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Shift Endomorphisms and Compact Lie Extensions

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—Dedicated to Ricardo Mañé

Abstract. We consider skew-products with an arbitrary compact Lie group, when the base map is a one-sided shift of finite type endowed with an equilibrium state of a Hölder continuous function. First we show that the weak-mixing property of the skew-product implies exactness and exponential mixing. Then we address the problem of classification under measure-theoretic isomorphisms. We show that for a generic set of equilibrium states the isomorphism class of the skew-products corresponds essentially to the cohomology classes of the defining skewing function and the isomorphism is essentially a homeomorphism.

Keywords: Exactness, Group Extensions, Equilibrium states, Shifts of finite type.

Introduction

For many problems associated with endomorphisms in Ergodic Theory it is appropriate to consider natural extensions and then to invoke, or prove, results for automorphisms. This is valid, for example, when considering certain ergodic or mixing properties or when considering certain entropy problems. However, this is not the case for, say, exactness, nor for classification theory. Endomorphism problems are not always reducible (extendable) to automorphism problems.

In this paper we restrict our attention to certain endomorphisms and consider their skew-products with compact Lie groups. Specifically, our endomorphisms will be one-sided aperiodic shifts of finite type equipped

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with an equilibrium state given by a Hölder continuous function. Such shifts are known to be exponentially mixing (i.e. there is exponential decay of correlations for appropriate functions) and exact [**9**]. Moreover, generically, these shifts can be classified by a countable number of invariants (multivariate characteristic functions) and two such shifts are measure-theoretically isomorphic if and only if the isomorphism (which is unique) is essentially a homeomorphism [**4**].

We consider these properties for skew-products of one-sided shifts with compact Lie groups and show that weak-mixing implies exponential mixing and exactness (see Siboni [**8**] for a special case). We then take up the problem of classification. We restrict our attention to shifts with a generic equilibrium state and associated (ergodic) skew-products with the same compact Lie group and clarify the problem of finding isomorphisms when they exist. Our main result is that, again, such isomorphisms are essentially homeomorphisms and are given as skew-products. This is a kind of rigidity result and is based on the 'super rigidity' of the shifts we consider and on the 'smoothing' of the solution to a Livsic type problem for certain functions appearing in a cocycle equation. This Livsic result will appear in a separate paper by the second author and M. Pollicott [**5**].

1. Preliminaries

Let A be an aperiodic $k \times k$ 0-1 matrix and define

$$X \;=\; \{x \in \prod_{n=0}^{\infty} \{1, \cdots, k\}\colon\ A(x_n, x_{n+1}) = 1,\ \text{for all}\ n = 0, 1, \cdots\}\,.$$

With respect to the Tychonov product topology, X is a zero dimensional, compact, metrisable space. The shift transformation σ given by $(\sigma x)_n = x_{n+1}$, for all $n = 0, 1, \cdots$ is a continuous surjective map of X onto itself, and is called a (one-sided) *shift of finite type*.

If $w\colon X \to \mathbb{C}^d$ is continuous and $\mathrm{var}_n w/\theta^n$ is a bounded sequence (for a fixed $0 < \theta < 1$) we define $|w|_\theta$ to be the least such bound. Here $\mathrm{var}_n w \;=\; \sup\{|w(x) - w(y)|\colon\ x_i = y_i\,, i \le n\}$, where $|\cdot|$ denotes Euclidean norm. We denote by $F_\theta = F_\theta(\mathbb{C}^d)$ the space of continuous functions w

with $|w|_\theta < \infty$ and equip F_θ with the norm $||w||_\theta = |w|_\theta + |w|_\infty$ ($|\cdot|_\infty$ is the supremum norm) making F_θ into a Banach space.

If $g \in F_\theta(\mathbb{R})$ satisfies $\sum_{\sigma y = x} e^{g(y)} = 1$ for all $x \in X$, we say that g is *normalised*. For such g we define the Ruelle operator $L: F_\theta(\mathbb{C}^d) \to F_\theta(\mathbb{C}^d)$ given by $(Lw)(x) = \sum_{\sigma y = x} e^{g(y)} w(y)$. The dependence of L on g is suppressed since g will usually be fixed in the discussion. For a normalised $g \in F_\theta(\mathbb{R})$ there is a unique σ-invariant probability m such that

$$0 = h(m) + \int g\, dm \geq h(\mu) + \int g\, d\mu$$

for all other σ-invariant probabilities μ. (Here, h denotes entropy.) Such a measure m is called the *equilibrium state* defined by g and we refer to m as an F_θ equilibrium state.

If $G \subseteq \mathcal{U}(d)$ is a compact Lie group and $f: X \to G$ is continuous we define $\mathrm{var}_n f = \sup\{|f(x) - f(y)|: x_i = y_i,\ i \leq n\}$, (where $|\cdot|$ denotes the Euclidean operator norm) and write $f \in F_\theta(G)$ if $\mathrm{var}_n f/\theta^n$ is a bounded sequence (for a fixed $0 < \theta < 1$). For a given normalised $g \in F_\theta(\mathbb{R})$ and $f \in F_\theta(G)$ we define the operator $L_f: F_\theta(\mathbb{C}^d) \to F_\theta(\mathbb{C}^d)$ by $L_f w = L(fw)$.

Let σ be a one-sided shift of finite type endowed with an F_θ equilibrium state m and let $f \in F_\theta(G)$. The *skew-product* transformation σ_f of $X \times G$ onto itself is defined as $\sigma_f(x, y) = (\sigma x, f(x)y)$. We note that σ_f preserves the measure $m \times m_G$, where m_G is the (normalised) Haar measure on G.

2. Exponential decay of correlations

Throughout we shall be concerned with an aperiodic shift of finite type $\sigma: X \to X$, a function $f \in F_\theta(G)$ and an F_θ equilibrium state m. Since the equilibrium state will be fixed, the corresponding Ruelle operator on $F_\theta(\mathbb{C}^d)$ will be understood, as will the operators $L_{R(f)}: F_\theta(\mathbb{C}^d) \to F_\theta(\mathbb{C}^d)$, for d dimensional unitary representations R.

Proposition 1. *If G is a compact Lie group and $f \in F_\theta(G)$ is such that $\sigma_f(x, y) = (\sigma x, f(x)y)$ is weak-mixing on $X \times G$ then the spectral* *radius of $L_{R(f)}$ is strictly smaller than 1 for each non-trivial irreducible representation R.*

Proof. One can show that the *essential* spectral radius of $L_{R(f)}$ is θ. (Cf. [**6**] for the main argument which generalises to our situation.) This means that the spectrum outside a disc of radius $\theta' > \theta$ is associated with a finite number of eigenvalues and the corresponding eigenspaces have finite dimension. Hence if the spectral radius of $L_{R(f)}$ is 1 (it cannot be larger since g is normalised) there must be an eigenvalue of modulus 1, i.e.

$$L_{R(f)}w \;=\; \alpha w \;, \quad |\alpha| = 1 \;.$$

Using the unitary character of R and the convexity properties of L (again since g is normalised) this equation can be rewritten as

$$w \circ \sigma \;=\; \overline{\alpha} R(f) w \;.$$

If we define $F(x,y) = R(y^{-1})w(x)$ we see that

$$\begin{aligned} F \circ \sigma_f(x,y) &= R(y^{-1})R(f(x))^{-1}w(\sigma x) \;=\; \overline{\alpha}R(y^{-1})w(x) \\ &= \overline{\alpha}F(x,y) \;. \end{aligned}$$

Since R is non-trivial this equation contradicts the weak-mixing hypothesis. Thus we see that the spectral radius of $L_{R(f)}$ is strictly less than 1.

From this we are able to deduce

Proposition 2. *The autocorrelations of functions of the form $R(y^{-1})$ $w(x)$, for $w \in F_\theta(\mathbb{C}^d)$ and R an irreducible representation on $\mathbb{C}^d$, converge to zero exponentially fast.*

Proof. We have to prove that

$$\int \langle F \circ \sigma_f^n, F \rangle \, dm \times m_G \;\longrightarrow\; 0$$

exponentially fast as $n \to \infty$, where $F(x,y) = R(y^{-1})w(x)$. This integral equals

$$\int \langle R(f^n(x))^{-1}w, w \rangle \, dm \;,$$

where $f^n(x) = f(\sigma^{n-1}x) \cdots f(\sigma x) f(x)$, and this in turn equals

$$\int \langle w, L^n_{R(f)} w \rangle \, dm \;.$$

Since the spectral radius of $L_{R(f)}$ is strictly less than 1, the result follows. □

Remark. When G is abelian the result shows that autocorrelations of functions of the form $w(x)\chi(y)$ for $w \in F_\theta(\mathbb{C})$ and χ a character in G, converge to zero exponentially fast.

3. Exactness

Two sided aperiodic shifts of finite type are known to be Bernoulli with respect to any F_θ equilibrium state [**1**]. Combining this with a general result of Rudolph's [**7**], it follows that weak-mixing compact group extensions of such shifts are also Bernoulli. However, this does not imply (a priori) that the same is true for one-sided shifts. Nevertheless we are able to prove

Theorem 3. *Let G be a compact Lie group and $f \in F_\theta(G)$. If σ_f is weak-mixing with respect to $m \times m_G$ where m is an F_θ equilibrium state on X, then σ_f is exact.*

Proof. Let $\mathcal{B} = \mathcal{B}(X) \times \mathcal{B}(G)$ and let $\mathcal{B}_\infty = \cap_n \sigma_f^{-n}\mathcal{B}$. Then we have to prove that $\mathcal{B}_\infty$ consists of sets of measure zero or one. We consider the action of G on $X \times G$ given by $g\colon (x,y) \to (x,y)g = (x,yg)$ which commutes with σ_f. The induced actions of G and of σ_f on $L^2(\mathcal{B})$ commute and this implies that G acts on each of $\sigma_f^n L^2(\mathcal{B}) = L^2(\sigma_f^{-n}\mathcal{B})$ and hence on $L^2(\mathcal{B})$. Since G is compact the Hilbert space $L^2(\mathcal{B})$ decomposes into a direct sum of finite dimensional subspaces V_R, each preserved by the action of G [**3**]. Here R is an irreducible representation and for an orthonormal basis $w_1, \cdots, w_d$ of V_R we have

$$w(x,yg) = R(g^{-1})w(x,y) ,$$

where w is the column vector col. $(w_1, \cdots, w_d)$. Let V denote the inner product space of all $\mathbb{C}^d$ valued square integrable functions w defined on $X \times G$ which are $\mathcal{B}_\infty$ measurable and satisfying the above equation. Note that for $w, v \in V$ we have (with respect to the Euclidean inner product) $\langle w, v\rangle$ is G-invariant and therefore it is measurable with respect to $\mathcal{B}_\infty \cap (\mathcal{B}(X) \times \mathcal{N})$ where $\mathcal{N}$ is the trivial σ-algebra of G. This intersection σ-algebra is easily shown to be $\cap_n \sigma^{-n}\mathcal{B}(X) \times \mathcal{N}$, which is trivial since, as we have said, σ is exact. Thus the function $\langle w, v\rangle$ is constant a.e. and equals $\int \langle w, v\rangle\, dm \times m_G = \langle\langle w, v\rangle\rangle$, the inner product of w, v in V. If we choose n vectors $v_1, \cdots, v_n$ in V we therefore have $\langle\langle v_i, v_j\rangle\rangle = \langle v_i(x_0, y_0), v_j(x_0, y_0)\rangle$ $i, j = 1, \cdots, n$ where x_0, y_0 are suitably chosen. Hence there is an isometry of the span of $v_1, \cdots, v_n$ into $\mathbb{C}^d$. Hence $n \leq d$ showing that V is at most d dimensional.

Returning to our original $w = \text{col.}\,(w_1, \cdots, w_d)$ we note that $w, w{\circ}\sigma_f$, $w{\circ}\sigma_f^2, \cdots \in V$, a finite dimensional space. Since σ_f is weak-mixing, by Lemma 4 below, this implies that w is constant a.e. and $d = 1$. Thus V_R contains only the constant functions and the same is true of $L^2(\mathcal{B}_\infty)$, i.e. $\mathcal{B}_\infty$ consists only of sets of measure zero or one. □

Lemma 4. *If σ is weak-mixing and w, $w{\circ}\sigma$, $w{\circ}\sigma^2, \cdots \in V$, where V is a finite dimensional vector space, then w is constant.*

Proof. By subtracting integrals there is no loss in assuming $\int w\, dm = 0$ and showing $w \equiv 0$ in this case. There is a largest n such that $w, w{\circ}\sigma, \cdots, w{\circ}\sigma^{n-1}$ are linearly independent. Hence there are constants a_i such that

$$w{\circ}\sigma^n = a_0 w + \cdots + a_{n-1} w{\circ}\sigma^{n-1} ,$$

and for some matrix A we have

$$v{\circ}\sigma = Av ,$$

where v is the column vector $\text{col.}\,(w, \cdots, w{\circ}\sigma^{n-1})$. Let λ be a non-zero eigenvalue of A with left eigenvector ξ, i.e. $\xi A = \lambda\xi$. Then

$$\langle \xi, v{\circ}\sigma\rangle = \lambda\langle \xi, v\rangle ,$$

and, by weak-mixing of σ and integration with respect to m, we conclude that $\lambda = 1$ and $\langle \xi, v\rangle = 0$. The latter shows that there are complex numbers ξ_i (not all zero) such that

$$\xi_0 w + \cdots + \xi_{n-1} w{\circ}\sigma^{n-1} = 0 ,$$

and this contradicts the fact that $w, \cdots, w{\circ}\sigma^{n-1}$ are l.i. Hence all eigenvalues of A must be zero, and there exists k such that $A^k \equiv 0$. Therefore $v{\circ}\sigma^k = A^k v = 0$ and since σ is onto it follows that $v \equiv 0$. □

4. Classification

In this section we are concerned with the measure-theoretic classification of skew-products σ_f where σ is a one-sided aperiodic shift of finite type and $f \in F_\theta(G)$, G a compact Lie group. The underlying probability measure, preserved by σ_f, will be taken to be $m \times m_G$ where m is the equilibrium state of a normalised $g \in F_\theta(\mathbb{R})$ and m_G is the normalised Haar measure of G.

As usual, we say that two such transformations σ_f, $\sigma'_{f'}$ are (measure-theoretically) isomorphic if there is an invertible measure-preserving map φ between their respective spaces $X \times G$, $X' \times G$ such that the diagram

$$\begin{array}{ccc} X \times G & \xrightarrow{\sigma_f} & X \times G \\ \varphi\downarrow & & \downarrow\varphi \\ X' \times G & \xrightarrow{\sigma'_{f'}} & X' \times G \end{array}$$

commutes a.e.

Our aim is to clarify this diagram, under the circumstances when the functions $g \in F_\theta$, $g' \in F_\theta$ satisfy the condition that they each separate points. We say that g separates points if, when $x, y \in X$ and $x \neq y$, there exists $n \in \mathbb{N}$ such that $g(\sigma^n x) \neq g(\sigma^n y)$. This condition, though simplifying, is generic in the relative F_θ topology for normalised functions [**4**]. In fact such functions form an open dense set. Our first step is the following

Proposition 5. **([4].)** *Let σ, σ' be two one-sided aperiodic shifts of finite type with equilibrium states m, m' corresponding, respectively, to the normalised F_θ functions g, g'. If g, g' separate points and if φ is a (measure-theoretic) isomorphism between σ and σ' then there exists a measure-preserving homeomorphism φ' such that $\varphi = \varphi'$ (a.e.). Moreover, φ' is unique.*

The proof of this result is based on the fact that the information functions for σ, σ' are related by

$$I(\mathcal{B}(X)|\sigma^{-1}\mathcal{B}(X)) = I(\mathcal{B}(X')|\sigma^{-1}\mathcal{B}(X'))\circ\varphi \quad \text{a.e.}$$

which simplifies to

$$g(x) = g'(\varphi(x))$$

and therefore

$$g(\sigma^n(x)) = g'((\sigma')^n(\varphi(x))) , \qquad n = 0, 1, \cdots$$

It is at this point that the separation condition is invoked to produce a unique homeomorphism $\varphi' = \varphi$ (a.e.).

If φ is an isomorphism between σ_f and $\sigma'_{f'}$ then exactly the same conclusion is reached, namely

$$g(\sigma_f^n(x, y)) = g'((\sigma'_{f'})^n(\varphi(x, y))) \qquad n = 0, 1, \cdots$$

where we interpret $g(x, y) = g(x)$ and $g'(x, y) = g'(x)$. Thus $g(\sigma^n x) = g'((\sigma')^n\varphi_1(x, y))$ where $\varphi(x, y) = (\varphi_1(x, y), \varphi_2(x, y))$. We conclude from this that φ_1 is independent of the second variable, i.e. $\varphi_1(x, y) = \varphi_1(x)$. In short we have

Proposition 6. *Let σ, σ' have equilibrium states m, m' corresponding to normalised F_θ functions g, g' which separate points, and let φ be a (measure-theoretic) isomorphism between σ_f and $\sigma'_{f'}$. Then φ has the form*

$$\varphi(x, y) = (\varphi_1(x), \varphi_2(x, y)) ,$$

where φ_1 is a measure-preserving homeomorphism.

We shall use this proposition in combination with the following to prove our main result.

Proposition 7. **([5].)** *Let σ have an equilibrium state m corresponding to a normalised F_θ function and suppose $f, f' \in F_\theta(G)$. If there exists a measurable $h: X \to G$ such that*

$$f = (h\circ\sigma)^{-1}.f'.h \quad a.e.$$

then there exists $h' \in F_\theta(G)$ such that $h = h'$ a.e. and

$$f(x) = h'(\sigma x)^{-1}.f'(x).h'(x)$$

everywhere.

Proposition 8. *If* $\varphi: X \times G \to G$ *is measurable and*

$$\varphi(\sigma x, f(x)y) = f'(x)\varphi(x, y) \quad a.e.$$

where $f, f' \in F_\theta(G)$. *Then* φ *has the form*

$$\varphi(x, y) = h(x)\alpha(y)$$

where $h: X \to G$ *is measurable and* α *is an automorphism of* G.

Proof. For each $g \in G$ we have

$$\varphi(\sigma x, f(x)yg) = f'(x)\varphi(x, yg) \quad \text{a.e.}$$

Inverting this equation and multiplying the original we get

$$(\varphi_g^{-1} \cdot \varphi)\circ\sigma_f = (\varphi_g^{-1} \cdot \varphi) \quad \text{a.e.}$$

where $\varphi_g(x, y) = \varphi(x, yg)$. Since σ_f is ergodic this means that $(\varphi_g^{-1} \cdot \varphi)$ is a constant depending on g, i.e.

$$\varphi(x, yg) = \varphi(x, y)\alpha(g) \quad \text{a.e.}$$

and it is clear that α is a continuous automorphism of G. This equation holds for all $g \in G$ and almost all $(x, y) \in X \times G$. Let $\Gamma \subseteq G$ be a countable dense subgroup. For each $g \in \Gamma$ there exists a null subset $N_g \subseteq X \times G$ such that, for all $(x, y) \notin N_g$,

$$\varphi(x, yg) = \varphi(x, y)\alpha(g) .$$

Defining $N = \cup_{g\in\Gamma} N_g$ we see that for all $(x, y) \notin N$ and for all $g \in \Gamma$ we have $\varphi(x, yg) = \varphi(x, y)\alpha(g)$. By Fubini there exists $y_0 \in G$ and a null subset $M \subseteq X$ (M independent of $g \in \Gamma$) such that

$$\varphi(x, y_0 g) = \varphi(x, y_0)\alpha(g) ,$$

for all $x \notin M$ and all $g \in \Gamma$. Since α is uniformly continuous on Γ we conclude by taking limits that $\varphi(x, y_0 g) = \varphi(x, y_0)\alpha(g)$, for all $x \notin M$ and all $g \in G$. Writing $y = y_0 g$ we have

$$\varphi(x, y) = \varphi(x, y_0)\alpha(y_0^{-1})\alpha(y) .$$

Hence we define $h(x) = \varphi(x, y_0)\alpha(y_0^{-1})$. □

We now proceed to our main result:

Theorem 9. *Let σ, σ' have equilibrium states m, m' corresponding to normalised F_θ functions g, g' which separate points, and suppose φ is an isomorphism between σ_f and $\sigma'_{f'}$, where $f, f' \in F_\theta(G)$. Then $\varphi = \varphi'$ a.e. where φ' is a measure-preserving homeomorphism between $X \times G$ and $X' \times G$ of the form*

$$\varphi'(x,y) \;=\; (\varphi_1(x), h(x)\alpha(y))$$

and where φ_1 is a homeomorphism, $h \in F_\theta(G)$ and α is an automorphism of G.

Proof. We use Proposition 6 to simplify the statement. In other words we compose φ with the homeomorphism $(x,y) \mapsto (\varphi_1^{-1}(x), y)$ where φ_1 is the measure-preserving homeomorphism between σ and σ'. In this situation we have an isomorphism $\varphi(x,y) = (x, \varphi_2(x,y))$ between two skew-products σ_f, $\sigma_{f'}$ with the same base transformation σ (preserving m). We need to prove that φ_2 takes the form $\varphi_2(x,y) = h(x)\alpha(y)$ where $h \in F_\theta(G)$ and α is an automorphism. The isomorphism gives a commutative diagram

$$\begin{array}{ccc} (x,y) & \longrightarrow & (\sigma x, f(x)y) \\ \downarrow & & \downarrow \\ (x, \varphi_2(x,y)) & \longrightarrow & (\sigma x, \varphi_2(\sigma_f(x,y))) \,. \end{array}$$

In other words

$$\varphi_2(\sigma x, f(x)y) \;=\; f'(x)\varphi_2(x,y) \,.$$

Here we use Proposition 8 to see that $\varphi_2(x,y) = h(x)\alpha(y)$ where $h: X \to G$ is measurable and α is an automorphism of G. Thus

$$h(\sigma x)\alpha(f(x))\alpha(y) \;=\; f'(x)h(x)\alpha(y) \,,$$

or equivalently

$$\alpha(f(x)) \;=\; h(\sigma x)^{-1} f'(x) h(x) \,.$$

By Proposition 7 we conclude that $h = h'$ a.e. where $h' \in F_\theta(G)$, and the theorem is proved. $\square$

5. Two examples

As an illustration of what can happen when the equilibrium state m corresponds to a normalised $g \in F_\theta$ where g does *not* separate points (an exceptional case), we take the most extreme example.

Let σ be the full two shift (i.e. $X = \prod_{n=0}^{\infty}(0,1)$) equipped with the Bernoulli $(1/2,1/2)$ measure m, which is the equilibrium state of the normalised function $g(x) = -\log(2)$. Clearly g does not separate points since it is constant.

Now let $G = N_n = N/nN = \{0,1,\cdots,n-1\}$ with addition modulo n, and define $f(x) = f(x_0)$ where $f(0) = 0$ and $f(1) = 1$. Then we have

Theorem 10. *The skew-product σ_f is measure-theoretically isomorphic to σ itself.*

Proof. Let $\mathcal{B} = \mathcal{B}(X) \times \mathcal{B}(N_n)$, then it suffices to produce a two set partition α of $X \times N_n$ with the properties

(i) $\mathcal{B} = \alpha \vee \sigma_f^{-1}\mathcal{B}$

(ii) α is independent of $\sigma_f^{-1}\mathcal{B}$ and

(iii) α is a strong-generator for σ_f.

If we have such an α then the map $\varphi(x,y) = (z_0, z_1, \cdots)$ where $\alpha = (A_0, A_1)$ and $\sigma_f^k(x,y) \in A_{z_k}$ for all k, will be an isomorphism between σ_f and σ.

Let $[i]$ denote the cylinder corresponding to the points of X with initial coordinate $x_0 = i$. Define

$$\begin{aligned} A_0 &= ([0] \times \{0\}) \cup ([1] \times \{0,1,\cdots,n-2\}) , \\ A_1 &= ([1] \times \{n-1\}) \cup ([0] \times \{1,2,\cdots,n-1\}) . \end{aligned}$$

This is illustrated in the case $n = 5$ by Figure 1 where the bold lines represent A_0 and the rest A_1.

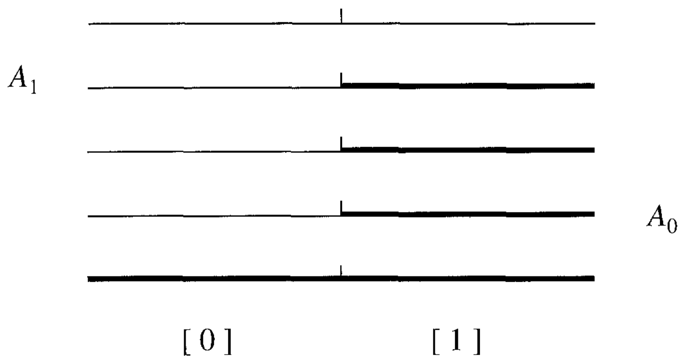

Figure 1

It is a simple matter to check that α satisfies (i) and (ii), from which it follows that $\alpha, \sigma_f^{-1}\alpha, \cdots$ are independent. To conclude the proof we have to show (iii), that α is a strong-generator. To do this it suffices to show there is a set $N \subseteq X \times N_n$ of measure zero such that if $(x,y),(x',y') \notin N$ and

$$\sigma_f^k(x,y), \sigma_f^k(x',y') \in A_{z_k}, \quad k = 0,1,\cdots$$

then $(x,y) = (x',y')$. We shall also consider the $2n$ set partition $\beta = (B_0, B_1, \cdots, B_{2n-1})$ where each B_i consists of points (x,y) all of which having the same y coordinate and the same x_0 coordinate.

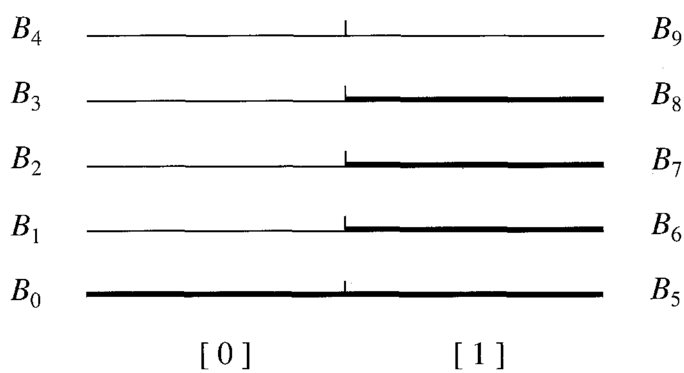

Figure 2

It is clear that $\beta \geq \alpha$. For ease of presentation we shall refer to a set of the form

$$\sigma_f^{-k} E_k \cap \sigma_f^{-(k+1)} E_{k+1} \cap \cdots \cap \sigma_f^{-\ell} E_\ell$$

as a word $(E_k, E_{k+1}, \cdots, E_\ell)$. Notice that if (x, y) begins with the word $(A_0, A_0, \cdots, A_0, A_1)$ (with n A_0's) then we know that $(x, y) \in B_0$ i.e. $x_0 = 0$, $y = 0$. And notice also that $\alpha \vee \sigma_f^{-1}\beta \geq \beta$. Therefore

$$\alpha \vee \sigma_f^{-1}\alpha \vee \sigma_f^{-2}\beta \geq \alpha \vee \sigma_f^{-1}\beta \geq \beta\,,$$

which implies

$$\alpha \vee \sigma_f^{-1}\alpha \vee \sigma_f^{-2}\beta \geq \beta \vee \sigma_f^{-1}\beta\,.$$

Iterating this inequality we get

$$\alpha \vee \sigma_f^{-1}\alpha \vee \cdots \vee \sigma_f^{-k}\alpha \vee \sigma_f^{-(k+1)}\beta \geq \beta \vee \sigma_f^{-1}\beta \vee \cdots \vee \sigma_f^{-k}\beta\,.$$

This means that if a point (x, y) has the word $(A_{z_0}, \cdots, A_{z_k}, B)$ then its β word $(B_{t_0}, B_{t_1}, \cdots, B_{t_k})$ (in the same position) is known. (Here $B \in \beta$.)

Let (x, y) be such that the word $(A_0, A_0, \cdots, A_0, A_1)$ (with n A_0's) occurs infinitely often in its α itinerary. Then, as we have said, B_0 occurs in the same position and knowing $(A_{z_0}, \cdots, A_{z_N}, A_0, A_0, \cdots, A_0, A_1)$ to be the initial α word for (x, y) implies that the initial β word for (x, y) of length $N+2$ is $(B_{t_0}, \cdots, B_{t_N}, B_0)$. Hence for such (x, y) the α itinerary for (x, y) determines the β itinerary for (x, y), i.e. (x, y) is determined. Let N be the null set of (x, y) for which $(A_0, \cdots, A_0, A_1)$ occurs only finitely often. If $(x, y) \notin N$ then the α sequence of (x, y) determines the β sequence of (x, y) and the theorem is proved. □

Remark. We note that the assertion in Theorem 10 cannot hold if m corresponds to a normalised $g \in F_\theta$ which separates points. The reason being that the property of separation of points is invariant under measure-theoretic isomorphism between the corresponding equilibrium states.

Corollary 11. *For every $n \geq 1$ there is a cyclic group N_n of invertible measure-preserving transformations commuting with the full one-sided two shift endowed with the Bernoulli* $(1/2, 1/2)$ *measure.*

This is because N_n commutes with σ_f and $\sigma_f \simeq \sigma$.

Corollary 12. *If $f(0) = 0$ and $f(1) = \ell$ (where $\gcd(\ell, n) = 1$) then $\sigma_f \simeq \sigma$.*

It is easy to show all such σ_f are mutually isomorphic using an isomorphism of the form $(x, y) \mapsto (x, \ell y)$. (Here we note that the condition $\gcd(\ell, n) = 1$ implies that $y \mapsto \ell y$ is an automorphism of N_n.)

We show next that Theorem 9 does not hold without the assumption of separation of points. Let σ_f be the skew-product in Theorem 10 and take $f'(x) = f'(x_0, x_1)$ where $f'(0,0) = f'(0,1) = f'(1,0) = 0$ and $f'(1,1) = 1$. Then $\sigma_{f'}$ is weak-mixing and it is not topologically conjugate to σ_f. The latter is because, for instance, the number of periodic points of period 2 for these maps are different. However, we have the following result:

Theorem 13. *The skew-product $\sigma_{f'}$ is measure-theoretically isomorphic to σ_f. (Therefore it is also measure-theoretically isomorphic to σ itself.)*

Proof. The strategy here is the same as in the proof of Theorem 10, i.e. we produce a two set partition α of $X \times N_n$ enjoying the properties (i), (ii), (iii) and then the corresponding map φ will be an isomorphism between $\sigma_{f'}$ and σ. Hence by Theorem 10 we obtain $\sigma_{f'} \simeq \sigma_f$.

Let $[ij]$ denote the cylinder corresponding to the points of X with initial coordinates $(x_0, x_1) = (i, j)$. Define the elements of α by

$$\begin{aligned} A_0 &= \big([00] \times \{0\}\big) \cup \big([01] \times \{0, 1, 2, \cdots, n-2\}\big) \\ &\quad \cup \big([10] \times \{n-1\}\big) \cup \big([11] \times \{0, 1, 2, \cdots, n-2\}\big)\ , \\ A_1 &= \big([00] \times \{1, 2, \cdots, n-1\}\big) \cup \big([01] \times \{n-1\}\big) \\ &\quad \cup \big([10] \times \{0, 1, 2, \cdots, n-2\}\big) \cup \big([11] \times \{n-1\}\big)\ . \end{aligned}$$

We illustrate this partition in the case $n = 5$ by Figure 3, where the bold lines represent A_0 and the rest A_1. Here again it is not difficult to check that (i) and (ii) are satisfied, and then it suffices to prove (iii).

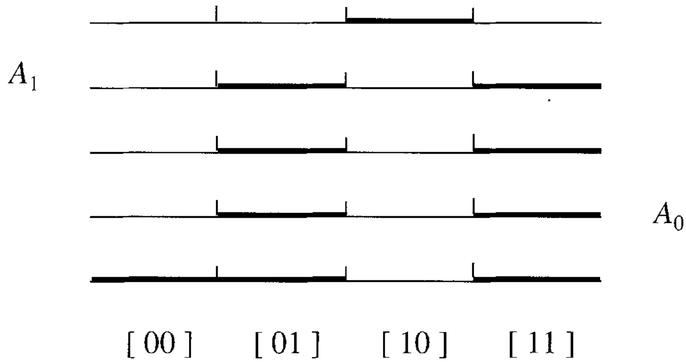

Figure 3

We consider the $4n$ set partition $\beta = \{B_{ij}\colon\ i = 0, 1,\ j = 0, 1, \cdots, 2n-1\}$ where each B_{ij} consists of points (x, y) with the same y coordinate and the same (x_0, x_1) coordinate.

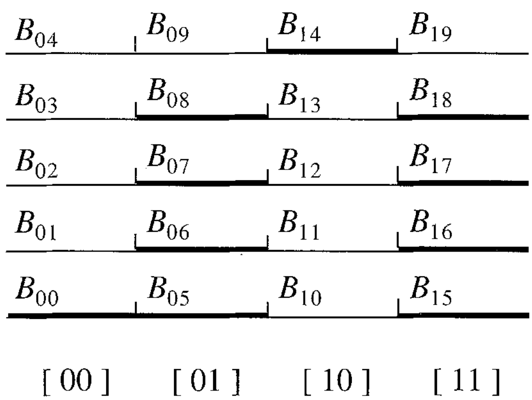

Figure 4

Then clearly $\beta \geq \alpha$ and it is not difficult to see that $\alpha \vee \sigma_{f'}^{-1}\beta \geq \beta$. From this it follows that

$$\alpha \vee \sigma_{f'}^{-1}\alpha \vee \cdots \vee \sigma_{f'}^{-k}\alpha \vee \sigma_{f'}^{-(k+1)}\beta \ \geq\ \beta \vee \sigma_{f'}^{-1}\beta \vee \cdots \vee \sigma_{f'}^{-k}\beta\,.$$

Now we note that if (x, y) has initial α word given by $w = (A_0, A_0, \cdots, A_0, A_1)$ (with $n+1$ A_0's) then necessarily $(x, y) \in B_{00}$. Therefore taking the set of (x, y) such that w appears infinitely often on its α itinerary, we conclude that the β itinerary is uniquely determined. Hence α is a strong-generator for $\sigma_{f'}$. $\square$

Problems

1. If σ is the full (one-sided) k shift with the Bernoulli $(1/k, \cdots, 1/k)$ measure and $f: \{1, 2, \cdots, k\} \to N_n$, is it true that $\sigma_f \simeq \sigma$ if σ_f is weak-mixing?

2. Can such results be achieved for $f: \{1, 2, \cdots, k\} \to [0, 1)$ (addition mod 1)? Even for $k = 2$ and $f(0) = 0$, $f(1) = \varepsilon$, ε being an irrational? This should be provable, in which case it would follow that there is a circle action commuting with σ.

3. What can be said about the centraliser of σ, i.e. the group of invertible measure-preserving transformations of the full 2-shift? What Lie groups does it contain? Which finite groups? (Compare Hedlund [**2**], for a discussion of *homeomorphisms* which commute with the *two-sided* full shift. There are only the obvious two.)

Postscript

After writing this paper we realised that our illustrations are subsumed by a much more general result of Adler, Goodwyn & Weiss (*Israel Journal of Maths* **27**, 49-63, 1977), who prove that any aperiodic shift of finite type with a constant number d of edges exiting each state is isomorphic to the full d shift (with respect to measures of maximal entropy). Although they are primarily interested in two-sided shifts, their result has the consequence that if each state has the same number d of entrances then the *one-sided* shift is isomorphic to the *one-sided* full d shift.

This leads immediately to:

Proposition. *If σ_f is a topologically mixing skew-product of the one-sided full d shift σ with a finite group, where f is a continuous function, then with respect to measures of maximal entropy σ_f is isomorphic to σ.*

This result only affects our illustrations and not the main body of our paper. It also solves problem 1 in the affirmative.

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