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Unconditional integrability for dual actions

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Abstract. The dual action of a locally compact abelian group, in the context of C*-algebraic bundles, is shown to satisfy an integrability property, similar to Rieffel's proper actions. The tools developed include a generalization of Bochner's integral as well as a Fourier inversion formula for operator valued maps.

Keywords: Dual actions, C*-algebraic bundles, integrability.

1. Introduction

The goal of this paper is to initiate a study of a new notion of integrability for an action α of a locally compact group Γ on a C^*-algebra B. In the literature, several notions of integrability can be found, and they primarily deal with the study of elements $b \in B$ for which one can make sense of the integral

$$\int_\Gamma \alpha_x(b)\, dx.$$

The main difficulty is, of course, that the integrand has constant norm and hence, when the group is not compact, the integral will not converge. In order to attribute meaning to this, authors have mostly resorted to the weak topology, in the spirit of Pettis integrals, and thus, one would speak of elements b for which

$$\int_\Gamma \phi(\alpha_x(b))\, dx$$

converges for all continuous linear functionals ϕ on B. See, for example, [**1**], [**2**], [**8**] and [**10**, 7.8.4].

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More recently, Rieffel [**11**] introduced a notion of "proper" actions in which a similar integrability condition plays a crucial role. There, Rieffel requires, among other things, that for all elements a and b of a fixed dense *-subalgebra B_0 of B, one has that

$$\int_\Gamma a\alpha_x(b)\,dx \quad \text{and} \quad \int_\Gamma \alpha_x(b)a\,dx$$

are integrable in the sense of Bochner. Under suitable extra hypothesis, he shows that one can then make sense of a "generalized fixed point algebra" and he obtains a powerful version of the Takai-Takesaki duality. Rieffel's condition have a strong smell of the strict topology, since integrability is obtained only after one performs a multiplication with another element of the algebra, although the role of that topology is not made explicit.

My interest in understanding Rieffel's notion of properness steams from a desire to study a rather general class of actions of locally compact abelian groups, obtained as dual actions on cross sectional algebras of C^*-algebraic bundles. However I have found it difficult to work with Rieffel's conditions in an abstract sense, mainly because the dense subalgebra B_0, mentioned above, comes about in a somewhat *ad hoc* way. The main motivational force behind the present work is, therefore, to attempt a reformulation of Rieffel's ideas in which the set of "integrable" elements arises in a more natural way. We think we have succeeded in doing so, although it is not clear for us, at the moment, what is the exact logical relationship between our notion of integrable actions and Rieffel's proper actions.

In developing our theory we have been forced to understand two critical additional phenomena. The first one is the concept of unconditional integration. This notion generalizes Bochner's theory of integration in the same way that unconditional summability for series in a Banach space generalizes the notion of absolute summability. The study of unconditional integrability is the content of section (2), below.

The second fundamental phenomena subjacent to our work is the Fourier inversion formula for operator valued maps, which states that if $p\colon G \to B(H)$ is a compactly supported, continuous, positive-type map, defined on a locally compact abelian group G, and taking values in the operators on a Hilbert space $\mathfrak{H}$, then

$$\int_{\Gamma} \overline{(t,x)}\widehat{p}(x)\,dx = p(t),$$

where Γ is the Pontryagin dual of G and "$\int$" is the unconditional integral mentioned above. The topology with respect to which the convergence of this integral takes place depends on the continuity properties of p. The proof of this result is accomplished in section (3) below.

In the next two sections, (4) and (5), we apply these results to show our main result (5.5), which we would now like to briefly describe. Given a C^*-algebraic bundle $\mathcal{B}$ over the locally compact abelian group G (see [4] for a comprehensive study of C^*-algebraic bundles), we consider a natural action of the dual group Γ on the C^*-cross sectional algebra $C^*(\mathcal{B})$, which we call the dual action. The content of our main theorem says, among other things, that there is a dense subset of the positive cone of $C^*(\mathcal{B})$, whose elements satisfy a certain integrability property. Namely, if p is in this subset then, for all a in $C^*(\mathcal{B})$, the maps $x \mapsto a\alpha_x(p)$ and $x \mapsto \alpha_x(p)a$ are unconditionally integrable.

In the following section (6), we conduct a brief study, from an abstract point of view, of the integrability property which turned out to be the conclusion of our main theorem. A few comments are then presented in section (7), and an open question is posed, with respect to the possible characterization of dual actions by means of integrability properties.

Last but not least, I would like to express my thanks to Beatriz Abadie, who took an active role in the early stages of the project which culminated with the present work.

2. Unconditional Integration

Let $(S, \mathfrak{M}, \mu)$ be a measure space and X be a Banach space. The well known Bochner's theory of integration (also referred to as Bochner–Dunford–Hildebrandt integration theory) discusses the conditions under which one can define the integral of functions $f\colon S \to X$. According to that theory, a strongly measurable f is integrable if and only if (see [**12**, V.5])

$$\int_S \|f(s)\| \, d\mu(s) < \infty.$$

In the special case of the counting measure on a set S, one therefore sees that a necessary and sufficient condition for f to be Bochner-integrable is that the series $\sum_{s\in S} f(s)$ be *absolutely* summable. However, in many situations, the point can be made that the most natural notion of summability for series in a Banach space is that of *unconditional* summability.

It is the goal of the present section to present an integration theory which generalizes Bochner's theory in the same way that unconditional summability generalizes the notion of absolute summability.

Let us start by considering a measure space $(S, \mathfrak{M}, \mu)$ where, as usual, $\mathfrak{M}$ is the σ-algebra of measurable subsets of S, and μ is a σ-additive positive measure defined on $\mathfrak{M}$.

An important ingredient of our theory is the notion of a local family, which we describe below.

2.1. Definition. *Given a measure space $(S, \mathfrak{M}, \mu)$, we say that a subset $\mathfrak{L} \subseteq \mathfrak{M}$ is a local family if the following conditions hold:*

i) *$\mathfrak{L}$ is closed under finite unions.*

ii) *If $L \in \mathfrak{L}$ and B is a measurable subset of L then $B \in \mathfrak{L}$ (that is, $\mathfrak{L}$ is hereditary).*

Once a local family is fixed we will say that its members are the local sets.

To avoid trivialities it is often interesting to assume that $\mathfrak{L}$ also satisfies

iii) If $L \in \mathfrak{L}$ then $\mu(L) < \infty$.

iv) For every measurable set B, one has that $\mu(B) = \sup\{\mu(L) \colon L \in \mathfrak{L}, L \subseteq B\}$.

However, we will find it unnecessary to assume that these last two properties hold for the local families in consideration in this section.

Note that (ii) implies that $\mathfrak{L}$ is closed under countable intersections.

An example of local family would be, of course, the collection of all sets of finite measure. In case we are speaking of a regular Borel measure on a locally compact topological space, a natural choice for a local family is the collection of all measurable, relatively compact subsets.

Throughout this chapter we will fix a measure space $(S, \mathfrak{M}, \mu)$ equipped with a fixed local family $\mathfrak{L}$.

Let X be a Banach space.

2.2. Definition. *We say that a function $f: S \to X$ is locally integrable (with respect to $\mathfrak{L}$) if f is Bochner-integrable over every local set.*

Observe that $\mathfrak{L}$ is an ordered set under set-inclusion, which is clearly directed in the sense that given L_1 and L_2 in $\mathfrak{L}$, there is an L in $\mathfrak{L}$, bigger than both L_1 and L_2 (namely their union). This allows us to use $\mathfrak{L}$ as the index set for nets. In particular, given a locally integrable function f, we can form the net

$$\left(\int_L f\, d\mu\right)_{L \in \mathfrak{L}}.$$

2.3. Definition. *We say that a function $f: S \to X$ is unconditionally integrable (with respect to $\mathfrak{L}$), or just u-integrable, if the above net converges in the norm topology of X. In this case we set*

$$(\mathrm{U})\int_S f\, d\mu = \lim_{L \in \mathfrak{L}} \int_L f\, d\mu.$$

The Cauchy condition for convergence of nets, when applied to ours, gives the following.

2.4. Proposition. *A function $f: S \to X$ is unconditionally integrable if and only if, for every $\varepsilon > 0$, there exists an L_0 in $\mathfrak{L}$ such that, given any D in $\mathfrak{L}$, which is disjoint from L_0, one has $\| \int_D f\, d\mu \| < \varepsilon$.*

It is easy to show that, for an unconditionally integrable f, one has that the supremum of $\| \int_L f\, d\mu \|$, as L ranges over all local sets, is finite. However, this condition does not imply unconditional integrability. Nevertheless, the functions satisfying this property are relevant for our study as well.

2.5. Definition. *A locally integrable function $f: S \to X$ is said to be* *pseudo-integrable if*

$$\sup_{L\in\mathfrak{L}} \|\int_L f\,d\mu\| < \infty.$$

For the special case of scalar valued functions we have the following.

2.6. Lemma. *If $f\colon S \to \mathbb{C}$ is pseudo-integrable then*

$$\sup_{L\in\mathfrak{L}} \int_L |f(s)|\,d\mu(s) < \infty.$$

Proof. Assume, initially, that f is real valued and let

$$M = \sup_{L\in\mathfrak{L}} |\int_L f\,d\mu|.$$

Given L in $\mathfrak{L}$ let $L_+ = \{s \in L\colon f(s) \geq 0\}$ and $L_- = \{s \in L\colon f(s) < 0\}$. Since f is locally integrable it is, in particular, measurable when restricted to local sets. Hence both L_+ and L_- are measurable sets, and therefore belong to $\mathfrak{L}$. We have

$$\begin{aligned}\int_L |f(s)|\,d\mu(s) &= \int_{L_+} f(s)\,d\mu(s) - \int_{L_-} f(s)\,d\mu(s)\\ &= |\int_{L_+} f(s)\,d\mu(s)| + |\int_{L_-} f(s)\,d\mu(s)| \leq 2M.\end{aligned}$$

In the general case, it is clear that both the real and imaginary part of f are pseudo-integrable and therefore the conclusion holds for them, and therefore also for f. □

Observe that we have entirely avoided the question of integrability of $|f|$. In fact, it's worth noticing that it is not even clear if, under the hypothesis above, f is measurable! However, it is one of the main features of our theory that only the local behavior of functions be under analysis.

In the following we let $L^\infty(S)$ be the classical space of bounded, measurable functions on S, with the essential supremum norm.

2.7. Proposition. *Let $f\colon S \to X$ be pseudo-integrable. Then there exists a positive constant M such that for all ϕ in $L^\infty(S)$ and, for all L in $\mathfrak{L}$,*

$$\|\int_L \phi f\,d\mu\| \leq M\|\phi\|.$$

Consequently ϕf *is also pseudo-integrable.*

Proof. Let x' be a continuous linear functional on X. Then, clearly, $x' \circ f$ is a pseudo-integrable scalar valued function on S, and hence, by Lemma (2.6), we have that

$$N := \sup_{L \in \mathfrak{L}} \int_L |x'(f(s))| \, d\mu(s) < \infty.$$

Let L be a local set and pick ϕ in $L^\infty(S)$ with $\|\phi\| \leq 1$. Then

$$\begin{aligned} |x' \left(\int_L \phi f \, d\mu \right)| &\leq \int_L |\phi(s)| \, |x'(f(s))| \, d\mu(s) \\ &\leq \|\phi\| \int_L |x'(f(s))| \, d\mu(s) \leq N. \end{aligned}$$

This shows that the set

$$\left\{ \int_L \phi f \, d\mu : L \in \mathfrak{L}, \phi \in L^\infty(S), \|\phi\| \leq 1 \right\}$$

is weakly bounded, and hence bounded in norm, from which the conclusion follows. □

We would like to thank Carmem Cardassi for a suggestion which helped simplify our original proof of (2.7).

2.8. Proposition. *If* f *is unconditionally integrable and* ϕ *is in* $L^\infty(S)$, *then* ϕf *is also unconditionally integrable.*

Proof. Assume, initially, that ϕ is the characteristic function of a measurable set B. By the Cauchy condition (2.4), for each $\varepsilon > 0$, let L_0 be a local set such that each D in $\mathfrak{L}$ which is disjoint from L_0, satisfies $\| \int_D f \, d\mu \| < \varepsilon$. Then

$$\| \int_D \phi f \, d\mu \| = \| \int_{D \cap B} f \, d\mu \| < \varepsilon,$$

which says that the Cauchy condition holds for ϕf as well. Hence ϕf is unconditionally integrable. If we now assume that ϕ is a linear combination of characteristic functions, i.e, a simple function, then the conclusion obviously holds, i.e, ϕf is unconditionally integrable.

To deal with the general ϕ, let M be such that

$$\| \int_L \psi f \, d\mu \| \leq M \|\psi\|, \quad \psi \in L^\infty(S), L \in \mathfrak{L},$$

as in (2.7). Now, given $\varepsilon > 0$, choose ϕ_0 in $L^\infty(S)$ to be a simple function satisfying $\|\phi - \phi_0\| < \varepsilon/2M$. Next, applying the Cauchy condition to $\phi_0 f$, which we already know is unconditionally integrable, pick a local set L_0 such that for any local set D, disjoint from L_0,

$$\|\int_D \phi_0 f\, d\mu\| < \varepsilon/2.$$

So, for all such D we have

$$\|\int_D \phi f\, d\mu\| \leq \|\int_D (\phi - \phi_0) f\, d\mu\| + \|\int_D \phi_0 f\, d\mu\| \leq M\|\phi - \phi_0\| + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the Cauchy condition holds for ϕf and hence that it is unconditionally integrable. □

2.9. Lemma. *If f is unconditionally integrable, then, for any positive ε, there exists a local set L_0, such that for all local sets D, disjoint from L_0,*

$$\|\int_D \phi f\, d\mu\| \leq \varepsilon\|\phi\|, \quad \phi \in L^\infty(S).$$

Proof. Arguing by contradiction, suppose that there exists $\varepsilon > 0$ such that for any local set L_0 there is a local set D, which does not intercept L_0, and a ϕ in $L^\infty(S)$ such that

$$\|\int_D \phi f\, d\mu\| > \varepsilon\|\phi\|.$$

Using this, pick a local set D_1 and a unit vector ϕ_1 in $L^\infty(S)$ such that $\| \int_{D_1} \phi_1 f\, d\mu\| > \varepsilon$. Then, letting D_1 play the role of L_0 above, pick a local set D_2, disjoint from D_1, and ϕ_2 in $L^\infty(S)$ with unit norm, such that $\| \int_{D_2} \phi_2 f\, d\mu\| > \varepsilon$. Continuing in this fashion, we obtain a pairwise disjoint sequence of local sets $(D_n)_n$ and a sequence $(\phi_n)_n$ of unit vectors in $L^\infty(S)$ with

$$\|\int_{D_n} \phi_n f\, d\mu\| > \varepsilon.$$

Now define $\phi = \sum_n \phi_n \chi_{D_n}$, where χ_{D_n} is the characteristic function of D_n. Clearly ϕ is in $L^\infty(S)$, so that, by (2.8), ϕf is unconditionally integrable and hence, by (2.4) there is a local set L_0 such that $\| \int_D \phi f\, d\mu\| < \varepsilon/2$ for any local set D, disjoint from L_0.

So, for all n we have

$$\varepsilon < \| \int_{D_n} \phi f \, d\mu \| \leq \| \int_{D_n \cap L_0} \phi f \, d\mu \| + \| \int_{D_n \setminus L_0} \phi f \, d\mu \| \leq$$

$$\leq \int_{D_n \cap L_0} \| \phi(s) f(s) \| \, d\mu(s) + \frac{\varepsilon}{2} \leq \int_{D_n \cap L_0} \| f(s) \| \, d\mu(s) + \frac{\varepsilon}{2}.$$

which implies that

$$\int_{D_n \cap L_0} \| f(s) \| \, d\mu(s) > \frac{\varepsilon}{2}.$$

Now, by the assumption that f is locally integrable, and hence Bochner-integrable over L_0, we have that $\int_{L_0} \| f(s) \| \, d\mu(s) < \infty$. So

$$\sum_{k=1}^{\infty} \int_{D_k \cap L_0} \| f(s) \| \, d\mu(s) < \infty,$$

which conflicts with the conclusion of the previous paragraph. □

Let $\mathcal{U}(S, X)$ denote the space of all unconditionally integrable functions from S to X. Observe that each f in $\mathcal{U}(S, X)$ defines a bounded linear transformation

$$T_f \colon \phi \in L^\infty(S) \mapsto (\mathrm{U})\int_S \phi f \, d\mu \in X.$$

(The boundedness of T_f is a consequence of (2.7)). In fact, since by definition,

$$(\mathrm{U})\int_S \phi f \, d\mu = \lim_{L \in \mathfrak{L}} \int_L \phi f \, d\mu$$

we see that $\|T_f\|$ is precisely given by

$$\|T_f\| = \sup \left\{ \| \int_L \phi f \, d\mu \| \colon L \in \mathfrak{L}, \|\phi\| \leq 1 \right\}.$$

This provides a way to equip $\mathcal{U}(S, X)$ with a norm, namely $\|f\| := \|T_f\|$, for $f \in \mathcal{U}(S, X)$. Actually, this is in general only a semi-norm, and hence, to form a normed space one needs to mod out the vectors of zero norm.

2.10. Proposition. *The subset of $\mathcal{U}(S, X)$ formed by the locally supported functions f (i.e vanishing outside some local set) is dense in $\mathcal{U}(S, X)$.*

Proof. Let f be in $\mathcal{U}(S,X)$. For $\varepsilon > 0$ let L_0 be as in Lemma (2.9). Then, if f_0 denotes the product of f by the characteristic function on L_0, we have for any ϕ in $L^\infty(S)$ and all local sets L,

$$\| \int_L \phi(f - f_0)\, d\mu \| = \| \int_{L \setminus L_0} \phi f \, d\mu \| \leq \varepsilon \|\phi\|.$$

This says that $\|f - f_0\| \leq \varepsilon$, concluding the proof. □

3. Fourier Inversion Theorem

Let G be a locally compact topological group. Also let $\mathfrak{H}$ be a Hilbert space, and denote by $\mathfrak{B}(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$.

3.1. Definition. *A function*

$$p\colon G \to \mathfrak{B}(\mathfrak{H})$$

is said to be of positive-type if, for every finite set $\{t_1, t_2, \ldots, t_n\} \subseteq G$ *one has that the* $n \times n$ *matrix* $\left(p(t_i^{-1}t_j)\right)_{i,j}$ *is a positive element of the* C^**-algebra* $M_n(\mathfrak{B}(\mathfrak{H}))$.

One of our main tools in dealing with positive-type maps is Naimark's theorem [**7**], [**9**, 4.8], *which we state below. We'd like to thank Fernando Abadie for having brought this result to our attention.*

3.2. Theorem. *If* $p\colon G \to \mathfrak{B}(\mathfrak{H})$ *is a positive-type, weakly continuous map then there is a strongly continuous unitary representation* u *of* G *on a Hilbert space* $\mathfrak{H}_1$*, and a bounded linear operator* $V\colon \mathfrak{H} \to \mathfrak{H}_1$ *such that*

$$p(t) = V^* u(t) V, \quad t \in G.$$

Observe that, as a consequence, any such p must necessarily be strongly continuous and bounded in norm.

Throughout this section we shall fix a positive-type, weakly continuous map

$$p\colon G \to \mathfrak{B}(\mathfrak{H}),$$

and we will let u, $\mathfrak{H}_1$ and V be as above.

Let us assume, from now on, that G is abelian. Also, let Γ be the Pontryagin dual of G. We will fix the Haar measures on G and Γ with the normalization convention [**5**, 31.1] which yields Plancherel's theorem [**5**, 31.18] as well as the Fourier inversion Theorem [**5**, 31.17]. The duality between G and Γ will be denoted by (t,x), for $t \in G$ and $x \in \Gamma$. That is, the value of the character x on the group element t is denoted (t,x). On the other hand, the inner product of vectors ξ and η in the Hilbert spaces under consideration will be denoted by $\langle \xi, \eta \rangle$.

By Stone's theorem on representations of locally compact abelian groups, [**6**, 36E] it follows that there exists a projection valued measure E on Γ such that

$$u(t) = \int_\Gamma (t,x)\, dE(x).$$

In the following result we will use the Fourier transform of a complex valued, integrable (with respect to the Haar measure) function g on G. Our convention for the Fourier transform will be

$$\widehat{g}(x) = \int_G (t,x) g(t)\, dt, \quad x \in \Gamma.$$

3.3. Proposition. *If g is an integrable function on G and ξ, η are in $\mathfrak{H}$, then*

$$\int_G g(t) \langle p(t)\xi, \eta \rangle \ dt = \left\langle \int_\Gamma \widehat{g}(x)\, dE(x) V\xi, V\eta \right\rangle.$$

Proof. We have

$$\begin{aligned}
\int_G g(t) \langle p(t)\xi, \eta \rangle \ dt &= \int_G g(t) \langle u(t)V\xi, V\eta \rangle \ dt \\
&= \int_G g(t) \left(\int_\Gamma (t,x)\, d\langle E(x)V\xi, V\eta \rangle \right) dt \\
&= \int_\Gamma \left(\int_G (t,x) g(t)\, dt \right) d\langle E(x)V\xi, V\eta \rangle \\
&= \int_\Gamma \widehat{g}(x)\, d\langle E(x)V\xi, V\eta \rangle \\
&= \left\langle \int_\Gamma \widehat{g}(x)\, dE(x) V\xi, V\eta \right\rangle.
\end{aligned}$$

□

From now on we will assume that p has compact support. We may, therefore, define its Fourier transform by

$$\widehat{p}(x) = \int_G (t,x) p(t)\, dt, \quad x \in \Gamma,$$

where we understand the integral with respect to the strong topology. By this we mean that $\widehat{p}(x)$ is the bounded linear operator on $\mathfrak{H}$ given by

$$\widehat{p}(x)\xi = \int_G (t,x)p(t)\xi\,dt, \quad \xi \in \mathfrak{H}.$$

Observe, in particular, that for fixed ξ in $\mathfrak{H}$, the map

$$x \in \Gamma \mapsto \widehat{p}(x)\xi \in \mathfrak{H}$$

is continuous, since it is the Fourier transform of $p(\cdot)\xi$. So $\widehat{p}$ is a strongly continuous map from Γ into $\mathfrak{B}(\mathfrak{H})$. It is also easy to see that $\widehat{p}$ is bounded in norm.

Given that p has compact support, it follows from the Plancherel Theorem that

$$x \in \Gamma \mapsto \langle \widehat{p}(x)\xi, \eta\rangle$$

is a function in $L^2(\Gamma)$. This fact is used below.

3.4 Proposition. *If g is in $L^1(\Gamma) \cap L^2(\Gamma)$ then, for all ξ and η in $\mathfrak{H}$,*

$$\int_\Gamma \widehat{g}(x)\left\langle \widehat{p}(x^{-1})\xi, \eta\right\rangle dx = \left\langle \int_\Gamma \widehat{g}(x)\,dE(x)V\xi, V\eta\right\rangle.$$

Proof. Let $h(x) = \overline{g(x)}$. Then it is easy to see that $\widehat{g}(x^{-1}) = \overline{\widehat{h}(x)}$. The left hand side above then equals

$$\int_\Gamma \widehat{g}(x^{-1})\,\langle \widehat{p}(x)\xi, \eta\rangle\,dx = \int_\Gamma \overline{\widehat{h}(x)}\,\langle \widehat{p}(x)\xi, \eta\rangle\,dx = \int_G \overline{h(t)}\,\langle p(t)\xi, \eta\rangle\,dt =$$

$$= \int_G g(t)\,\langle p(t)\xi, \eta\rangle\,dt \overset{(3.3)}{=} \left\langle \int_\Gamma \widehat{g}(x)\,dE(x)V\xi, V\eta\right\rangle. \qquad \square$$

3.5. Corollary. *For every ξ and η in $\mathfrak{H}$, $\left\langle \widehat{p}(x^{-1})\xi, \eta\right\rangle dx$ and $d\langle E(x)V\xi, V\eta\rangle$ agree as measures on Γ.*

Proof. Note that the map $t \mapsto \langle p(t)\xi, \xi\rangle$ is a continuous scalar valued, positive-type map of compact support. So, by the scalar Fourier inversion Theorem [**5**, 31.17], its Fourier transform is in $L^1(\Gamma)$. By the polarization formula it follows that the first measure cited in the statement is of finite total variation, the same being true with respect to $d\,\langle E(x)V\xi, V\eta\rangle$.

Both measures then define continuous linear functionals on $C_0(\Gamma)$. Now, if g is in $L^1(\Gamma) \cap L^2(\Gamma)$, we have seen that

$$\int_\Gamma \widehat{g}(x) \left\langle \widehat{p}(x^{-1})\xi, \eta \right\rangle \, dx = \int_\Gamma \widehat{g}(x) \, d \left\langle E(x)V\xi, V\eta \right\rangle .$$

So, our measures coincide on a dense subset of $C_0(\Gamma)$ and hence everywhere. □

3.6. Theorem. *If p is a weakly continuous, compactly supported, positive -type function from G to $\mathfrak{B}(\mathfrak{H})$ then, for every t in G and every measurable subset L of the dual group Γ, with finite measure, one has*

$$\int_L \overline{(t, x)} \widehat{p}(x) \, dx = V^* E(L^{-1}) u(t) V. \qquad \square$$

Proof. We have already observed that $\widehat{p}$ is strongly continuous and norm-bounded. Therefore, since L has finite measure, the integral on the left hand side above is well defined with respect to the strong operator topology. Fix ξ and η in $\mathfrak{H}$. Then

$$\left\langle \int_L \overline{(t, x)} \widehat{p}(x) \, dx \; \xi, \eta \right\rangle = \int_L (t, x^{-1}) \left\langle \widehat{p}(x)\xi, \eta \right\rangle \, dx \; =$$

$$= \int_{L^{-1}} (t, x) \left\langle \widehat{p}(x^{-1})\xi, \eta \right\rangle \, dx \overset{(3.5)}{=} \int_{L^{-1}} (t, x) \, d \left\langle E(x)V\xi, V\eta \right\rangle =$$

$$= \left\langle \int_{L^{-1}} (t, x) \, dE(x) V\xi, V\eta \right\rangle = \left\langle V^* \int_{L^{-1}} (t, x) \, dE(x) V \; \xi, \eta \right\rangle .$$

Next observe that for any measurable set $B \subseteq \Gamma$

$$\int_B (t, x) \, dE(x) = E(B) u(t).$$

Using this in the above calculation we obtain

$$\left\langle \int_L \overline{(t, x)} \widehat{p}(x) \, dx \; \xi, \eta \right\rangle = \left\langle V^* E(L^{-1}) u(t) V \; \xi, \eta \right\rangle .$$

Now, since ξ and η are arbitrary, we obtain the desired conclusion. □

From this point on we will let $\mathfrak{L}$ be the local family of all measurable, relatively compact subsets of Γ (see (2.1)). This said, a local set will henceforth mean any measurable, relatively compact subset of Γ. As before we will consider $\mathfrak{L}$ as a directed set.

This brings us to the Fourier inversion Theorem for weakly continuous, positive-type, operator valued maps.

3.7. Theorem. *It p is a weakly continuous, compactly supported, positive-type function from G to* $\mathfrak{B}(\mathfrak{H})$ *then, for every* ξ *in* $\mathfrak{H}$,

$$(\mathrm{U})\int_\Gamma \overline{(t,x)}\widehat{p}(x)\xi\, dx = p(t)\xi.$$

Proof. Follows from (3.6) and the fact that $(E(L))_{L\in\mathfrak{L}}$ converges to the identity in the strong operator topology, a fact which will be proved below. □

3.8. Lemma. *Let u be a strongly continuous unitary representation of the locally compact group G on the Hilbert space* $\mathfrak{H}_1$, *with corresponding spectral measure E, on the dual group* Γ. *Let V be a bounded operator from* $\mathfrak{H}$ *to* $\mathfrak{H}_1$. *If*

$$\lim_{t\to e} \|V - u(t)V\| = 0$$

Then

$$\lim_{L\in\mathfrak{L}} \|V - E(L)V\| = 0.$$

In addition, if ξ *is in* $\mathfrak{H}_1$, *then*

$$\lim_{L\in\mathfrak{L}} \|\xi - E(L)\xi\| = 0.$$

Proof. Let $f \in L^1(G)$ and set

$$\pi(f) = \int_G f(t)u(t)\, dt = \int_\Gamma \widehat{f}(x)\, dE(x).$$

Note that for any measurable $B \subseteq \Gamma$

$$\|\pi(f) - E(B)\pi(f)\| = \|\int_{\Gamma\setminus B} \widehat{f}(x)\, dE(x)\| = \sup_{x\in\Gamma\setminus B} |\widehat{f}(x)|.$$

This, together with the Riemann-Lebesgue Lemma [**5**, 28.40], which says that $\widehat{f}$ is in $C_0(\Gamma)$, implies that

$$\lim_{L\in\mathfrak{L}} \|\pi(f) - E(L)\pi(f)\| = 0.$$

Let

$$\mathfrak{S} = \left\{T \in \mathfrak{B}(\mathfrak{H},\mathfrak{H}_1) : \lim_{L\in\mathfrak{L}} \|T - E(L)T\| = 0\right\}.$$

The argument above gives that any operator of the form $T = \pi(f)S$, with f in $L^1(G)$ and S in $\mathfrak{B}(\mathfrak{H}, \mathfrak{H}_1)$, is in $\mathfrak{S}$. On the other hand it is easy to show that $\mathfrak{S}$ is norm-closed. So, out strategy for proving that V is in $\mathfrak{S}$ will be to show that V is the norm limit of operators of the form $\pi(f)V$, with f in $L^1(G)$. Pick any f such that $f \geq 0$ and $\int_G f(t)\,dt = 1$. For any neighborhood U of the unit in G we have

$$\begin{aligned}\|V - \pi(f)V\| &= \|\int_G f(t)\,(V - u(t)V)\;dt\| \\ &\leq \int_U f(t)\|V - u(t)V\|\,dt + \int_{G\setminus U} f(t)\|V - u(t)V\|\,dt \\ &\leq \sup_{t\in U}\|V - u(t)V\| + 2\|V\| \int_{G\setminus U} f(t)\,dt.\end{aligned}$$

which can be made arbitrarily small, under a suitable choice of f.

The last part of the statement is a consequence of what we have already done, for the special case of the operator $V\colon \mathfrak{H}_1 \to \mathfrak{H}_1$ defined by $V(\eta) = \langle \eta, \xi\rangle\,\xi$. □

Our previous result is used below, to prove the Fourier inversion Theorem for norm-continuous, positive-type, operator valued maps.

3.9. Theorem. *If p is a positive-type, compactly supported function from G to $\mathfrak{B}(\mathfrak{H})$, which is norm-continuous then*

$$(\mathrm{U})\!\int_\Gamma \overline{(t,x)}\widehat{p}(x)\,dx = p(t), \qquad t \in G.$$

Proof. Representing $p(t) = V^*u(t)V$ as we have been doing, observe that

$$\begin{aligned}\|V - u(t)V\|^2 &= \|(V^* - V^*u(t)^*)(V - u(t)V)\| = \\ &= \|V^*V - V^*u(t)V - V^*u(t)^*V + V^*V\| \leq \\ &\leq \|p(e) - p(t)\| + \|p(t)^* - p(e)\|,\end{aligned}$$

which converges to zero, as $t \to e$, in virtue of the fact that p is norm-continuous at e. So (3.8) applies and thus $E(L)V \to V$ in norm. Hence

$$\begin{aligned}(\mathrm{U})\!\int_\Gamma \overline{(t,x)}\widehat{p}(x)\,dx &= \lim_{L\in\mathfrak{L}} \int_L \overline{(t,x)}\widehat{p}(x)\,dx \overset{(3.6)}{=} \lim_{L\in\mathfrak{L}} V^*u(t)E(L^{-1})V = \\ &= V^*u(t)V = p(t).\end{aligned}$$

□

4. Multiplier valued positive-type functions

Throughout this section we will let G be a locally compact abelian group. Like before, we will denote by Γ its dual, and by $\mathfrak{L}$ the local family of measurable, relatively compact subsets of Γ, which will again be viewed as a directed set. Also, let A be a C^*-algebra, considered fixed throughout.

The main object of study in this section will be a function p from G into the multiplier algebra $\mathcal{M}(A)$, which will be assumed to have compact support and to be continuous with respect to the strict topology of $\mathcal{M}(A)$. Given such a p, we can define its Fourier transform by

$$\widehat{p}(x) = \int_G (t,x)p(t)\,dt, \quad x \in \Gamma,$$

which should be understood with respect to the strict topology. Precisely, for each a in A, we have that both

$$\int_G (t,x)ap(t)\,dt \quad \text{and} \quad \int_G (t,x)p(t)a\,dt$$

are well defined Bochner integrals, and hence define the left and right action, respectively, of the multiplier $\widehat{p}(x)$. It is easy to see that the map

$$x \in \Gamma \mapsto \widehat{p}(x) \in \mathcal{M}(A)$$

is continuous with respect to the strict topology.

4.1. Definition. The function $p\colon G \to \mathcal{M}(A)$ is said to be of *positive-type* if, for every finite set $\{t_1, t_2, \dots, t_n\} \subseteq G$ one has that the $n \times n$ matrix $\left(p(t_i^{-1}t_j)\right)_{i,j}$ is a positive element of $M_n(\mathcal{M}(A))$.

4.2. Proposition. *Given a strictly continuous, compactly supported function $p\colon G \to \mathcal{M}(A)$ of positive-type then, for any t in G,*

$$\sigma\text{-}\lim_{L \in \mathfrak{L}} \int_L \overline{(t,x)}\widehat{p}(x) = p(t).$$

where σ-lim stands for strict-limit.

Proof. Let's suppose that A is represented as a non-degenerated C^*-algebra of operators in $\mathfrak{B}(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$. It is then clear that p becomes a weakly continuous, operator valued positive-type map.

We may then apply (3.6) to conclude that

$$\int_L \overline{(t,x)}\widehat{p}(x)a\,dx = V^*u(t)E(L^{-1})Va$$

and

$$\int_L \overline{(t,x)}a\widehat{p}(x)\,dx = aV^*E(L^{-1})u(t)V$$

for each a in A, where u, V and E are as in (3.2).

Proving the statement, thus amounts to showing that

$$\lim_{L\in\mathfrak{L}} E(L)Va = Va,$$

in norm. This will follow from (3.8) once we show that $\lim_{t\to e}\|Va - u(t)Va\| = 0$. For that purpose note that

$$\begin{aligned}\|Va - u(t)Va\|^2 &= \|a^*V^*Va - a^*V^*u(t)Va - a^*V^*u(t)^*Va + a^*V^*Va\| \\ &\leq \|a^*p(e)a - a^*p(t)a\| + \|a^*p(t)^*a - a^*p(e)a\|\end{aligned}$$

which converges to zero, as $t\to e$, because p is strictly continuous. □

5. C*-Algebraic Bundles

This is the main section of the present work. The goal which we will reach here is the proof that the dual action of a locally compact abelian group satisfies an integrability property related to certain conditions which have often appeared in the literature, as, for example in [**1**], [**2**], [**8**], [**10**, 7.8.4] and [**11**].

The most general context in which the concept of dual action of an abelian group can be defined is that of C^*-algebraic bundles. The reader interested in reading about C^*-algebraic bundles is referred to Fell and Doran's book [**4**], which is also our main reference in what follows.

Let $\mathcal{B}$ be a C^*-algebraic bundle over the locally compact abelian group G. The fiber of $\mathcal{B}$ over each t will be written B_t. We shall denote by $C^*(\mathcal{B})$ its cross sectional C^*-algebra [**4**, VIII.17.2], by $L^1(\mathcal{B})$ the Banach *-algebra of the integrable sections [**4**, VIII.5.2], and by $C_c(\mathcal{B})$ the dense sub-algebra of $L^1(\mathcal{B})$ formed by the continuous, compactly supported sections [**4**, II.14.2]. We remark that our notation differs from [**4**] with respect to $C_c(\mathcal{B})$. Under the usual identifications, we will regard $L^1(\mathcal{B})$ as a subalgebra of $C^*(\mathcal{B})$.

As before, let us denote the dual of G by Γ, and the local family of measurable, relatively compact subsets of Γ, by $\mathfrak{L}$. For each x in Γ, let α_x be the transformation of $L^1(\mathcal{B})$ given by the formula

$$\alpha_x(f)|_t = (t,x)f(t), \quad f \in L^1(\mathcal{B}), t \in G.$$

It is easy to see that α_x is a well defined automorphism of $L^1(\mathcal{B})$, which therefore extends to an automorphism, also denoted by α_x, of its enveloping C^*-algebra, namely $C^*(\mathcal{B})$. In addition it is clear that the map

$$\alpha: x \in \Gamma \mapsto \alpha_x \in \mathrm{Aut}(C^*(\mathcal{B}))$$

is a strongly continuous group action of Γ on $C^*(\mathcal{B})$.

5.1. Definition. The *dual action* of Γ on $C^*(\mathcal{B})$ is that which has just been defined.

Observe that, in case $\mathcal{B}$ is the semi-direct product bundle [**4**, VIII.4.2] constructed from an action τ of G on a C^*-algebra A, then $C^*(\mathcal{B})$ is isomorphic to $A \times_\tau G$, in such a way that the dual action we have defined corresponds to the usual dual action [**10**, 7.8.3] on the crossed product.

Let f be in $C_c(\mathcal{B})$. It will be fruitful to view f both as an element of $C^*(\mathcal{B})$ and as a map from G into the multiplier algebra of $C^*(\mathcal{B})$ in a way we will now describe.

Initially note that each element u in B_t (recall that this means the fiber of over t) defines [**4**, VIII.5.8] a multiplier of the algebra $L^1(\mathcal{B})$, by the formulas

$$(ug)|_s = ug(t^{-1}s), \quad s \in G,$$

and

$$(gu)|_s = g(st^{-1})u, \quad s \in G,$$

for each g in $L^1(\mathcal{B})$. Now, by [**4**, VIII.1.15] one can extend the above to a multiplier of $C^*(\mathcal{B})$. Thus, a function f in $C_c(\mathcal{B})$ defines a map

$$F: G \to \mathcal{M}(C^*(\mathcal{B}))$$

which is given, for g in $L^1(\mathcal{B})$, by

$$(F(t)g)|_s = f(t)g(t^{-1}s), \quad s \in G$$

and

$$(gF(t))|_s = g(st^{-1})f(t), \quad s \in G.$$

5.2. Proposition. *If f is in $C_c(\mathcal{B})$, then the corresponding F is continuous with respect to the strict topology.*

Proof. Fixing g in $C_c(\mathcal{B})$ and t_0 in G, consider the map

$$\lambda\colon (t,s) \in G \times G \mapsto \|f(t)g(t^{-1}s) - f(t_0)g(t_0^{-1}s)\|,$$

where the norm used is that of the fiber B_s. It is a consequence of the continuity of the norm and the other bundle operations, that λ is continuous.

Let V be a compact neighborhood of t_0. It is easy to see that, for t in V, one has that $\lambda(t,s) = 0$ unless $s \in V \cdot \mathrm{supp}(g)$, which is a compact subset of G. An often used topological argument now shows that $\lim_{t\to t_0} \sup_{s\in G} \lambda(t,s) = 0$, which, combined with the fact that g has compact support, implies that

$$\lim_{t\to t_0} \int_G \|f(t)g(t^{-1}s) - f(t_0)g(t_0^{-1}s)\| = 0,$$

which, in turn, can be interpreted as saying that the map

$$t \in G \mapsto F(t)g \in L^1(\mathcal{B})$$

is continuous at t_0.

Since the inclusion of $L^1(\mathcal{B})$ in $C^*(\mathcal{B})$ is continuous, we have that $\lim_{t\to t_0} F(t)g = F(t_0)g$ in the norm of $C^*(\mathcal{B})$. A similar reasoning shows that $\lim_{t\to t_0} gF(t) = gF(t_0)$. Finally, observing that f is bounded, we can show the above continuity, even if g is replaced by an arbitrary element of $C^*(\mathcal{B})$, thus proving F to be strictly continuous. □

5.3. Lemma. *Let f be in $C_c(\mathcal{B})$, and denote by F the corresponding map into $\mathcal{M}(C^*(\mathcal{B}))$. Since we now know that F is strictly continuous, we may define the Fourier transform $\widehat{F}$ of F as in the beginning of section (4). Then, for all x in Γ, we have that $\widehat{F}(x) \in C^*(\mathcal{B})$ and*

$$\widehat{F}(x) = \alpha_x(f).$$ □

Proof. Viewing both $\widehat{F}(x)$ and $\alpha_x(f)$ as elements of $\mathcal{M}(C^*(\mathcal{B}))$, all we need to do is show that, for every g in $C_c(\mathcal{B})$ and x in Γ, one has $\widehat{F}(x)g = \alpha_x(f) * g$. We have, for t in G,

$$(\alpha_x(f) * g)|_t = \int_G \alpha_x(f)(s)g(s^{-1}t)\,ds = \int_G (s,x)f(s)g(s^{-1}t)\,ds =$$

$$= \int_G (s,x)\big(F(s)g\big)(t)\,ds = \Big(\int_G (s,x)F(s)g\,ds\Big)(t).$$

The last equality following from [4, II.15.19]. This shows that

$$\alpha_x(f) * g = \int_G (s,x)F(s)g\,ds = \widehat{F}(x)g. \qquad \square$$

A last preparatory result, before we can prove our main theorem, is in order.

5.4. Lemma. *Let f be in $C_c(\mathcal{B})$ and put $p = f^* * f$. Denote by P the corresponding map into $\mathcal{M}(C^*(\mathcal{B}))$. Then P is of positive-type.*

Proof. Let $C^*(\mathcal{B})$ be faithfully represented on a Hilbert space $\mathfrak{H}$ under a non-degenerated representation. Choose finite sets $\{t_1, t_2, \ldots, t_n\} \subseteq G$, $\{a_1, a_2, \ldots, a_n\} \subseteq C_c(\mathcal{B})$ and $\{\xi_1, \xi_2, \ldots, \xi_n\} \subseteq \mathfrak{H}$. Then

$$\sum_{i,j} \left\langle P(t_i^{-1}t_j)a_j\xi_j, a_i\xi_i \right\rangle = \sum_{i,j} \int_G \langle F(st_i)^* F(st_j)a_j\xi_j, a_i\xi_i\rangle \,ds =$$

$$= \int_G \left\langle \sum_j F(st_j)a_j\xi_j, \sum_i F(st_i)a_i\xi_i \right\rangle ds \geq 0. \qquad \square$$

We are now ready to present our main result.

5.5 Theorem. *Let $\mathcal{B}$ be a C^*-algebraic bundle over the locally compact abelian group G with dual Γ, and let p be of the form $p = f^* * f$, where $f \in C_c(\mathcal{B})$. Then, for all a in $C^*(\mathcal{B})$, the maps*

$$x \in \Gamma \mapsto a\alpha_x(p) \in C^*(\mathcal{B})$$

and

$$x \in \Gamma \mapsto \alpha_x(p)a \in C^*(\mathcal{B})$$

are unconditionally integrable. Moreover, for each t in G one has

$$(\mathrm{U})\int_\Gamma \overline{(t,x)}\,a\alpha_x(p)\,dx = aP(t)$$

and

$$(\mathrm{U})\int_\Gamma \overline{(t,x)}\alpha_x(p)a\,dx = P(t)a,$$

where P is the corresponding map into $\mathcal{M}(C^*(\mathcal{B}))$.

Proof. By (5.2) we know that P is strictly continuous, while the Lemma above tells us that P is of positive-type. Hence we are allowed to employ (4.2), and conclude that

$$\sigma\text{-}\lim_{L\in\mathfrak{L}}\int_L \overline{(t,x)}\widehat{P}(x)\,dx = P(t),$$

which implies, for all a in $C^*(\mathcal{B})$, that

$$(\mathrm{U})\int_\Gamma \overline{(t,x)}a\widehat{P}(x)\,dx = aP(t).$$

Now, (5.3) tells us that $\widehat{P}(x) = \alpha_x(p)$, which, substituted in the above formula brings us to the conclusion. The case in which a is taken to multiply $\alpha_x(p)$ on the right is treated similarly. □

6. Unconditional Integrability for Group Actions

In this section we will conduct a brief study of abelian group actions on C^*-algebras, from a point of view motivated by theorem 5.5. Our results in this section will be mostly of an exploratory nature, possibly paving the way for a future, more comprehensive study of the present phenomenon.

Let us keep the notation of the previous section and hence G and Γ will be locally compact abelian groups, each being the other's dual. We will also retain the use of $\mathfrak{L}$, the local family of of all measurable, relatively compact subsets of Γ, with respect to which we will speak of unconditional integration.

Let us also fix a C^*-algebra B and a strongly continuous action

$$\alpha: \Gamma \to \mathrm{Aut}(B).$$

6.1. Definition. Let b be in B. We will say that b is *α-integrable* if, for all a in B, the maps

$$x \in \Gamma \mapsto \alpha_x(b)a \in B \quad \text{and} \quad x \in \Gamma \mapsto b\alpha_x(b) \in B$$

are unconditionally integrable.

Employing the terminology just introduced, Theorem 5.5 is seen to state that the elements of the form $p = f^* * f$ (notation as in (5.5)) are α-integrable. Since the linear combinations of such elements form a dense subset of of $C^*(\mathcal{B})$, we get an abundance of α-integrable elements.

Let b be an α-integrable element of B. Observe that, by (2.8), for any ϕ in $L^\infty(\Gamma)$ we may define

$$L(a) = (\mathrm{U})\!\int_\Gamma \phi(x)\alpha_x(b)a\,dx, \quad a \in A$$

and

$$R(a) = (\mathrm{U})\!\int_\Gamma \phi(x)a\alpha_x(b)\,dx, \quad a \in A,$$

both of which are well defined elements of B. It is clear that the pair (L, R) is then a multiplier of B, which we will denote, simply, by

$$(L, R) = \int_\Gamma \phi(x)\alpha_x(b)\,dx.$$

It is worth noticing that any α-integrable element satisfies the more usual notion of integrability (see the references given in the introduction), namely that, given an α-integrable element b, there exists an element b_0 in $\mathcal{M}(B)$ such that, for any continuous linear functional f on B, one has

$$\int_\Gamma f(\alpha_x(b))\,dx = f(b_0).$$

To see this, note that by the Cohen-Hewitt factorization theorem [**5**, 32.22], any continuous linear functional f is of the form $f(b) = g(ab)$ for some functional $g \in B'$ and a in B. Then, letting $b_0 = \int_\Gamma \alpha_x(b)\,dx$ we have

$$\int_\Gamma f(\alpha_x(b))\,dx = \int_\Gamma g(a\alpha_x(b))\,dx = g(ab_0) = f(b_0).$$

6.2. Definition. Let b be an α-integrable element of B. The *Fourier transform* of b is the map $\widehat{b}\colon G \to \mathcal{M}(B)$ defined by

$$\widehat{b}(t) = \int_\Gamma \overline{(t, x)}\alpha_x(b)\,dx.$$

6.3. Proposition. *The Fourier transform of each α-integrable element b is continuous as a map from G into $\mathcal{M}(B)$, with the strict topology.*

Proof. Given a in B, we know that $\alpha_x(b)a$ is unconditionally integrable and so, by (2.7), there exists a constant $M > 0$ such that

$$\|\int_L \phi(x)\alpha_x(b)a\,dx\| \leq M\|\phi\|, \quad \phi \in L^\infty(\Gamma), L \in \mathfrak{L}.$$

On the other hand, using (2.9), there exists, for each $\varepsilon > 0$, a local set $L_0 \subseteq \Gamma$ such that

$$\|\int_D \phi(x)\alpha_x(b)a\,dx\| \leq \varepsilon\|\phi\|, \quad \phi \in L^\infty(\Gamma), D \cap L_0 = \varnothing.$$

Let t_0 be in G and denote by V the neighborhood of t_0 consisting of those t in G such that $|(t,x) - (t_0,x)| < \varepsilon$, for all x in L_0. The reason why V is a neighborhood of t_0 is precisely the Pontryagin–van Kampen duality Theorem [**5**, 24.8].

For each t in V we have

$$\|\int_L (\overline{(t,x)} - \overline{(t_0,x)})\alpha_x(b)a\,dx\| \leq$$

$$\leq \|\int_{L\cap L_0} (\overline{(t,x)} - \overline{(t_0,x)})\alpha_x(b)a\,dx\| + \|\int_{L\setminus L_0} (\overline{(t,x)} - \overline{(t_0,x)})\alpha_x(b)a\,dx\| \leq$$

$$\leq M \sup_{x\in L\cap L_0} |(t,x) - (t_0,x)| + \varepsilon \sup_{x\in L\setminus L_0} |(t,x) - (t_0,x)| \leq M\varepsilon + 2\varepsilon.$$

Therefore, taking limit as $L \in \mathfrak{L}$, we conclude that

$$\|\widehat{b}(t)a - \widehat{b}(t_0)a\| \leq M\varepsilon + 2\varepsilon.$$

A similar argument shows that $\|a\widehat{b}(t) - a\widehat{b}(t_0)\|$ also tends to zero as t approaches t_0. □

Every automorphism of B has a unique extension to an automorphism of the multiplier algebra $\mathcal{M}(B)$. This implies that there is an extension of the action α to an action of Γ on $\mathcal{M}(B)$ (which may not be strongly continuous). For simplicity we will denote that action by α as well. For each t in G, let $\mathcal{M}(B)_t$ be the subspace of $\mathcal{M}(B)$ given by

$$\mathcal{M}(B)_t = \{m \in \mathcal{M}(B) : \alpha_x(m) = (t,x)m\}.$$

6.4. Proposition. *For each α-integrable element b in B, and each t in G, one has that $\widehat{b}(t) \in \mathcal{M}(B)_t$.*

Proof. The proof is a simple change of variable in the definition of $\widehat{b}(t)$. □

6.5. Lemma. *Let $a, b \in B$, let $m, n \in \mathcal{M}(B)$ and let L be a local set. Then*

$$\|\int_L m^*\alpha_x(a^*b)n\,dx\| \le \|\int_L m^*\alpha_x(a^*a)m\,dx\| \; \|\int_L n^*\alpha_x(b^*b)n\,dx\|.$$

Proof. Let us assume that B is faithfully represented on a Hilbert space $\mathfrak{H}$, and let ξ and η be unit vectors in $\mathfrak{H}$. We then have

$$|\left\langle \int_L m^*\alpha_x(a^*b)n\,dx\ \xi, \eta \right\rangle| = |\int_L \langle \alpha_x(b)n\xi, \alpha_x(a)m\eta\rangle\ dx| \le$$

$$\le \int_L \|\alpha_x(b)n\xi\|\,\|\alpha_x(a)m\eta\|\,dx \le \left(\int_L \|\alpha_x(b)n\xi\|^2\,dx\right)^{\frac{1}{2}} \left(\int_L \|\alpha_x(a)m\eta\|^2\,dx\right)^{\frac{1}{2}} =$$

$$= \left(\int_L \left\langle n^*\alpha_x(b^*b)n\xi, \xi\right\rangle\,dx\right)^{\frac{1}{2}} \left(\int_L \left\langle m^*\alpha_x(a^*a)m\eta, \eta\right\rangle\,dx\right)^{\frac{1}{2}} \le$$

$$\le \|\int_L n^*\alpha_x(b^*b)n\,dx\|^{\frac{1}{2}}\ \|\int_L m^*\alpha_x(a^*a)m\,dx\|^{\frac{1}{2}}.$$

Since ξ and η are arbitrary, the proof is concluded. □

6.6. Proposition. *The subset of B_+ consisting of the positive, α-integrable elements is a hereditary cone in B.*

Proof. Let $0 \le h \le k$ where k is α-integrable. Given c in B, using 6.5, where we set $a = b = h^{\frac{1}{2}}$, $m = 1$ and $n = c$, we obtain, for every local set L,

$$\|\int_L \alpha_x(h)c\,dx\| \le \|\int_L \alpha_x(h)\,dx\|^{\frac{1}{2}}\ \|\int_L c^*\alpha_x(h)c\,dx\|^{\frac{1}{2}} \le$$

$$\le \|\int_L \alpha_x(k)\,dx\|^{\frac{1}{2}}\ \|\int_L c^*\alpha_x(k)c\,dx\|^{\frac{1}{2}}.$$

Next observe that the term $\|\int_L \alpha_x(k)\,dx\|$ is bounded with respect to L, because k is α-integrable. This said, we see that the Cauchy condition (2.4) for the integrability of $c^*\alpha_x(k)c$ implies the Cauchy condition for $\alpha_x(h)c$. This concludes the proof. □

7. Concluding Remarks

We have seen in Theorem (5.5) that, for the case of the dual action of Γ on $C^*(\mathcal{B})$, the set of α-integrable elements is dense. This would seem to indicate that dual actions could be characterized via some sort of integrability condition, a question that Rieffel once suggested to us in private communication. Precisely, we feel that it would be interesting to be able to determine conditions over a given action of Γ on a C^*-algebra B which would imply that B is isomorphic to the cross sectional C^*-algebra for some C^*-algebraic bundle, under an isomorphism which puts in correspondence the given action on B and the dual action on $C^*(\mathcal{B})$.

Consider, for example, an action for which the set of α-integrable elements is dense. For each t in G, define B_t to be the subset of $\mathcal{M}(B)$ formed by the elements of the form $\widehat{b}(t)$, where b ranges over the set of α-integrable elements. It is not hard to see that the B_t's form a C^*-algebraic bundle over the group obtained by giving G the discrete topology.

In particular, if the action we are talking of happens to be a dual action, it would be interesting to decide what is the relationship between the bundle constructed from the action and the bundle which originated it. For example, consider the semi-direct product bundle obtained from the action of the circle group S^1 on $C(S^1)$ by translation. The C^*-cross sectional algebra [4, VIII.17.2] turns out to be isomorphic to the algebra of compact operators on $l^2(\mathbb{Z})$, with the dual action of $\mathbb{Z}$ being the action obtained by conjugation by the powers of the bilateral shift. It can be proved, in this case, that B_0 is precisely the set of Laurent operators with symbol in $L^\infty(S^1)$, while the fiber of the original bundle corresponds to symbols in $C(S^1)$. That is, the hope that the bundle be exactly recovered via the α-integrable elements is not a reasonable one. However, Theorem (5.5) implies that, in the case of a general dual action, we always get the original fibers as a subspace of B_t. The problem would then be to decide a selection criteria to determine which elements in B_t correspond to the elements of the original fiber. This should, quite likely, resemble the Landstad conditions [**10**, 7.8.2].

Among other things, the interest in being able to show an action to be equivalent to a dual action, is that C^*-algebraic bundles can be characterized via twisted partial actions [**3**]. The achievement of this goal would allow one to gain a deep understanding of the action in question.

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