

On the asymptotics of a 1-parameter family of infinite measure preserving transformations

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— *Dedicated to R. Mañé*

Abstract. We estimate various aspects of the growth rates of ergodic sums for some infinite measure preserving transformations which are not rationally ergodic.

Keywords: Infinite measure preserving transformation, logarithmic ergodic theorem, asymptotics, rationally ergodic.

0. Ergodic sums of infinite measure preserving transformations

Let $T = (X_T, \mathcal{B}_T, m_T, T)$ be a conservative, ergodic measure preserving transformation of a σ -finite, infinite, nonatomic standard measure space. It is known ([Ho], see also [A],[Kr]) that for

$$f \in L^1(m_T)_+ := \{f \in L^1(m_T) : f \geq 0, \int_X f dm_T > 0\},$$

$$S_n(f)(x) = S_n^T(f) := \sum_{k=0}^{n-1} f(T^k x) \rightarrow \infty \text{ for a.e. } x \in X,$$

and for $f, g \in L^1_+$:

$$\frac{S_n(f)(x)}{S_n(g)(x)} \rightarrow \frac{\int_X f dm}{\int_X g dm} \text{ for a.e. } x \in X,$$

whence,

$$S_n(f) = o(n) \text{ a.e.}$$

On the other hand, for any sequence of constants $(a_n)_{n \in \mathbb{N}}$,

$$S_n(f) \not\asymp a_n \text{ a.e.}$$

as was shown in [A2] (see also [A]).

A rationally ergodic transformation $T = (X_T, \mathcal{B}_T, m_T, T)$ satisfies a kind of ergodic theorem :

$$\forall n_k \rightarrow \infty, \exists m_\ell = n_{k_\ell} \rightarrow \infty \ni \frac{1}{N} \sum_{\ell=1}^N \frac{S_{m_\ell}(f)}{a_{m_\ell}} \rightarrow \int_X f dm \text{ a.e. } \forall f \in L^1 \quad (1)$$

where $a_n = a_n(T)$ are constants ([A1], see also [A]). This sequence of constants, called the *return sequence*, is determined by (1) uniquely up to asymptotic equality, and can therefore be considered to represent the absolute rate of growth of $S_n(f)$ as $n \rightarrow \infty$ for $f \in L^1_+$.

In order to study the rate of growth of $S_n^T(f) \rightarrow \infty$ for general T , define as in [A3] the *median sequences* $\alpha_n(P, f, \theta)$ for P a m_T -absolutely continuous probability on X_T , $f \in L^1(m_T)_+$, $0 < \theta < 1$ by

$$\alpha_n(P, f, \theta) := \max \{t \geq 0 : P([S_n(f) \geq t]) \geq \theta\}.$$

For example if $T : \mathbb{R} \rightarrow \mathbb{R}$ is Boole's transformation defined by $Tx = x - \frac{1}{x}$, then T is a conservative, ergodic, measure preserving transformation of \mathbb{R} equipped with Lebesgue measure (see [Ad-W]) and is rationally ergodic with return sequence $a_n(T) \sim \frac{\sqrt{2n}}{\pi}$ ([A3], see also [A]).

It is also shown in [A3] that

$$P\left(\left[\sum_{k=0}^{n-1} f \circ T^k \geq \frac{\sqrt{2n}}{\pi} t\right]\right) \rightarrow \frac{2}{\pi} \int_t^\infty e^{-\frac{s^2}{\pi}} ds$$

as $n \rightarrow \infty$ for $t \geq 0$ and $f \in L^1_+$, $\int_X f dm = 1$; whence

$$\alpha_n(P, f, \theta) \sim \frac{\sqrt{2n}}{\pi} \eta(\theta) \int_X f dm$$

where

$$\frac{2}{\pi} \int_{\eta(\theta)}^\infty e^{-\frac{s^2}{\pi}} ds = \theta.$$

A different kind of behaviour is exhibited by a conservative, ergodic, measure preserving transformation $T = (X_T, \mathcal{B}_T, m_T, T)$ which is *squashable* (see [A]) in the sense that it commutes with a non singular transformation Q which is not measure preserving).

In this case (as shown in [A3]) there is no ergodic theorem of type (1), and moreover $\frac{\alpha_n(P,f,\theta)}{\alpha_n(Q,g,\theta')} \rightarrow 0$ as $n \rightarrow \infty \forall P, Q$ m_T -absolutely continuous probabilities on $X_T, f, g \in L^1(m_T)_+, 0 < \theta' < \theta < 1$.

Suppose that $R : W \rightarrow W$ is a non-singular transformation of the probability space (W, \mathcal{B}, μ) and that

$$\frac{d\mu \circ R}{d\mu} = c^\phi$$

where $0 < c < 1$ and $\phi : W \rightarrow \mathbb{Z}$.

The Maharam \mathbb{Z} -extension of R is the skew product transformation $T : W \times \mathbb{Z} \rightarrow W \times \mathbb{Z}$ defined by $T(x, n) = (Rx, n - \phi(x))$ considered with respect to the invariant measure m_T defined by $m_T(A \times \{n\}) = \mu(A)c^n$. The Maharam \mathbb{Z} -extension of R is ergodic if, and only if R is of type III_c (see [A], [W]); and in this case it is squashable commuting with the transformation $Q(x, n) = (x, n + 1)$ (for which $m_T \circ Q = cm_T$).

In this paper we look at the 1-parameter family of Maharam \mathbb{Z} -extensions considered in [H-I-K] proving a logarithmic pointwise ergodic theorem as in [Fi] and evaluating their median sequences.

It turns out that a limiting transformation of our 1-parameter family is actually boundedly rationally ergodic with return sequence

$$a_n \asymp \frac{n}{\sqrt{\log n}}$$

This latter phenomenology was also obtained for some analogous transformations in [A-K], but by rather different methods.

1. The 1-parameter family

Let $\Omega = \{0, 1\}^{\mathbb{N}}$, and \mathcal{B} is the σ -algebra generated by cylinders. Define the adding machine $\tau : \Omega \rightarrow \Omega$ by

$$\tau(1, \dots, 1, 0, \epsilon_{n+1}, \epsilon_{n+2}, \dots) = (0, \dots, 0, 1, \epsilon_{n+1}, \epsilon_{n+2}, \dots).$$

For $p \in (0, 1)$, define a probability μ_p on Ω by

$$\mu_p([\epsilon_1, \dots, \epsilon_n]) = \prod_{k=1}^n p(\epsilon_k)$$

where $p(0) = 1 - p$ and $p(1) = p$.

It is not hard to show that $\mu_p \circ \tau \sim \mu_p$, and

$$\frac{d\mu_p \circ \tau}{d\mu_p} = \left(\frac{1-p}{p}\right)^\phi$$

where

$$\phi(x) = \sum_{n=1}^{\infty} (x_n - (\tau x)_n) = \min\{n \in \mathbb{N} : x_n = 0\} - 2.$$

This means that τ is an invertible non-singular transformation of $(\Omega, \mathcal{B}, \mu_p)$ and a measure preserving transformation of $(\Omega, \mathcal{B}, \mu_{1/2})$.

It is well known that τ is ergodic on $(\Omega, \mathcal{B}, \mu_p)$, (indeed, τ -invariant sets are tail-measurable and hence trivial by the Kolmogorov 0 – 1 law).

Set,

$$X = \Omega \times \mathbb{Z}, \quad T(x, n) = (\tau x, n - \phi(x)),$$

and, for $p \in (0, 1)$,

$$m_p(A \times \{n\}) = \mu_p(A) \left(\frac{1-p}{p}\right)^n.$$

Our 1-parameter family is $\{T_p : p \in (0, 1), 0 < p \leq \frac{1}{2}\}$ where

$$T_p := (X, \mathcal{B}, m_p, T).$$

Even though T_p is defined for $\frac{1}{2} < p < 1$, we "stop" at $p = \frac{1}{2}$ because T_p^{-1} is isomorphic with T_{1-p} by $(x, n) \leftrightarrow (\pi x, -n)$ where $(\pi x)_n := 1 - x_n$.

As above, $m_p \circ T^{-1} = m_p$ and $TQ = QT$ where $Q(x, n) = (x, n + 1)$.

It was shown in [H-I-K] (see also [A]) that T_p is ergodic $\forall p \in (0, 1)$, whence T_p , being an ergodic Maharam \mathbb{Z} -extension, is squashable for $p \neq \frac{1}{2}$.

It follows from results in [A4] (see [A]) that the representation of T_p for $p \neq \frac{1}{2}$ as a Maharam \mathbb{Z} -extension of a transformation of type $III_{\frac{p}{1-p}}$ is unique (up to isomorphism of the type $III_{\frac{p}{1-p}}$ transformation).

2. The results

Theorem 1. *For every $p \in (0, 1)$,*

$$\frac{\log S_n(f)}{\log n} \rightarrow \hat{H}(p) \text{ } m_p\text{-a.e. } \forall f \in L^1_+(m_p) \quad (2)$$

where $H(p) := -p \log p - (1 - p) \log(1 - p)$ and $\hat{H}(p) := \frac{H(p)}{\log 2}$.

Theorem 2. For $p \neq \frac{1}{2}$:

$$\alpha_n(P, f, \theta) = n^{\hat{H}(p)} e^{c_p \xi(\theta) \sqrt{\log n} (1+o(1))} \tag{3}$$

as $n \rightarrow \infty \forall P$ a m_p -absolutely continuous probability on X , $f \in L^1(m_p)_+$ and $0 < \theta < 1$ where

$$c_p = \sqrt{\frac{p(1-p)}{\log 2}} \log \frac{1-p}{p} \quad \text{and} \quad \int_{\xi(\theta)}^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt = \theta;$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{S_n(f)}{n^{\hat{H}(p)} e^{t \sqrt{\log n \log^{(3)} n}}} = \begin{cases} 0 & t > -c_p \\ \infty & t < -c_p \end{cases}$$

and

$$\overline{\lim}_{n \rightarrow \infty} \frac{S_n(f)}{n^{\hat{H}(p)} e^{t \sqrt{\log n \log^{(3)} n}}} = \begin{cases} 0 & t > c_p \\ \infty & t < c_p \end{cases} \tag{4}$$

a.e. $\forall f \in L^1(m_p)_+$ where $\log^{(3)} n := \log \log \log n$.

Theorem 3. For $p = \frac{1}{2}$, T is boundedly rationally ergodic, and

$$a_n(T_{\frac{1}{2}}) \asymp \frac{n}{\sqrt{\log n}}.$$

3. The Main Lemma

For $x = (x_1, x_2, \dots) \in \Omega$, and $n \in \mathbb{N}$, let

$$\rho_n(x) = \min\{1 \leq r \leq n : x_{n-r} = 0\}, \quad \sigma_n(x) = \min\{s \geq 1 : x_{n+s} = 0\},$$

$$s_n(x) = \sum_{k=1}^n x_k, \quad p_n = \frac{s_n}{n}, \quad N_n(x) = S_{2n}(1_{\Omega \times \{0\}})(x, 0).$$

Note that

$$s_n \sim np, \quad \& \quad \limsup_{n \rightarrow \infty} \frac{\rho_n}{\log n} = \limsup_{n \rightarrow \infty} \frac{\sigma_n}{\log n} = \frac{1}{\log \frac{1}{p}} \quad \mu_p - \text{a.e.}$$

Main Lemma.

$$N_n(x) = \Phi_n(x) \binom{n}{s_n(x)}$$

where

$$|\log \Phi_n| = O(\log n) \quad \mu_p - a.e.,$$

and

$$\forall \epsilon > 0 \exists M = M_\epsilon, n_\epsilon \ni \mu_p(|\log \Phi_n| \geq M) \leq \epsilon \quad \forall n \geq n_\epsilon.$$

Sublemma 1.

$$\begin{aligned} & \binom{n - \rho_n(x) - 1}{s_{n-\rho_n(x)-1}(x) - 1} \leq N_n(x) \leq \\ & \leq \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1}. \end{aligned}$$

Proof. We first establish the lower bound. Letting

$$k_n(x) = 2^{n-\rho_n(x)} - \sum_{k=1}^{n-\rho_n(x)} 2^{k-1} x_k,$$

we see that

$$(\tau^{k_n(x)} x)_j = \begin{cases} 0 & 1 \leq j \leq n - \rho_n(x) - 1, \\ 1 & n - \rho_n(x) \leq j \leq n + \sigma_n(x) - 1, \\ x_k & \text{else.} \end{cases}$$

It follows that

$$\begin{aligned} N_n(x) & \geq \# \left\{ k_n(x) \leq j \leq k_n(x) + 2^{n-\rho_n(x)-1} - 1 : \sum_{t=1}^{\infty} ((\tau^j x)_t - x_t) = 0 \right\} = \\ & = \# \left\{ (\epsilon_1, \dots, \epsilon_{n-\rho_n(x)-1}) \in \{0, 1\}^{n-\rho_n(x)-1} : \sum_{k=1}^{n-\rho_n(x)-1} \epsilon_k = s_{n-\rho_n(x)-1}(x) - 1 \right\} \\ & = \binom{n - \rho_n(x) - 1}{s_{n-\rho_n(x)-1}(x) - 1}. \end{aligned}$$

To check the upper bound, set $K_n(x) = k_n(x) + 2^{n-\rho_n(x)-1}$, and note

that

$$\begin{aligned}
 N_n(x) &= \#\{0 \leq j \leq K_n(x) - 1 : \phi_j(x) = 0\} + \\
 &\quad + \#\{K_n(x) \leq j \leq 2^n - 1 : \phi_j(x) = 0\} \\
 &\leq \#\{\underline{\epsilon} \in \{0, 1\}^{n-\rho_n(x)} : s_{n-\rho_n(x)}(\underline{\epsilon}) = s_{n-\rho_n(x)}(x)\} \\
 &\quad + \#\{\underline{\epsilon} \in \{0, 1\}^n : s_n(\underline{\epsilon}) = s_n(x) + \rho_n(x) + \sigma_n(x) - 1\} \\
 &= \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1}.
 \end{aligned}$$

□

Sublemma 2. *Suppose that $0 \leq k \leq n$, and $0 \leq k + b \leq n + a$, then*

$$\begin{aligned}
 &\left| \log \binom{n+a}{k+b} - \log \binom{n}{k} \right| \leq \\
 &\leq (|a| + |b|) \left(\left| \log \left(p - \frac{|a| + |b|}{n} \right) \right| + \left| \log \left(1 - p - \frac{|a| + |b|}{n} \right) \right| \right)
 \end{aligned}$$

where $p := \frac{k}{n}$.

The proof of sublemma 2 is straightforward, and is left to the reader.

Proof of the main lemma. Define Φ_n by

$$N_n = \Phi_n \binom{n}{s_n}.$$

By sublemma 1,

$$N_n \geq \binom{n - \rho_n - 1}{s_{n-\rho_n-1} - 1}$$

and by sublemma 2,

$$\binom{n - \rho_n - 1}{s_{n-\rho_n-1} - 1} \geq \left[\left(p_n - \frac{a_n + b_n}{n} \right) \left(1 - p_n - \frac{a_n + b_n}{n} \right) \right]^{a_n + b_n} \binom{n}{s_n}$$

where $a_n = \rho_n + 1$, and $b_n = s_n - s_{n-\rho_n-1} + 1 \leq \rho_n + 2$, whence

$$\Phi_n \geq \left[\left(p_n - \frac{2\rho_n + 3}{n} \right) \left(1 - p_n - \frac{2\rho_n + 3}{n} \right) \right]^{2\rho_n + 3} \tag{5}$$

Again by sublemma 1,

$$N_n(x) \leq \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1},$$

and again by sublemma 2,

$$\binom{n - \rho_n}{s_{n-\rho_n}} \leq \left[\frac{1}{(p_n - \frac{a_n+b_n}{n})(1 - p_n - \frac{a_n+b_n}{n})} \right]^{a_n+b_n} \binom{n}{s_n}$$

where $a_n = \rho_n$, and $b_n = s_n - s_{n-\rho_n} \leq \rho_n$,

$$\binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1} \leq \left[\frac{1}{(p_n - \frac{b_n}{n})(1 - p_n - \frac{b_n}{n})} \right]^{b_n} \binom{n}{s_n}$$

where $b_n = \sigma_n + \rho_n$, and it follows that

$$\Phi_n \leq 2 \left[\frac{1}{(p_n - \frac{2(\rho_n+\sigma_n)}{n})(1 - p_n - \frac{2(\rho_n+\sigma_n)}{n})} \right]^{2(\rho_n+\sigma_n)}. \tag{6}$$

It follows from (5) and (6) that

$$\begin{aligned} & |\log \Phi_n| \leq \\ & \leq (2(\rho_n + \sigma_n) + 3) \left| \log \left((p_n - \frac{2(\rho_n + \sigma_n) + 3}{n})(1 - p_n - \frac{2(\rho_n + \sigma_n) + 3}{n}) \right) \right|. \end{aligned}$$

By the SLLN, μ_p -a.s.,

$$(p_n - \frac{2(\rho_n + \sigma_n) + 3}{n})(1 - p_n - \frac{2(\rho_n + \sigma_n) + 3}{n}) \rightarrow p(1 - p),$$

also,

$$(\rho_n + \sigma_n) = O(\log n),$$

whence

$$|\log \Phi_n| = O(\log n).$$

Also, given $\epsilon > 0$, if $K = 2|\log p(1 - p)|$, and $p^{L-2} < \frac{\epsilon}{4}$, then,

$$\mu_p([2(\rho_n + \sigma_n) + 3 \geq 2L + 3]) \leq \mu_p([\rho_n \geq L]) + \mu_p([\sigma_n \geq L]) \leq 2p^{L-2} < \frac{\epsilon}{2},$$

and by the WLLN, for n large enough,

$$\mu_p \left(\left[\log \left((p_n - \frac{2(\rho_n + \sigma_n) + 3}{n})(1 - p_n - \frac{2(\rho_n + \sigma_n) + 3}{n}) \right) \geq K \right] \right) < \frac{\epsilon}{2}.$$

It follows that, for n large enough, $\mu_p(|\log \Phi_n| \geq K^{2L+3}) < \epsilon$. □

4. Proofs of the Results

By Stirling’s formula, and the SLLN, we have that

$$\binom{n}{s_n} \sim \frac{C_p}{\sqrt{n}} \frac{1}{p_n^{np_n}(1-p_n)^{n(1-p_n)}} = \frac{C_p}{\sqrt{n}} e^{nH(p_n)} \quad \mu_p - \text{ a.e. as } n \rightarrow \infty,$$

where

$$C_p = \frac{1}{\sqrt{2\pi p(1-p)}}, \text{ and } H(p) = -p \log p - (1-p) \log(1-p).$$

Combining this with the main lemma, we obtain that

$$N_n = \Psi_n \frac{e^{nH(p_n)}}{\sqrt{n}}, \tag{*}$$

where

$$|\log \Psi_n| = O(\log n) \quad \mu_p - \text{ a.e.,}$$

and

$$\forall \epsilon > 0 \exists M = M_\epsilon, n_\epsilon \ni \mu_p(|\log \Psi_n| \geq M) \leq \epsilon \quad \forall n \geq n_\epsilon.$$

Proof of theorem 1. It follows from (*) that

$$\frac{\log_2 N_n}{n} = H(p_n) + O(1) \rightarrow H(p) \text{ a.s. as } n \rightarrow \infty,$$

whence, since $N_n = S_{2n}(1_\Omega)$,

$$\frac{\log S_n(1_\Omega)}{\log n} \rightarrow H(p) \text{ a.s. as } n \rightarrow \infty,$$

and theorem 1 follows from the ratio ergodic theorem. □

The other results are established by considering the Taylor expansion of H around p , and the asymptotic behaviour of $p_n - p$ as $n \rightarrow \infty$.

Let

$$s_n^* = s_n^{*,p} = \frac{s_n - np}{\sqrt{p(1-p)n}},$$

then

$$p_n - p = \sqrt{p(1-p)} \frac{s_n^*}{\sqrt{n}}.$$

By the central limit theorem (CLT),

$$\mu_p([s_n^* \geq \xi(\theta)]) \rightarrow \theta \quad \forall 0 < \theta < 1,$$

and by the law of the iterated logarithm (LIL)

$$\lim_{n \rightarrow \infty} \frac{s_n^*}{\sqrt{\log^{(2)} n}} = -1, \quad \overline{\lim}_{n \rightarrow \infty} \frac{s_n^*}{\sqrt{\log^{(2)} n}} = 1 \quad \mu_p \text{ - a.e..}$$

Expanding H around p , we obtain that

$$\begin{aligned} H(p_n) &= H(p) + (p_n - p)H'(p) + \frac{(p_n - p)^2 H''(y)}{2} \\ &\quad \text{for some } y \text{ between } p \text{ and } p_n; \\ &= H(p) + \log \frac{1-p}{p} \sqrt{p(1-p)} \frac{s_n^*}{\sqrt{n}} - \frac{p(1-p)}{2y(1-y)} \frac{s_n^{*2}}{n}. \end{aligned}$$

Proof of theorem 2. It follows from the Taylor expansion of H around p , (*) and LIL that

$$\begin{aligned} \log N_n &= nH(p_n) + O(\log n) \\ &= nH(p) + \log \frac{1-p}{p} \sqrt{p(1-p)} n s_n^* + O(\log n). \end{aligned} \tag{\dagger}$$

From (\dagger) and the CLT, we obtain that

$$\alpha_{2n}(m_p|_{\Omega \times \{0\}}, 1_{\Omega \times \{0\}}, \theta) = e^{nH(p) + c_p \sqrt{n} \xi(\theta)(1+o(1))}$$

as $n \rightarrow \infty$, whence

$$\alpha_n((m_p|_{\Omega \times \{0\}}, 1_{\Omega \times \{0\}}, \theta) = n^{\hat{H}(p)} e^{c_p \xi(\theta) \sqrt{\log n}(1+o(1))}.$$

and (3) follows from lemma 1 of [A3].

To establish (4), choose $t \in \mathbb{R}$ and note that by (\dagger),

$$R(n, t) := \frac{N_n}{e^{nH(p) + t \sqrt{n \log^{(2)} n}}} = e^{\sqrt{n}(c_p s_n^* - t \sqrt{\log^{(2)} n}) + O(\log n)}$$

μ_p -a.e. as $n \rightarrow \infty$.

It now follows from LIL that

$$\lim_{n \rightarrow \infty} R(n, t) = \begin{cases} 0 & t > -c_p \\ \infty & t < -c_p \end{cases}, \quad \& \quad \overline{\lim}_{n \rightarrow \infty} R(n, t) = \begin{cases} 0 & t > c_p \\ \infty & t < c_p \end{cases}$$

Statement (4) follows from this and the ratio ergodic theorem. □

Proof of theorem 3. The proof of theorem 3 is slightly different.

To prove bounded rational ergodicity, we show that $\exists M > 0$ such that

$$S_n(1_{\Omega \times \{0\}}) \leq M \int_{\Omega \times \{0\}} S_n(1_{\Omega \times \{0\}}) dm_{\frac{1}{2}}$$

for $n \geq 1$ and to obtain the return sequence, we show that

$$\int_{\Omega \times \{0\}} S_n(1_{\Omega \times \{0\}}) dm_{\frac{1}{2}} \asymp \frac{n}{\sqrt{\log n}}.$$

These follow from

$$N_n \leq 2 \binom{n}{\lfloor \frac{n}{2} \rfloor} \asymp \frac{2^n}{\sqrt{n}} \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\sqrt{n} E(N_n)}{2^n} > 0,$$

which latter we prove.

By sublemma 1,

$$\begin{aligned} N_n(x) &\leq \binom{n - \rho_n(x)}{s_{n-\rho_n(x)}(x)} + \binom{n}{s_n(x) + \rho_n(x) + \sigma_n(x) - 1} \\ &\leq \binom{n - \rho_n(x)}{\lfloor \frac{n - \rho_n(x)}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &\leq 2 \binom{n}{\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

To conclude, by (*) and the Taylor expansion of H around $\frac{1}{2}$,

$$\begin{aligned} \log N_n &= nH(p_n) - \log \sqrt{n} + \log \Psi_n \\ &= n \log 2 - \log \sqrt{n} + \log \Psi_n - s_n^{*2} + o\left(\frac{s_n^{*3}}{\sqrt{n}}\right), \end{aligned}$$

whence $\liminf_{n \rightarrow \infty} \frac{\sqrt{n} E(N_n)}{2^n} > 0$. □

We conclude with the remark that there is no sequence of constants $a_n \rightarrow \infty$ such that

$$\frac{T_{\frac{1}{2}}(f)}{a_n}$$

converges in measure on sets of finite measure. If there were such a sequence, then for some $n_k \rightarrow \infty$,

$$a_{2^{n_k}} \propto \frac{2^{n_k}}{\sqrt{n_k}}$$

and

$$\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k}$$

would converge in probability to a constant.

However

$$\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k} = \log \Psi_{n_k} - s_{n_k}^{*2} + o\left(\frac{s_{n_k}^{*3}}{\sqrt{n_k}}\right)$$

whence by CLT,

$$\lim_{k \rightarrow \infty} \mu_{\frac{1}{2}}([\log N_{n_k} - n_k \log 2 + \log \sqrt{n_k} < -M]) > 0 \quad \forall M > 0.$$

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