mathcal{U}$ a small neighborhood of X, by $\mathcal{N}$ the local codimension one submanifold on the space of vector fields defined by the unique degeneracy in Γ, and by U a small neighborhood of Γ. We may characterize $\mathcal{N}$ by

$$\mathcal{N} = \{Y \in \mathcal{U} \ / \ Y \text{ has a ssc } \Gamma_Y, \Gamma_Y \text{ the analytic continuation of } \Gamma\}.$$

The submanifold $\mathcal{N}$ divides $\mathcal{U}$ into two components. The *analytic continuation of a ssc* Γ of a vector field X are the ssc's of nearby vector fields formed by identical number and type of critical elements as in Γ, with similar connections between critical elements and which are close to Γ in the Hausdorff metric of compact subsets.

For every vector field $Y \in \mathcal{U}$ let $\Lambda_Y = \cap_{t\in\mathbb{R}}\varphi_Y(t, U)$, where $\varphi_Y(t, x)$ denotes the flow of Y, be the maximal invariant set of Y in U. By $\Omega(Y)$ we denote the non-wandering set of Y.

We say that a vector field X with a ssc Γ belongs to the *boundary of the Morse-Smale systems* if $\Omega(Y) \cap U$ is trivial for every vector field Y in one of the components of $\mathcal{U} \setminus \mathcal{N}$. We say that $\Omega(Y) \cap U$ is trivial if it only contains the analytic continuation of the critical elements in Γ. (By the analytic continuation of a saddle-node critical element we mean the two critical elements unfolding from it).

We now characterize and classify the ssc of vector fields that are in the boundary of the Morse-Smale systems. As we will see, this depends both on standard generic conditions and on local properties to be satisfied by the ssc. The description of the generic conditions is done in Section 3. For the moment, a ssc that verifies these generic conditions is said to be a *generic* simple singular cycle (gssc).

We say that a gssc Γ of a vector field X is *isolated* if $\Lambda_X = \Gamma$ for every U small enough. Later on we will prove that Γ being isolated is a necessary condition for X to be in the boundary of the Morse-Smale systems.

Let X be a vector field with a hyperbolic gssc Γ. Let γ_j be the non-transversal orbit of $\Gamma, \gamma_j \subseteq W^u(\sigma_j) \cap W^s(\sigma_{j+1})$. We say that γ_j is *adjacent* to the singularity if either σ_j or σ_{j+1} is the singularity.

In Section 3 we define a *signature* for the non-transversal orbit γ_j of a hyperbolic gssc. The non-transversal orbit is the intersection of a stable and an unstable manifold whose tangent spaces generically span a codimension one subspace (they can't intersect transversally). These manifolds are the boundary of a center-stable and center-unstable manifold respectively. Roughly, the signature of the non-transversal intersection tells as about how these center-stable and center-unstable manifolds are relatively located. For the moment we o state Theorem 1 below.

The following two conditions for the non-transversal orbit γ_j of a hyperbolic gssc are relevant for the statement of our results.
(C1) The weakest contracting (resp. expanding) eigenvalue of σ_j (resp. σ_{j+1}) is real. Further, it is positive if σ_j (resp. σ_{j+1}) is a periodic orbit.
(C2) The non-transversal orbit γ_j has negative signature.

Theorem 1. *Let X be a vector field with a heteroclinic hyperbolic gssc Γ. Let γ_j be the non transversal orbit of $\Gamma, \gamma_j \subseteq W^u(\sigma_j) \cap W^s(\sigma_{j+1})$. Then:*

(a) *X is in the boundary of the Morse-Smale systems if and only if conditions (C1) and (C2) hold.*
(b) *If (C1) and (C2) hold then Γ is isolated.*
(c) *If γ_j is adjacent to the singularity the converse of b) is true: if Γ is isolated then conditions (C1) and (C2) hold.*

Remark. If γ_j is not adjacent to the singularity then there are *subordinated cycles* of Γ (that is, cycles whose critical elements are some of the ones in Γ) containing only periodic orbits as critical elements. This kind of non-singular cycles can be considered as a diffeomorphism's cycle. In our case we arrive exactly to those cycles studied in [**DR**] and [**D**]. In this way we find isolated singular cycles where either (C1) or (C2) (or both) doesn't hold.

Consider a hyperbolic singularity σ of a vector field X. Let $a < 0$ (resp. $b > 0$) be the real part of the weakest contracting (resp. expanding) eigenvalue of DX_σ. That is, $\Re\lambda \leq a$ for every eigenvalue of DX_σ with negative real part. Similarly for eigenvalues with positive real parts. We say that σ is *centrally contracting* (resp. *centrally expanding*) if $a + b < 0$ (resp. $a + b > 0$). Generically, these weakest eigenvalues are simple and either real or complex.

Now consider condition (C) defined for a homoclinic gssc $\Gamma = \{\sigma\} \cup \{\gamma\}$.

(C) Either σ is centrally contracting with a real weakest expansion or σ is centrally expanding with a real weakest contraction.

Theorem 2. *Let X be a vector field with a homoclinic hyperbolic gssc Γ. They are equivalent:*
(a) *X belongs to the boundary of the Morse-Smale systems.*
(b) *Condition (C) holds.*
(c) *Γ is isolated.*

Theorem 3. *Every vector field with a non-hyperbolic gssc is in the boundary of the Morse-Smale systems. Moreover, it is isolated.*

3. Simple singular cycles and genericity

3.1. Simple singular cycles. Proof of Lemma 1.

Let σ be a critical element of a vector field X, and let $\bar{s}, \bar{u}$ and $\bar{c}$ be, respectively, the number of contracting, expanding and neutral eigenvalues of DX_σ in case σ is a singularity, and of $D\pi_p$ ($p \in \sigma$, π the Poincare map of σ) in case σ is a periodic orbit. In the first case $\bar{s} + \bar{u} + \bar{c} = n$, while in the second $\bar{s} + \bar{u} + \bar{c} = n - 1$.

For critical elements of vector fields in generic one-parameter families of vector fields, the sets $W^s(\sigma)$ and $W^u(\sigma)$ are smooth submanifolds. If either $W^s(\sigma)$ or $W^u(\sigma)$ has a boundary (for σ non-hyperbolic), we denote it (the boundary) by $W^{ss}(\sigma)$ or $W^{uu}(\sigma)$ respectively. Let s and u be the dimensions of $W^s(\sigma)$ and $W^u(\sigma)$ respectively.

Table 1 in the next page describes all the possible cases of critical elements (hyperbolic and non-hyperbolic) of vector fields in generic one-parameter families. The non-hyperbolic critical elements in this table are termed *simple non-hyperbolic* critical elements.

Let W^s and W^u be two invariant submanifolds of dimensions s and u respectively. If $s+u > n+1$ (resp. $s+u < n+1$) then we say that *there is* (resp. *there is not*) *enough dimension for transversality.* The number of exceeding (missing) dimensions is $s+u-(n+1)$ $(resp. n+1-(s+u))$.

For a singular cycle $\Gamma = (\cup\sigma_i)\cup(\cup\gamma_i)$ with either hyperbolic or simple non-hyperbolic critical elements, we denote by s_i and u_i the dimensions of $W^s(\sigma_i)$ and $W^u(\sigma_i)$ respectively.

Table 1

Critical element	$s, u, \bar{c}$	$u+s$
Hyperbolic singularity	$s=\bar{s}; u=\bar{u}; \bar{c}=0$	n
Hyperbolic periodic orbit	$s=\bar{s}+1; u=\bar{u}+1; \bar{c}=0$	$n+1$
Saddle-node singularity	$s=\bar{s}+1; u=\bar{u}+1; \bar{c}=1$ $W^{ss}\neq\varnothing, W^{uu}\neq\varnothing$	$n+1$
Hopf singularity	$\left\{\begin{array}{l} s=\bar{s}+2 \\ \text{and} \\ u=\bar{u} \end{array}\right\}$ or $\left\{\begin{array}{l} u=\bar{u}+2 \\ \text{and} \\ s=\bar{s} \end{array}\right\}$ $\bar{c}=2$	n
Saddle-node periodic orbit	$s=\bar{s}+2; u=\bar{u}+2; \bar{c}=1$ $W^{ss}\neq\varnothing, W^{uu}\neq\varnothing$	$n+2$
Flip periodic orbit	$\left\{\begin{array}{l} s=\bar{s}+2 \\ \text{and} \\ u=\bar{u}+1 \end{array}\right\}$ or $\left\{\begin{array}{l} s=\bar{s}+1 \\ \text{and} \\ u=\bar{u}+2 \end{array}\right\}$ $\bar{c}=1$	$n+1$
Elliptic periodic orbits	$\left\{\begin{array}{l} s=\bar{s}+3 \\ \text{and} \\ u=\bar{u}+1 \end{array}\right\}$ or $\left\{\begin{array}{l} s=\bar{s}+1 \\ \text{and} \\ u=\bar{u}+3 \end{array}\right\}$ $\bar{c}=2$	$n+1$

The proof of Lemma 1 is based on the following result:

Lemma 2. *(a) In every hyperbolic singular cycle Γ there is some γ_j which is not a transversal intersection between $W^s(\sigma_{j+1})$ and $W^u(\sigma_j)$. In fact, there is a j such that $s_{j+1}+u_j < n+1$. The total sum of missing dimensions equals the number of singularities in Γ plus the total sum of exceeding dimensions.*
(b) If only one critical element in a singular cycle Γ is non-hyperbolic and of saddle-node type, then the total sum of missing dimensions equals the number of singularities in Γ plus the total sum of exceeding dimensions plus one.

(c) If only one critical element in a singular cycle Γ is non-hyperbolic, and it is simple but not of saddle-node type, then the total sum of missing dimensions is equal to the number of singularities in Γ plus the total sum of exceeding dimensions.

Proof. Consider a singular cycle Γ with k critical elements and l singularities, $1 \leq l \leq k$. From the table above, in cases a) or c) we have $s_i + u_i = n$ if σ is a singularity and $s_i + u_i = n + 1$ if σ_i is a periodic orbit. Then

$$\sum_{i=0}^{k-1}(s_{i+1} + u_i) = \sum_{i=0}^{k-1}(s_i + u_i) = kn + k - \ell < k(n+1).$$

Hence there exists $j \in \{0, \cdots, k-1\}$ such that $s_{j+1} + u_j < n + 1$ and $W^s(\sigma_{j+1})$ is not transversal to $W^u(\sigma_j)$. Moreover, the total sum of missing dimensions equal ℓ plus the total sum of exceeding dimension.

Case (b) is proved similarly. □

We now prove Lemma 1. The result of part (a) of Lemma 1 is nowadays a standard fact in bifurcation theory.

Let prove part (b). If a hyperbolic singular cycle Γ contains more than one singularity then there are at least two missing dimensions for transversality. The same occurs if Γ contains only one singularity and there is some exceeding dimension at some γ_i. Vector fields with such singular cycles lie in higher codimensional submanifolds in the space of vector fields.

If a hyperbolic singular cycle Γ has only one singularity and there are no exceeding dimensions, then we are only one dimension short for transversality. Assuming general standard conditions for the intersection of the invariant manifolds (the sum of tangent spaces as big as possible), we have a hyperbolic simple singular cycle. If not, we increase the number of degeneracies.

Similar easy arguments can be given for non-hyperbolic singular cycles that are non-simple: they necessarily contain more than one degeneracy. This proves Lemma 1. □

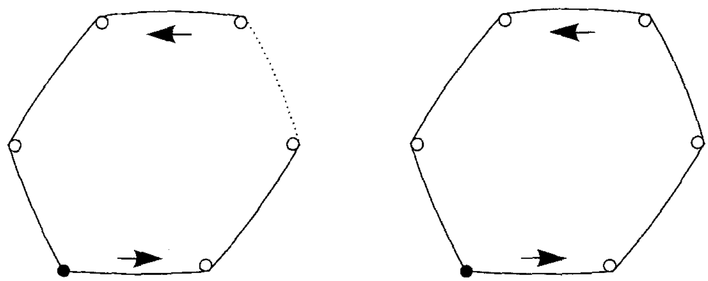
3.2. Classification of simple singular cycles.

We first classify all the hyperbolic ssc's.

Let $\Gamma = (\cup_{i=0}^{k-1}\sigma_i) \cup (\cup_{i=0}^{k-1}\gamma_i)$ be a hyperbolic ssc. Assume that σ_0 is the singularity and that γ_j, $j \in \{0, \cdots, k-1\}$, is the non-transversal intersection of Γ. Recall that $s_i = \dim W^s(\sigma_i)$ and $u_i = \dim W^u(\sigma_i)$ for every i.

The hyperbolic ssc are classified by the four-tuple $(n, k, j, s_0) \in \mathbb{N}^4$ that verifies $n \geq 2$; $k \geq 1$; $0 \leq j \leq k-1$; $1 \leq s_0 \leq n-1$. If $s_0 = 1$ then $j = k-1$ and if $s_0 = n-1$ then $j = 1$.

In fact, for such a four-tuple, $u_0 = n - s_0$ and:
$s_i = s_0 + 1, u_i = u_0 \ \forall i = 1, \ldots, j$ and
$s_i = s_0, u_i = u_0 + 1 \ \forall i = j+1, \ldots, k-1$.



(a) (b)

Figure 1

Hyperbolic ssc are graphically represented as in Figure 1(a). The black point is the singularity, the circles represent the periodic orbits. The dotted line represents the non-transversal orbit.

We now classify the non-hyperbolic ssc.

Let $\Gamma = (\cup_{i=0}^{k-1}\sigma_i) \cup (\cup_{i=0}^{k-1}\gamma_i)$ be a non hyperbolic ssc. Assume that σ_0 is the singularity of Γ and that σ_j, $j \in \{0, \cdots k-1\}$, is the saddle-node type critical element in Γ. As before $s_i = \dim W^s(\sigma_i)$ and $u_i = \dim W^u(\sigma_i)$. Recall that $s_j + u_j = n+1$ if $j = 0$ and $s_j + u_j = n$ otherwise.

The non-hyperbolic ssc are also classified by the four-tuple $(n, k, j, s_0) \in \mathbb{N}^4$ that verifies: $n \geq 2$; $k \geq 1$; $0 \leq j \leq k-1$; If $j = 0$ then $1 \leq s_0 \leq n$ and if either $s_0 = 1$ or $s_0 = n$ then $k = 1$. If $j \neq 0$ then $2 \leq s_0 \leq n-2$.

For such a four-tuple:
If $j=0$ then $u_0 = n+1-s_0$ and $s_i = s_0, u_i = u_0 \ \forall i = 0, \dots, k-1$.
If $j \neq 0$ then $u_0 = n - s_0$ and:
$s_i = s_0 + 1,\ u_i = u_0 \ \ \forall i = 1, \dots, j-1,$
$s_j = s_0 + 1,\ u_j = u_0 + 1,$ and
$s_i = s_0,\ u_i = u_0 + 1,\ \ \forall i = j+1, \dots, k-1.$
This situation is illustrated in Figure 1(b). The double circle indicates the non-hyperbolic critical element.

3.3. Generic conditions for simple singular cycles.

Let $\lambda_1, \dots, \lambda_{\bar{s}}$ and $\mu_1, \dots, \mu_{\bar{u}}$ be the contracting and the expanding eigenvalues of a hyperbolic critical element σ of a vector field X.

In case σ is a hyperbolic singularity ($\bar{s} + \bar{u} = n$), the real part of these eigenvalues may be ordered as follows:

$$\Re\lambda_{\bar{s}} \leq \Re\lambda_{\bar{s}-1} \leq \dots \Re\lambda_1 < 0 < \Re\mu_1 \leq \cdots \leq \Re\mu_{\bar{u}-1} \leq \Re\mu_{\bar{u}}.$$

We know that generically either λ_1 is real with $\Re\lambda_2 < \lambda_1$ or λ_1 is complex with $\Re\lambda_3 < \Re\lambda_2 = \Re\lambda_1$. Further, generically either μ_1 is real with $\mu_1 < \Re\mu_2$ or μ_1 is complex with $\Re\mu_1 = \Re\mu_2 < \Re\mu_3$.

In case σ is a hyperbolic periodic orbit ($\bar{s} + \bar{u} = n-1$) there are similar generic properties for the norm of its eigenvalues.

The first generic property for a ssc is:
(G1) The eigenvalues of the vector field at the singularity (if it is hyperbolic), and of the return map at every hyperbolic periodic orbit, verify the above generic property.

The eigenvalue λ_1 (resp. μ_1) is called the *weakest contracting* (resp. *expanding*) eigenvalue of the hyperbolic critical element. This weakest eigenvalue has naturally associated a weakest contracting (expanding) direction which is either one or two dimensional.

Let u (resp. s) be the dimension of $W^u(\sigma)$ (resp. $W^s(\sigma)$) and let c be the dimension of the weakest contracting (resp. expanding) direction ($c = 1$ or 2). Any $u+c$ (resp. $s+c$) dimensional invariant submanifold containing $W^u(\sigma)$ (resp. $W^s(\sigma)$) which is tangent at σ to the weakest contracting (resp. expanding) direction will be denoted by $W^{cu}(\sigma)$

(resp. $W^{cs}(\sigma)$) and called the *center-unstable* (resp. *center-stable*) manifold of σ.

The last two generic conditions are:

(G2) At every hyperbolic critical element σ_i of a ssc Γ, the regular orbit γ_{i-1} (resp. γ_i) reaches (resp. leaves) σ_i tangentially to the weakest contracting (resp. expanding) direction of σ_i. At the saddle-node type critical element σ_j (if it exists) the regular orbit γ_{j-1} (resp. γ_j) reaches (resp. leaves) σ_j tangentially to the 1-dimensional neutral direction.

(G3) If γ_j is the non-transversal intersection between $W^u(\sigma_j)$ and $W^s(\sigma_{j+1})$ of a hyperbolic ssc, then $W^{cu}(\sigma_j) \pitchfork W^s(\sigma_{j+1})$ and $W^u(\sigma_j) \pitchfork W^{cs}(\sigma_{j+1})$.

At every hyperbolic critical element σ_i, $W^{cu}(\sigma_i)$ and $W^{cs}(\sigma_i)$ can be chosen so that $\gamma_{i-1} \subseteq W^{cs}(\sigma_i)$ and $\gamma_i \subseteq W^{cu}(\sigma_i)$.

Finally, for technical reasons we will assume the following linearization condition. It is not really essential for our results.

(L) There is a neighborhood $\mathcal{U}$ of X in the C^3-topology such that the analytic continuations of the hyperbolic critical elements of X are C^2-linearizable for every $Y \in \mathcal{U}$.

A ssc verifying these generic conditions will be called a *generic ssc* (gssc).

Finally, we define the signature of the non-transversal orbit of a hyperbolic gssc.

Consider a hyperbolic gssc Γ and assume that

$$\gamma_j \subseteq W^u(\sigma_j) \cap W^s(\sigma_{j+1})$$

is the non-transversal intersection of Γ. Assume further that the weakest contracting eigenvalue of σ_j and the weakest expanding eigenvalue of σ_{j+1} are real. In addition, suppose that whenever σ_j (resp. σ_{j+1}) is a periodic orbit this eigenvalue is positive. Then $W^u(\sigma_j)$ (resp. $W^s(\sigma_{j+1})$) separates $W^{cu}(\sigma_j)$ (resp. $W^{cs}(\sigma_{j+1})$) into two invariant submanifolds, one of which contains γ_{j-1} (γ_{j+1}). Call this manifold $W^{cu}_{+}(\sigma_j)$ (resp. $W^{cs}_{+}(\sigma_{j+1})$). At each point $p \in \gamma_j$ choose vectors v_p and w_p with v_p (resp. w_p) tangent to $W^{cu}(\sigma_j)$ (resp. $W^{cu}(\sigma_{j+1})$) and pointing towards $W^{cu}_{+}(\sigma_j)$.

By (G3) neither v_p nor w_p are contained in the hyperspace $T_pW^u(\sigma_j) + T_pW^s(\sigma_{j+1})$. We say that γ_j has *positive* (*negative*) *signature* if v_p and w_p points towards the same (resp. different) side of such a hyperplane.

4. Proofs of Theorems.

The crucial point for the proof of Theorem 1 is the transition map of orbits close to critical elements. We start recalling the information about these transition maps.

4.1. Local properties of the transition maps.

In this section we describe properties of the local transition map near hyperbolic critical elements. Generic conditions are assumed throughout.

Let σ be a hyperbolic critical element of a vector field X. Choose generic points $p \in W^s(\sigma)$ and $q \in W^u(\sigma)$ inside a linearization neighborhood of σ. The positive (resp. negative) orbit of p (resp. q) reaches σ tangentially to the weak stable (resp. unstable) direction. Consider a central unstable (resp. stable) manifold $W^{cu}(\sigma)$ (resp.$W^{cs}(\sigma)$) of σ such that $p \in W^{cu}(\sigma)$(resp.$q \in W^{cs}(\sigma)$). Consider some small transversal sections Σ_p and Σ_q at p and q respectively. In case σ is a periodic orbit choose Σ_p and Σ_q inside a bigger transversal section to σ. See Figure 2.

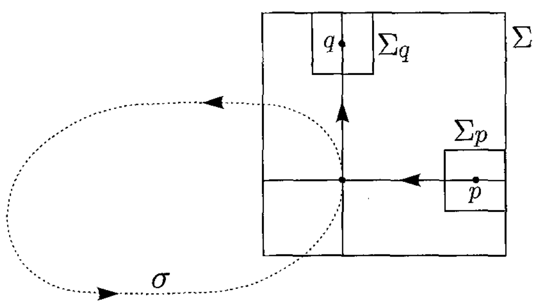


Figure 2

Let $\varphi(t,x)$ be the flow of X and define $D_\sigma = \{x \in \Sigma_p / \exists\ T > 0$ with $\varphi(T,x) \in \Sigma_q$ and $\varphi(t,x)$ remains in the linearization neighborhood $\forall t \in [0,T]\}$.

Consider the *transition map* $\pi_\sigma : D_\sigma \to \Sigma_q$ and let $T_\sigma = \pi_\sigma(D_\sigma)$ be the *target* of π_σ. The set D_σ is the *domain* of π_σ.

When the time is reversed, T_σ becomes the domain of the transition map, while D_σ becomes its target.

We now describe the geometry of the sets $\overline{D_\sigma}$ and $\overline{T_\sigma}$. We first introduce some terminology.

Let $N_0 \subseteq N_1$ be proper submanifolds of a manifold M. We consider the possibility that N_0 be the boundary of N_1. In this case $T_x N_1$ is a semi space for every $x \in N_0$. We say that a set $A \subseteq M$, with $N_0 \subseteq A$, is *tangent* to N_1 at N_0 if $\alpha'(0) \in T_{\alpha(0)} N_1$ for every regular curve $\alpha : [0, \varepsilon[\to A$ with $\alpha(0) \in N_0$. Moreover, for every $p_0 \in N_0$ there is such a regular curve with $\alpha(0) = p_0$ and $\alpha'(0) \in T_{p_0} N_1 \setminus T_{p_0} N_0$. If $\dim N_1 = \dim N_0 + 1$ and N_0 separates N_1 into two components, we say that a set A with $N_0 \subseteq A$ is *one-sided tangent* to N_1 at N_0 if it is tangent to one of the components of $N_1 \setminus N_0$ at its boundary N_0.

With no possible confusion we make no distinction between $\overline{D_\sigma}$ and D_σ. They only differs on orbits in $W^s(\sigma)$.

Lemma 3. *Assume that σ is a hyperbolic singularity. Then:*

(a) *D_σ and T_σ are connected sets.*

(b) *D_σ (resp. T_σ) is contained in a set which is tangent to $W^{cs}(\sigma) \cap \Sigma_p$ (resp. $W^{cu}(\sigma) \cap \Sigma_q$) at $W^s(\sigma) \cap \Sigma_p$ (resp. $W^u(\sigma) \cap \Sigma_q$).*

(c) *If the weakest expansion (resp. contraction) is real, then D_σ (resp. T_σ) is itself one-sided tangent to the submanifolds described in (b).*

Assume that σ is a hyperbolic periodic orbit. Then:

(d) *D_σ (resp. T_σ) has enumerable many connected components accumulating in $W^s(\sigma) \cap \Sigma_p$ (resp. $W^u(\sigma) \cap \Sigma_q$). The distances of consecutive components of D_σ (resp. T_σ) to $W^s(\sigma) \cap \Sigma_p$ (resp. $W^u(\sigma) \cap \Sigma_q$) is a geometrical sequence of ratio the modulus of the weakest expansion (resp. contraction) of σ.*

(e) *Same statement as (b).*

(f) *If the weakest expansion (resp. contraction) is real and positive, then D_σ (resp. T_σ) is contained in a set which is one-sided tangent to the submanifolds described in (b).*

Proof. For a linear Jordan form vector field the proof follows from direct calculations. We carry these out in Section 5.

For a C^2-linearizable critical element the Lemma follows from the invariance of the statement under a C^1 change of coordinates. □

From the proof of Lemma 3 we can deduce further geometrical properties of D_σ and T_σ:

Corollary 1. *In linear coordinates, the domain D_σ (resp. target T_σ) is the intersection of Σ_p (resp. Σ_q) with a cylinder of ax $W^s(\sigma)$ (resp. $W^u(\sigma)$) over a set in $W^u(\sigma)$ (resp. $W^s(\sigma)$). This set is contained in a set that is tangent to the weakest expansion (resp. contraction).*

If $\mathbb{R}^n = W^s \times W^u$ we say that a set of the form $A \times W^s$ with $A \subset W^u$ is a *cylinder with ax W^s over the set A.*

Remark. We claim that Lemma 3 holds even for non-linearizable critical elements, but its proof may involve some complicated nonlinear estimations. We have adopted a generic view point instead.

4.2. Proof of Theorem 1(a) (suff. condition) and 1(b).

Let X be a vector field with a hyperbolic gssc Γ. Let γ_j be the non transversal orbit of $\Gamma, \gamma_j \subseteq W^u(\sigma_j) \cap W^s(\sigma_{j+1})$. Assume that conditions $(C1)$ and $(C2)$ holds.

At a point $p \in \gamma_j$, $T_pW^u(\sigma_j) + T_pW^s(\sigma_{j+1})$ is a hyperspace of T_pM. Moreover, $W^u(\sigma_j)$ is the boundary of $W^{cu}_+(\sigma_j)$ and $W^s(\sigma_j)$ is the boundary of $W^{cs}_+(\sigma_{j+1})$. By hypothesis, in a small neighborhood of p we have that $W^{cu}_+(\sigma_j) \cap W^{cs}_+(\sigma_{j+1}) = \{p\}$. See Figure 3 below.

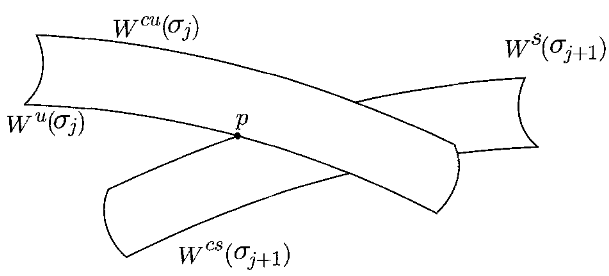


Figure 3

Let U be a small neighborhood of Γ. Let Σ be a small transversal section to X at p and let $D = \{x \in \Sigma / \exists t_0 > 0$ with $\varphi(t_0, x) \in \Sigma$ and $\varphi(t, x) \in U \; \forall t \in [0, t_0]\}$ be the domain of the return map $\pi : D \to \Sigma$. Let $T = \pi(D)$ be its target.

Let T_j (resp.D_{j+1}) be the target (resp. domain) of the local transition map in a neighborhood of σ_j (resp.σ_{j+1}). Then the flow sends D into D_{j+1} and T is sent by the reverse flow into T_j. By condition (C1) and Lemma 3 we see that D is contained in a set which is tangent to $W^{cs}_{+}(\sigma_{j+1}) \cap \Sigma$ at $W^s(\sigma_{j+1}) \cap \Sigma$ and that T is contained in a set which is tangent to $W^{cu}_{+}(\sigma_j) \cap \Sigma$ at $W^u(\sigma_j) \cap \Sigma$. From condition (C2) we now conclude that $D \cap T = \{p\}$. Hence Γ is isolated. This proves part (b).

For $Y \in \mathcal{U}$ define D_Y and T_Y to be the domain and the target of the return map $\pi_Y : \Sigma \to \Sigma$. By arguments analogous to those used above we see that D_Y is contained in a set which is tangent to $W^{cs}_{+}(\sigma_{j+1}(Y)) \cap \Sigma$ at $W^s(\sigma_{j+1}(Y)) \cap \Sigma$ and that T_Y is contained in a set which is tangent to $W^{cu}_{+}(\sigma_j(Y)) \cap \Sigma$ at $W^u(\sigma_j(Y)) \cap \Sigma$. For vector fields on one of the components of $\mathcal{U} \setminus \mathcal{N}$, we have that $D_Y \cap T_Y = \phi$. Hence, for a vector field Y in this component Λ_Y contains only the analytic continuation of the critical elements of X in Γ. This means that X is in the boundary of the Morse-Smale systems. Thus we have prove the sufficient part of Theorem 1(a). □

4.3. Theorem 1(a) (necessary condition).

We prove that if either condition (C1) or (C2) is false then in both components of $\mathcal{U} \setminus \mathcal{N}$ there are vector fields Y with a transversal homoclinic orbit $\tilde{\gamma}_Y$ to either $\sigma_j(Y)$ or $\sigma_{j+1}(Y)$ contained in a small neighborhood of Γ.

By transversality and the λ-Lemma (see [**PdM**]), $W^u(\sigma_{j+1})$ intersects transversally $W^s(\sigma_j)$. Since $\dim(W^u(\sigma_j)) + \dim(W^s(\sigma_{j+1})) = n+2$ this intersection generically contains a continuous set of orbits.

It follows that there are subordinated cycles of the form

$$\tilde{\Gamma} = \{\sigma_j, \sigma_{j+1}\} \cup \{\gamma_j, \tilde{\gamma}_{j+1}\}$$

contained in U.

The analysis now follows very similar to that in [**DR**] where heterodimensional cycles for diffeomorphisms are considered.

Consider $p \in \gamma_j$ and Σ as before. For every $Y \in \mathcal{U}$ define D_Y, T_Y and $\pi_Y : D_Y \to T_Y$ relative to this subordinated cycle. Both D_Y and T_Y has infinitely many connected components accumulating in $W^s(\sigma_{j+1})(Y) \cap \Sigma$ and $W^u(\sigma_j)(Y) \cap \Sigma$ respectively.

On the other hand, every component of D_Y (resp. T_Y) contains in its boundary a piece of $W^s(\sigma_j)(Y) \cap \Sigma$ (resp. $W^u(\sigma_j)(Y) \cap \Sigma$).

By hypothesis $W^u(\sigma_j)$ intersects $W^s(\sigma_{j+1})$ non-transversally, but in general position, along γ_j. Even if (C1) and (C2) is false it may occurs, depending on the geometry of the intersection of $W^u(\sigma_{j+1})$ with $W^s(\sigma_j)$), that $D \cap T = \{p\}$. In this case the cycle Γ is isolated.

However, if either (C1) or (C2) is false, in both components of $\mathcal{U} \setminus \mathcal{N}$ there are vector fields Y with a homoclinic orbit $\tilde{\gamma}_Y$ to either $\sigma_j(Y)$ or $\sigma_{j+1}(Y)$.

This proves that Γ is not in the boundary of the Morse-Smale systems. The necessary condition in Theorem 1(a) is now proved. □

4.4. Proof of Theorem 1(c).

Without loose of generality assume that σ_0 is the singularity and that γ_0 is the non-transversal orbit in Γ. We prove that if either conditions (C1) or (C2) fails to hold then Γ is not isolated. To this end we prove that $W^u(\sigma_{k-1})$ intersects $W^s(\sigma_2)$ transversally along an orbit $\tilde{\gamma}$ contained in U and close to $\gamma_{k-1} \cup \sigma_0 \cup \gamma_0 \cup \sigma_1$. This proves the existence of a transversal heteroclinic cycle of periodic orbits, thus proving that Γ is not isolated. Here $k \geq 2$ is the number of critical elements in Γ.

Consider

$$p \in \gamma_0, \Sigma, D \subset \Sigma, \pi : D \to \Sigma \quad \text{and} \quad T = \pi(D)$$

as in the previous section. Since σ_0 is the singularity, it follows from Lemma 3 that T is a connected set with $W^u(\sigma_0) \cap \Sigma$ contained in its boundary.

Since $W^u(\sigma_{k-1})$ intersects $W^s(\sigma_0)$ transversally along γ_{k-1} we have that $W^u(\sigma_{k-1}) \cap \Sigma$ is connected, contained in T and with $W^u(\sigma_0) \cap \Sigma$

contained in its boundary. Indeed, $W^u(\sigma_{k-1})$ overlaps with $W_+^{cu}(\sigma_0)$ in a relatively open set.

On the other hand, since $W^u(\sigma_1)$ intersects $W^s(\sigma_2)$ transversally along γ_1 it follows that $W^s(\sigma_2) \cap \Sigma$ is contained in D and accumulates on $W^s(\sigma_1)$.

Since by hypothesis $W^u(\sigma_0)$ intersects $W^s(\sigma_1)$ non-transversally, but in general position along γ_0, it follows that if either (C1) or (C2) is false then $W^u(\sigma_{k-1})$ intersects $W^s(\sigma_2)$ transversally along an orbit $\tilde{\gamma}$ as indicated.

This proves that Γ is not isolated. The proof of Theorem 1 is now complete. □

4.5. Proof of Theorems 2 and 3.

For the proof of Theorem 2 we refer the reader to [**A**], page 117. Specially for the equivalence $(a) \Leftrightarrow (b)$. The other equivalences are already contained in Theorem 1.

The proof of Theorem 3 follows from some straightforward geometric arguments. In fact, when the saddle-node type critical element splits into two hyperbolic critical elements, the maximal invariant set becomes trivial.

5. Local transition maps.

In this Section we prove Lemma 3. As it was explained it is enough to consider linear vector fields. In fact, the generic conditions given in Section 3 guarantees smooth linearization and the results to be proved are invariant under C^1 change of coordinates.

5.1. Local transition maps through singularities.

Let X be a hyperbolic linear vector field on $\mathbb{R}^n$. Assume that X is given by:

$$\begin{aligned} \dot{x} &= Ax \qquad \Re B < \beta < 0 < \alpha < \Re A \\ \dot{y} &= By \qquad x \in \mathbb{R}^u, \;\; y \in \mathbb{R}^s \end{aligned}$$

An inequality of the type $\Re M < c$ (resp. $\Re M > c$) means that the real part $\Re\mu$ of every eigenvalue μ of the matrix M is less (resp. greater) than c.

Let $<\ ,\ >$ be the scalar product in $\mathbb{R}^n$ such that $< Ax, x > \geq \alpha\|x\|^2 \quad \forall x \in \mathbb{R}^u$ and $< Ay, y > \leq \beta\|x\|^2 \quad \forall y \in \mathbb{R}^s$. In what follows every scalar product will be of this kind.

Let $\varphi(t, x, y) = (e^{tA}x, e^{tB}y)$ be the flow of X. Consider $p = (0, y_0) \in W^s(0)$ and $q = (x_0, 0) \in W^u(0)$. For an arbitrarily small number $\epsilon > 0$ consider the transversal sections:

$$\Sigma_p = \{(x,y)/\|x\| \leq \epsilon, \|y\| = \|y_0\|, \|y - y_0\| \leq \epsilon\},$$

and

$$\Sigma_q, = \{(x,y)/\|x\| = \|x_0\|, \|x - x_0\| \leq \epsilon, \|y\| \leq \epsilon\}.$$

As already defined $D_0 = \{(x,y) \in \Sigma_p/\varphi(t,x,y) \in \Sigma_q \text{ for some } t > 0\}$. Let prove that D_0 is a connected set. Recall that since we are not doing any distinction between $\overline{D_0}$ and D_0, this last is a closed set.

Consider $(\overline{x}, \overline{y}) \in D_0$. It exists $T > 0$ such that $\varphi(T, \overline{x}, \overline{y}) \in \Sigma_q$. By an easy verification $(e^{-sA}\overline{x}, \overline{y}) \in D_0 \quad \forall s \geq 0$. Since $(e^{-sA}\overline{x}, \overline{y}) \to (0, \overline{y})$ as $s \to 0$ it follows that D_0 is an arc-wise connected set. Hence it is connected.

By duality, reversing time, we see that $T_0 = \pi_0(D_0)$ is also a connected set. Part (a) of Lemma 3 is now proved.

Let now prove that D_0 is contained in a set which is tangent to $W^{cs}(0) \cap \Sigma_p$ at $W^s(0) \cap \Sigma_p$. By duality it immediately follows that T_0 is contained in a set which is tangent to $W^{cu}(0) \cap \Sigma_q$ at $W^u(0) \cap \Sigma_q$.

Assume that X is given by:

$$\begin{aligned} \dot{x} &= A_0 x \qquad && \Re B < 0 < \Re A_1 < \Re A_0 \\ \dot{w} &= A_1 w \qquad && (x, w) \in \mathbb{R}^u, \ \ y \in \mathbb{R}^s \\ \dot{y} &= By && \end{aligned}$$

As usual $\varphi(t, x, w, y)$ denotes the flow of X. For this linear vector field $W^s(0) = \{(x,w,y)/x = w = 0\}$ and $W^{cs}(0) = \{(x,w,y)/x = 0\}$. We will prove that D_0 is contained in a set of the form $\{(x,w,y)/\|x\| \leq K\|w\|^\alpha\}$ with $\alpha > 1$.

Consider $p = (0,0,y_0) \in W^s(0)$, $q = (x_0, w_0, 0) \in W^u(0), w_0 \neq 0$. For $\epsilon > 0$ consider the transversal sections:

$$\Sigma_p = \{(x,w,y)/\|x\| \leq \epsilon, \|w\| \leq \epsilon, \|y\| = \|y_0\|, \|y - y_0\| \leq \epsilon\},$$

and

$$\Sigma_q = \{(x,w,y)/\|(x,w)\| = \|(x_0,w_0)\|, \|(x,w) - (x_0,w_0)\| \leq \epsilon, \|y\| \leq \epsilon\}.$$

Recall that $D_0 = \{(x,w,y) \in \Sigma_p/\varphi(t,x,w,y) \in \Sigma_q$ for some $t > 0\}$. We look for $(x,w,y) \in \Sigma_p$ such that $(e^{t_0 A_0}x, e^{t_0 A_1}w, e^{t_0 B}y) \in \Sigma_q$ for some $t_0 > 0$.

Since e^{-tA_0}, e^{-tA_1} and e^{tB} goes to zero as t goes to ∞, there exists $T > 0$ such that $\|e^{tB}y\| \leq \epsilon \; \forall t \geq T$, $\forall y$ with $\|y\| = \|y_0\|, \|y - y_0\| \leq \epsilon$, and

$$\|(e^{-tA_0}\overline{x}, e^{-tA_1}\overline{w})\| \leq \epsilon \; \forall t \geq T, \forall(\overline{x},\overline{w})$$

with

$$\|(\overline{x},\overline{w})\| = \|(x_0,w_0)\|, \|(\overline{x},\overline{w}) - (x_0,w_0)\| \leq \epsilon.$$

Consider

$$(x,w,y) = (e^{-tA_0}\overline{x}, e^{-tA_1}\overline{w}, y)$$

with $t > T$, $\|y\| = \|y_0\|$, $\|y - y_0\| \leq \epsilon$, $\|(\overline{x},\overline{w})\| = \|(x_0,w_0)\|$ and $\|(\overline{x},\overline{w}) - (x_0,w_0)\| \leq \epsilon$. It follows that $(x,w,y) \in D_0$ and that every point in D_0 can be written in this form.

We now prove that there are positive real numbers K_-, K_+ and real numbers α_-, α_+ with $1 < \alpha_+ < \alpha_-$ such that:

$$K_- \|w\|^{\alpha_-} \leq \|x\| \leq K_+ \|w\|^{\alpha_+} \;\; \forall(x,w,y) \in D_0.$$

This will prove part (b).

Let $\underline{\alpha_i}, \overline{\alpha_i}$, $i = 0,1$ be positive real numbers such that $0 < \underline{\alpha_1} < \Re A_1 < \overline{\alpha_1} < \underline{\alpha_0} < \Re A_0 < \overline{\alpha_0}$. We now that:

$$e^{-t\overline{\alpha_0}}\|\overline{x}\| \leq \|x\| = \|e^{-tA_0}\overline{x}\| \leq e^{-t\underline{\alpha_0}}\|\overline{x}\| \;\forall t \geq 0 \;\forall \overline{x},$$

and

$$e^{-t\overline{\alpha_1}}\|\overline{w}\| \leq \|w\| = \|e^{-tA_1}\overline{w}\| \leq e^{-t\underline{\alpha_1}}\|\overline{w}\| \;\forall t \geq 0 \;\forall \overline{w}.$$

It easily follows that:

$$\left(\frac{\|w\|}{\|\overline{w}\|}\right)^{\frac{1}{\alpha_1}} \leq e^{-t} \leq \left(\frac{\|x\|}{\|\overline{x}\|}\right)^{\frac{1}{\alpha_0}},$$

and

$$\left(\frac{\|x\|}{\|\overline{x}\|}\right)^{\frac{1}{\alpha_0}} \leq e^{-t} \leq \left(\frac{\|w\|}{\|\overline{w}\|}\right)^{\frac{1}{\alpha_1}}.$$

From this we deduce the existence of K_-, K_+, α_- and α_+ as desired.

We now turn to prove part (c). It is enough to prove our thesis for D_0. The proof for T_0 follows by duality.

By hypothesis $W^s(\sigma) \cap \Sigma_p$ is of codimension one in $W^{cs}(\sigma) \cap \Sigma_p$. Hence it separates $W^{cs}(\sigma) \cap \Sigma_p$ in two components. By (a), D_0 is connected. Further, as it is easy to verify, $D_0 \setminus W^s(\sigma) \cap \Sigma_p$ is also connected. Using (b) we conclude that D_0 is tangent $W^{cs}(\sigma) \cap \Sigma_p$ at $W^s(\sigma) \cap \Sigma_p$.

This proves Lemma 3 when σ is a singularity.

5.2. Local transition through fixed points

We will only prove part (d) of Lemma 3. Parts (e) and (f) are proved following estimations similar to thats in (b) and (c).

Let L be a hyperbolic linear automorphism of $\mathbb{R}^n$. Assume that L has the form

$$\begin{aligned} x' &= Ax \qquad & |B| < 1 < |A| \\ y' &= By \qquad & x \in \mathbb{R}^u,\ \ y \in \mathbb{R}^s \end{aligned}$$

The notation $|M| < 1$ means that the modulus of every eigenvalue of the matrix M is less than 1. Let $\|\ \|$ be a norm in $\mathbb{R}^n$ such that A^{-1} and B are contractions.

Consider $p = (0, y_0) \in W^s(0)$ and $q = (x_0, 0) \in W^u(0)$. Let

$$\Sigma_p = \{(x,y)/\|x\| \leq \epsilon,\ \|y - y_0\| \leq \epsilon\}$$

and

$$\Sigma_q = \{(x,y)/\|x - x_0\| \leq \epsilon,\ \|y\| \leq \epsilon\}$$

be neighborhoods of p and q respectively.

Then $D_0 = \{(x,y) \in \Sigma_p / L^n(x,y) \in \Sigma_q$ for some $n \in \mathbb{N}\}$. Let $D_n = L^{-n}(\Sigma_q) \cap \Sigma_p$. There is a smallest $N_0 \in \mathbb{N}$ such that $D_n \neq \varnothing\ \ \forall n \geq N_0$.

It then follows that $D_0 = \cup_{n \geq N_0} D_n$ is a disjoint union of enumerable components.

Since by generic assumption q has a non-zero component along the weakest expansion, the sequence of distances of $\{\text{dist}(D_n, 0)\}_{n \geq N_0}$ is a geometrical one with ratio the modulus of the weakest expansion.

By dual arguments, the same results are proved for T_0. This proves Lemma 3.

6. Final Remarks. A conjecture.

Once the gssc of vector fields in the boundary of the Morse-Smale systems are classified, it becomes a natural problem to study the 1-parameter unfolding of each of such cycles.

In this Section we present some general results in this direction. Consider a vector field X with a hyperbolic gssc $\Gamma = (\bigcup_{i=0}^{k-1} \sigma_i) \cup (\bigcup_{i=0}^{k} \gamma_i)$. Assume that γ_0 is the non-transversal orbit of Γ. On any neighborhood of Γ there are subordinated cycles of the form $\Gamma_0 = (\sigma_0 \cup \sigma_1) \bigcup (\gamma_0 \cup \gamma_1')$. Such a cycle Γ_0 may be non-singular. However, in this case it also exists a subordinated gssc $\tilde{\Gamma}_0$ of Γ of the form $(\sigma_0 \cup \sigma_1 \cup \sigma_j) \bigcup (\gamma_0 \cup \tilde{\gamma}_1 \cup \tilde{\gamma}_j)$ where σ_j is a singularity.

A generic 1-parameter family that unfolds Γ also unfolds Γ_0 (resp. $\tilde{\Gamma}_0$). Thus, in the unfolding of Γ we will see all the dynamical phenomena of the unfolding of Γ_0 (resp. $\tilde{\Gamma}_0$). This motivates us to first consider the unfolding of gssc with two or three critical elements, and in this last case to even consider the unfolding of the subordinated cycle with two periodic orbits.

Let us consider a gssc Γ with two critical elements. We represent it graphically as follows:

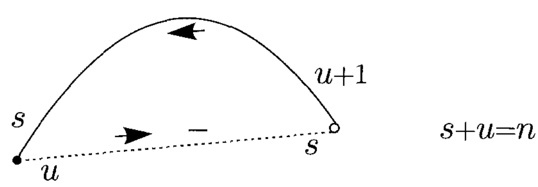


Figure 4

The weakest contraction of σ_0 is real and the weakest expansion of σ_1 is real and positive. The non-transversal orbit has negative curvature. The *central directions* of σ_0 and σ_1 are the directions of the weakest contraction and the weakest expansions. The weakest expansion (resp. contraction) of σ_0 (resp. σ_1) may be real or complex. The regular orbits γ_0 and γ_1 tend tangentially (as $t \to \pm\infty$) to the central directions. Consider the non-central directions as directions of normal hyperbolicity. Reducing the cycle to the central directions we then have the following four situations:

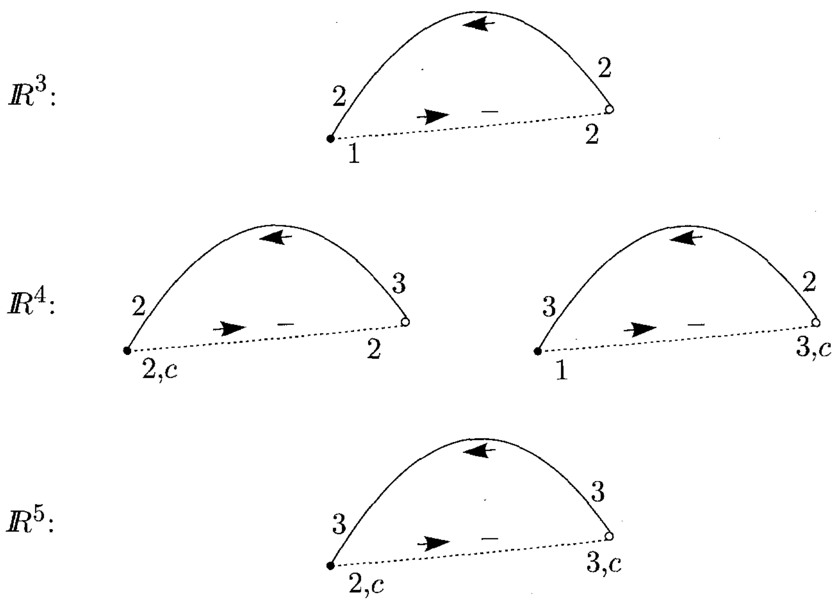


Figure 5

where " c " means complex eigenvalues, " − " means negative signature and the numbers are the dimensions of the corresponding invariant manifolds.

For a hyperbolic gssc Γ with three critical elements (with the non-transversal orbit between the two periodic orbits) the reduction to central directions gives the following three cases:

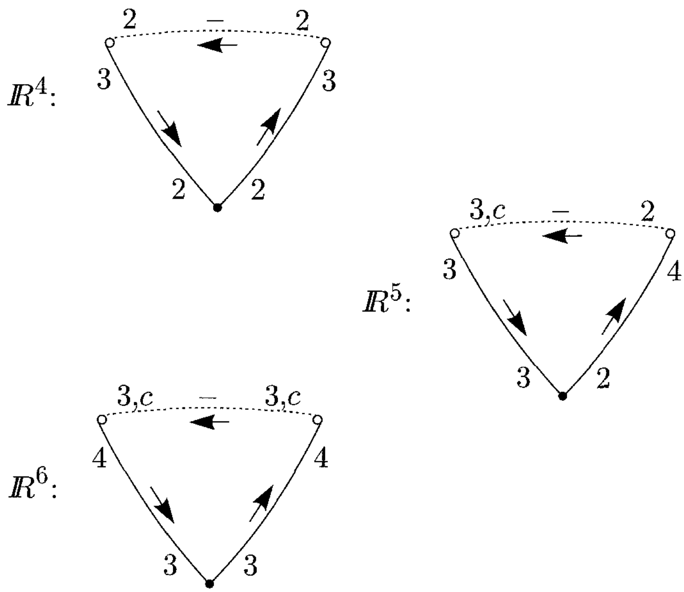


Figure 6

Finally, there are hyperbolic gssc Γ with only one critical element (a singularity). Reducing to central directions we obtain the following two cases:

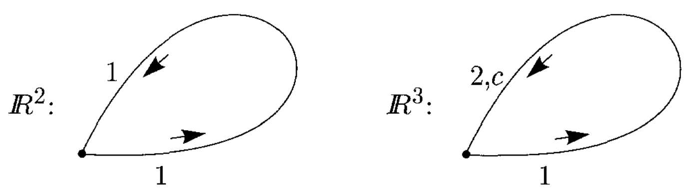


Figure 7

We conclude this work by posing the following conjecture:

Conjecture. Every dynamically significant phenomenon to be found in the unfolding of a hyperbolic gssc is already present in the unfolding of one of the 9 cycles given above.

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References

[**A**] V.I. Arnold (Ed.), *Dynamical Systems V*, Encyclopedia of Mathematical Sciences Vol. 5, Springer Verlag, 1994.

[**BLMP**] R. Bamón, R. Labarca, R. Mañe, M.J. Pacífico, *The explosion of singular cycles*, Publ. Math. I.H.E.S. **78:** (1993), 207-232.

[**D**] L. Diaz, *Persistence of cycles and nonhyperbolic dynamics*, Nonlinearity **8:** (1995), 693-713.

[**DR**] L. Díaz, J. Rocha, *Non-connected heterodimensional cycles: bifurcation and stability*, Nonlinearity **5:** (1992), 1315-1341.

[**L**] R. Labarca, *Bifurcation of contracting singular cycles*, Ann. Ec. Nor. Sup. T. **28:** (1995), 705-745.

[**PdM**] J. Palis, W. de Melo, *Geometric Theory of Dynamical Systems*, Springer Verlag (1982).

[**PR**] M.J. Pacífico, A. Rovella, *Unfolding contracting singular cycles*, Ann. Scient. Ec. Norm. Sup. 4 Serie, T.**26:** (1993), 691-700.

[**Ri**] M.R. Richlik, *Lorenz attractor through Sil'nikov-type bifurcation I*, Erg. Th. and Dyn. Sys. **10, 4:** (1990), 793-822.

[**Ro**] A. Rovella, *The dynamics of perturbations of the contracting Lorenz attractor*, Bol. Soc. Bras. Mat. **24, 2:** (1993), 233-259.

[**S**] L.P. Sil'nikov, *Generation of periodic motions from a trajectory exiting from an equilibrium of saddle-node and re-entering it*, Sov. Math. Dokl. **7:** (1966), 1155-1158.

Rodrigo Bamón
Departamento de Matemática
Facultad de Ciencias
Universidad de Chile
Casilla 653 Santiago
Chile

E-mail: rbamon@abello.dic.uchile.cl