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Cubic Tangencies and Hyperbolic Diffeomorphisms

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— Dedicated to the memory of R. Mañé

Abstract. We show that the loss of hyperbolicity of an Anosov diffeomorphism of the torus T^2 can be produced by a cubic tangency at a heteroclinic point. Such a first bifurcation is generic for 3-parameters families of diffeomorphisms. Our construction may also be applied to any basic set Λ of a surface diffeomorphism. Moreover, if the point q of cubic tangency corresponds to a lateral point of Λ then the bifurcation is generic for two parameters. In this case the point q may be a homoclinic intersection.

Keywords: Anosov diffeomorphism, Axiom A, cubic tangency, conefields

Introduction

One of the basic problems in bifurcation theory is to understand explicitly how structurally stable systems become unstable. So far the examples for which such a loss of stability is understood fall into one of the two categories:

(1) *loss of hyperbolicity:* one of the periodic points fails to be hyperbolic (or some new nonhyperbolic periodic orbit is created) by an elementary bifurcation (*saddle-node, flip or Hopf*), and all other periodic orbits are hyperbolic and their invariant manifolds intersect transversely,

(2) *loss of transversality:* all periodic orbits are hyperbolic and their invariant manifolds meet transversely except along one orbit. Then we

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say that f *exhibits a homoclinic or heteroclinic tangency*. For surface diffeomorphisms Palis and Takens conjectured that such (nontransverse) intersection is generically quadratic and associated to a lateral (see definition below) periodic point, see [**PT**].

Related to this conjecture, our aim in this paper is to understand *bifurcations of low codimension* of Anosov diffeomorphisms of the two dimensional torus T^2, or more generally of hyperbolic sets of surface diffeomorphisms.

Since Smale ([**Sm**]) it is known that there is a submanifold of codimension one (in the space of diffeomorphisms) in the boundary of the Anosov systems consisting of diffeomorphisms with a saddle-node. To get such a submanifold one considers the splitting of a hyperbolic periodic point of an Anosov diffeomorphism into a saddle and a source *via* a generic saddle-node bifurcation. Under suitable hypotheses, after such a bifurcation one gets a hyperbolic diffeomorphism (so-called *Derived from Anosov diffeomorphism*, see [**W**]) with a source and a global transitive attractor.

Now the question of the existence of diffeomorphisms exhibiting homoclinic or heteroclinic tangencies and lying in the closure of the Anosov ones arises naturally.

Heuristically, one is led to believe that such a tangency could not be quadratic. This can be seen as follows: On one hand, Anosov diffeomorphisms have global invariant stable and unstable foliations which are transverse. On the other hand, if one deformes by an isotopy these foliations to obtain a tangency, one gets a point in which the foliations are tangent but topologicaly transverse, and so (assuming enough differentiability) the tangency is of odd order. We observe that in this paper we do not deal with the case in which the invariant foliations are not globally defined at the bifurcation.

In the case of Anosov diffeomorphisms our results can be formulated (avoiding technical details) as follows: First, denote by $\mathrm{Diff}^3(T^2)$ the space of C^3-diffeomorphisms in the two-dimensional torus T^2 endowed with the usual topology.

Theorem A. *Let f be any Anosov diffeomorphism of the torus T^2. There exists a submanifold $\mathcal{C}$ of* $\mathrm{Diff}^3(T^2)$ *of codimension 3 contained in the boundary of the Anosov diffeomorphisms such that:*

(1) *Every diffeomorphism $g \in \mathcal{C}$ has a heteroclinic cubic tangency associated to hyperbolic periodic points,*
(2) *Every 3-dimensional submanifold $\Sigma \subset \mathrm{Diff}^3(T^2)$ transverse to $\mathcal{C}$ at some g_0 contains a smooth arc $\{g_t\}_{t\in[0,1]}$ such that, for any $t > 0$, g_t is an Anosov diffeomorphism isotopic to f (within the class of the Anosov diffeomorphisms).*

We say that the submanifold $\mathcal{C}$ satisfying (2) in the theorem is *well located in the boundary of the Anosov diffeomorphisms.*

In view of the theorem, a natural question is to determine if the $g \in \mathcal{C}$ are topologically conjugate to the Anosov diffeomorphism f.

To give an intuitive explanation of why the natural codimension of the bifurcation is 3 we only need the invariant foliations induced by the dynamical systems. Given two global transverse foliations the creation of a tangency by an isotopy of the foliations is a phenomenon of codimension 1. This first tangency can not be quadratic and is generically cubic. Moreover, the tangency is *a priori* associated to any pair of leaves. In our construction the tangency is related to two prescribed leaves of the foliation: the stable and the unstable manifolds of some periodic orbits. In this way we gain two new restrictions, bringing the codimension of the bifurcation to 3.

Our construction also applies to any surface diffeomorphism f having a hyperbolic basic set Λ_f: by a generic bifurcation of codimension 3, one creates a cubic tangency between the stable and unstable manifolds of two prescribed (different) periodic orbits of Λ_f.

In [**NP**], Newhouse and Palis proved that any basic set Λ_f of a surface diffeomorphism f (except if f is Anosov) has a *lateral point.* Roughly speaking, a lateral point P is a point of the basic set Λ_f such that Λ_f accumulates on the local stable/unstable manifold of P from one side only. These lateral points are in the unstable (or stable) manifolds of finitely many periodic points.

Let q be a lateral homoclinic point and consider a deformation of f by an isotopy creating a homoclinic cubic tangency at q. Since q is lateral the points at one side (say the right) of (for example) the local unstable manifold of q ($W^u_{\text{loc}}(q)$) do not belong to the locally maximal set. Thus to control the hyperbolicity of the diffeomorphisms before the creation of the tangency it is enough to focus our attention on the points at the left of $W^u_{\text{loc}}(q)$. This allows us to perturb f to obtain a nondegenerate cubic tangency: all the turning points of the perturbed unstable foliation stay at the right of $W^u_{\text{loc}}(q)$. Thus we get a bifurcation of codimension 2.

Theorem B. *Let f be a diffeomorphism defined on a surface M and Λ_f a basic set of f different from M. There exists a submanifold $\mathcal{C}_\ell$ of* $\text{Diff}^3(M)$ *of codimension 2 in the boundary of the set of hyperbolic diffeomorphisms satisfying:*

(1) *Every diffeomorphism $g \in \mathcal{C}_\ell$ has a homoclinic cubic tangency associated to a lateral periodic point,*
(2) *The submanifold $\mathcal{C}_\ell$ is well located: every 2-dimensional submanifold $\Sigma \subset \text{Diff}^3(M)$ transverse to $\mathcal{C}_\ell$ at some g_0 contains a smooth arc $\{g_t\}_{t\in[0,1]}$ such that, for any $t > 0$, g_t is isotopic to f (within the class of hyperbolic diffeomorphisms) and has a basic set that is the continuation of Λ_f.*

Related to our results Lewowicz exhibited in [**L**] a class of transitive diffeomorphisms at the boundary of the Anosov ones, that do not satisfy the strong transversality condition: each diffeomorphism is conjugate to an Anosov one and has a nonhyperbolic fixed point P so that its stable and unstable sets exhibit a cubic tangency at P itself.

In view of all these results let us pose some questions. To state the first one let us observe that to to prove Theorem A we use estimates on the products of the eigenvalues of the periodic points P_1 and P_2 involved in the creation of the heteroclinic cycle, see Proposition 3.5 for details. Actually, each inequality is the converse of the other one, thus they do not hold when $P_1 = P_2$ (i.e. in the homoclinic case). So the first question is:

Question 1. *Is there a submanifold of codimension 3 in the closure of the Anosov diffeomorphisms consisting of diffeomorphisms with a cubic homoclinic tangency?*

In our construction an essential property of the point q of cubic tangency is that it is nonrecurrent: its ω and α-limit sets are periodic orbits. Thus a natural question is

Question 2. *Construct arcs of diffeomorphisms (of low codimension) bifurcating from Anosov systems through a tangency between invariant manifolds at some point with a dense orbit.*

Finally, there is the natural problem:

Question 3. *Is there a submanifold $\mathcal{C}$ in the boundary of the Anosov diffeomorphisms consisting of diffeomorphisms having cubic tangencies such that the codimension of $\mathcal{C}$ is (strictly) less than 3?*

We end this introduction by saying a few words about the organization of this article and the main steps of our proofs. The precise statements of results are in Section 1.

On one hand, the proofs of Theorems A and B (corresponding to the nonlateral and lateral cases, respectively) are conceptually similar. On the other hand, the Anosov case presents some extra technical difficulties. So we first give all the details of the Anosov case (see Sections 2–4). In Section 5 we briefly explain how to get advantage from the laterality in order to decrease the codimension of the bifurcation.

The structure of the proof in the Anosov case is the following. In Section 2 for each $\mu > 0$ we define a family of diffeomorphisms $\{\theta_{t,\mu}\}$ of the square $[-\mu, \mu]^2$ so that $\theta_{1,\mu}$ is the identity. Moreover, the image by $\theta_{0,\mu}$ of the vertical foliation in $[-\mu, \mu]^2$ has a unique degenerate cubic tangency with the horizontal foliation at the origin. We estimate the action of the derivative of $\theta_{t,\mu}$ on a conefield around the vertical direction. In Section 3 given a heteroclinic point q of an Anosov diffeomorphism f we perturb f to get a cubic tangency at q. More precisely, using a chart at q we consider the family of diffeomorphisms $f_{t,\mu} = f \circ \theta_{t,\mu}$. Then $f_{0,\mu}$ has a heteroclinic cubic tangency at q for each $\mu > 0$. To see that $f_{t,\mu}$ is hyperbolic for $t > 0$ we combine the hyperbolicity of f (existence of invariant conefields) with the estimates in Section 2. Finally, in Section 4 we show that the diffeomorphisms of the form $g \circ \theta_{0,\mu}$, where g is $\mathcal{C}^3$-close to f, define locally a submanifold of codimension 3 of diffeomorphisms with a cubic tangency in the boundary of the Anosov diffeomorphisms.

Finally, let us observe that simultaneously and independently H. Henrich has announced similar results for the boundary of the Anosov diffeomorphisms, see [**H**].

1. Basic definitions and statement of the results

Before presenting more precisely our results let us recall some definitions.

An f-invariant set Λ is *hyperbolic* if there exist a continuous splitting of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus E^u$, a constant λ, $0 < \lambda < 1$, and a norm $|\cdot|$, such that for every $x \in \Lambda$ there is n so that $f_*(E^s_x) \subset E^s_{f(x)}$, $(f^{-1})_*(E^u_x) \subset E^u_{f^{-1}(x)}$, $|(f^n)_*(v)| \leq \lambda |v|$ if $v \in E^s_x$, and $|(f^{-n})_*(v)| \leq \lambda |v|$ if $v \in E^u_x$, where f_* denotes the derivative of f at x.

An f-invariant (*a priori* noncompact) set Λ is *nonuniformly hyperbolic* if there are a continuous splitting of the tangent bundle over Λ, $T_\Lambda M = E^s \oplus E^u$, a constant λ, $0 < \lambda < 1$, and a norm $|\cdot|$, such that for every $x \in \Lambda$ there is $n(x)$ so that $f_*(E^s_x) \subset E^s_{f(x)}$, $(f^{-1})_*(E^u_x) \subset E^u_{f^{-1}(x)}$, $|(f^{n(x)})_*(v)| \leq \lambda |v|$ if $v \in E^s_x$, and $|(f^{-n(x)})_*(v)| \leq \lambda |v|$ if $v \in E^u_x$.

We define the *stable*, $W^s(x)$, and *unstable*, $W^u(x)$, *sets* of a point x by $W^s(x) = \{y \in M \colon d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}$ and $W^u(x) = \{y \in M \colon d(f^n(x), f^n(y)) \to 0 \text{ as } n \to -\infty\}$, where d denotes the distance induced by the norm $|\cdot|$.

A point x is *nonwandering* if for every neighbourhood $\mathcal{U}$ of x there exists $m, m \neq 0$, such that $f^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. These points form the *nonwandering set* $\Omega(f)$. We say that a compact subset Λ_f of $\Omega(f)$ is a *basic set* if it is hyperbolic, transitive (i.e. it has a dense orbit) and locally maximal (i.e. there exists a neighbourhood $\mathcal{U}$ of Λ_f such that $\Lambda_f = \bigcap_{\mathbb{Z}} f^n(\mathcal{U})$). Basic sets are *locally stable*, meaning that there is a neighbourhood $\mathcal{W}$ of Λ_f such that for every diffeomorphism g $\mathcal{C}^1$-close to f the set $\Lambda_g = \bigcap_{\mathbb{Z}} g^i(\mathcal{W})$ (called the *continuation of* Λ_f) is a basic set *conjugate to* Λ_f (i.e. there is a homeomorphism h from Λ_f to Λ_g so that $h \circ f(x) = g \circ h(x)$ for all $x \in \Lambda_f$). We say that a basic set Λ_f is nontrivial if it contains more than one orbit.

A diffeomorphism f defined on a compact manifold is *Anosov* if its nonwandering set $\Omega(f)$ is hyperbolic and coincides with the whole manifold. We observe that the only surface supporting Anosov diffeomorphisms is the torus.

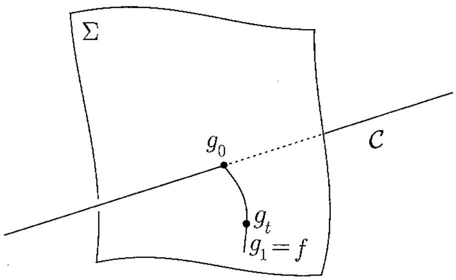

Figure 1

We are now ready to state precisely our results.

Theorem A. *Let M be compact surface and f be a C^3-diffeomorphism on M with a nontrivial basic set $\Lambda_f = \bigcap_{i\in\mathbb{Z}} f^i(\mathcal{W})$, where $\mathcal{W}$ is a neighbourhood of Λ_f. Denote by $\mathcal{H}_f(M) \subset \mathrm{Diff}^3(M)$ the arc-connected component of f in the set of diffeomorphisms g for which $\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(\mathcal{W})$ is a basic set.*

Let $q \in \Lambda_f$ be a point such that $q \in W^s(P_f, f) \pitchfork W^u(Q_f, f)$ for some hyperbolic periodic points P_f and Q_f of Λ_f with disjoint orbits.

Then there is a submanifold $\mathcal{C} \subset \mathrm{Diff}^3(M)$ of codimension 3 such that every 3-manifold $\Sigma \subset \mathrm{Diff}^3(M)$ transverse to $\mathcal{C}$ at g_0 contains an arc $\{g_t\}_{t\in[0,1]}$ with $g_1 = f$ such that

(1) *$g_t \in \mathcal{H}_f(M)$ for every $t \in]0,1]$,*

(2) *the continuations $\{q(g_t)\}_{t\in]0,1]}$ of the point q form a path in M that converges to the point $q(g_0)$. Moreover, the point $q(g_0)$ is a (cubic) nontransverse intersection between the invariants manifolds of the continuations P_{g_0} and Q_{g_0} of P_f and Q_f, respectively,*

(3) *the set $\tilde{\Lambda}_{g_0} = \bigcap_{i\in\mathbb{Z}} g_0^i(\mathcal{W} \setminus \{q\})$ is nonuniformly hyperbolic.*

Let f be a surface diffeomorphism with a nontrivial basic set Λ_f. We say that $P \in \Lambda_f$ is a *lateral point of* Λ_f if for every small enough neighbourhood $\mathcal{U}$ of P either $(\mathcal{U} \setminus W^s_{\mathrm{loc}}(P,f))$ or $(\mathcal{U} \setminus W^u_{\mathrm{loc}}(P,f))$ has a connected component that does not intersect Λ_f. If Λ_f is different from the ambient manifold the set of lateral points is nonempty. Newhouse and Palis proved that the set of lateral points of Λ_f is contained in the union of the invariant manifolds of finitely many lateral periodic points, see [**NP**].

Theorem B. *Let M be a compact surface and f be a C^3-diffeomorphism on M with a nontrivial basic set $\Lambda_f = \bigcap_{i\in\mathbb{Z}} f^i(\mathcal{W})$, where $\mathcal{W}$ is a neighbourhood of Λ_f, and Λ_f is different from M. Denote by $\mathcal{H}_f(M) \subset$ $\mathrm{Diff}^3(M)$ the arc-connected component of f in the set of diffeomorphisms g such that $\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(\mathcal{W})$ is a basic set.*

Let $q \in \Lambda_f$ be a point such that $q \in W^s(P_f,f) \pitchfork W^u(P_f,f)$ for some lateral hyperbolic periodic point P_f.

Then there is a submanifold $\mathcal{C}_\ell \subset \mathrm{Diff}^3(M)$ of codimension 2 such that every 2-manifold $\Sigma \subset \mathrm{Diff}^3(M)$ transverse to $\mathcal{C}_\ell$ at g_0 contains an arc $\{g_t\}_{t\in[0,1]}$ with $g_1 = f$ such that

(1) *$g_t \in \mathcal{H}_f(M)$ for every $t \in]0,1]$,*
(2) *the continuations $\{q(g_t)\}_{t\in]0,1]}$ of the point q form a path in M that converges to the point $q(g_0)$. Moreover, the point $q(g_0)$ is a (cubic) nontransverse intersection between the invariants manifolds of the continuation P_{g_0} of P_f,*
(3) *the set $\tilde{\Lambda}_{g_0} = \bigcap_{i\in\mathbb{Z}} g_0^i(\mathcal{W} \setminus \{q\})$ is nonuniformly hyperbolic.*

We observe that Theorem B can also be stated in the heteroclinic case, for that one takes the two points in the cycle being lateral of the same type, i.e. the stable (resp. unstable) manifold of the hyperbolic set does not accumulate on both sides of the stable (resp. unstable) manifolds of the two points in the cycle.

2. Families of local deformations

We deal with diffeomorphisms θ defined on the cube $[-1,1]^2 \times [0,1]$ verifying:

(H0) $\theta(x, y, t) = (\theta_t(x, y), t)$, where θ_t is a diffeomorphism of the square $[-1, +1]^2$ depending smoothly on t such that θ_1 is the identity. In other words, θ is an isotopy between θ_0 and the identity.

For each $t \geq 0$ let

$$(\theta_t)_*(\frac{\partial}{\partial x}) = a_t(x, y) \cdot \frac{\partial}{\partial x} + b_t(x, y) \cdot \frac{\partial}{\partial y},$$
$$(\theta_t)_*(\frac{\partial}{\partial y}) = c_t(x, y) \cdot \frac{\partial}{\partial x} + d_t(x, y) \cdot \frac{\partial}{\partial y}.$$

The diffeomorphisms θ also satifies the following hypotheses:

(H1) For every t the map θ_t is the identity on a neighbourhood (independent of t) of the boundary of the square $[-1, 1]^2$,

(H2) $\theta_0(0, 0) = (0, 0)$,

(H3) $d_0(0, 0) = 0$, for every t the function d_t is strictly positive on $([-1, 1]^2 \setminus \{(0, 0)\})$ and $\frac{\partial}{\partial t} d_t(0, 0) > 0$,

(H4) $d_0(x, y)$ is a Morse function at $(0, 0)$.

We denote by Θ the set of C^3-diffeomorphisms defined on the cube $[-1, 1]^2 \times [0, 1]$ satisfying the hypotheses (H0–H4).

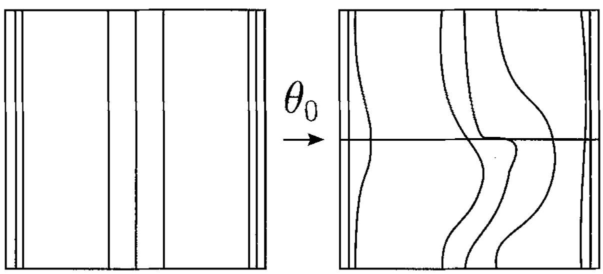

Figure 2

Lemma 2.1. *For every $\theta \in \Theta$ there are a C^3-neighbourhood $\mathcal{D}$ of θ in Θ and strictly positive constants α, β and γ such that for every $\tilde{\theta}$ in $\mathcal{D}$ and $(x, y, t) \in [-1, 1]^2 \times [0, 1]$ we have*

$$(\tilde{\theta}_t)_*(\mathcal{C}_t(x, y)) \subset \overline{\mathcal{C}}_t(\tilde{\theta}_t(x, y)),$$

where $\mathcal{C}_t$ and $\overline{\mathcal{C}}_t$ are the conefields defined on $[-1,1]^2$ *by*

$$\mathcal{C}_t(x,y) = \{(v_1,v_2)\colon |v_1| < \alpha \cdot x^2 \cdot |v_2|\},$$
$$\overline{\mathcal{C}}_t(x,y) = \{(v_1,v_2)\colon |v_1| < \frac{\beta}{t+y^2} \cdot |v_2|\}.$$

Proof. Since d_0 is a Morse function at $(0,0)$, $\frac{\partial}{\partial t} d_t(0,0) > 0$ and d_t is positive outside of $(0,0)$ (see (H3–H4)) there are strictly positive constants α, β' and γ with

$$d_t(x,y) \geq \gamma \cdot t + |b_t(x,y)| \cdot \alpha \cdot x^2 + \beta' \cdot \overline{y}^2, \quad \text{where } \theta_t(x,y) = (\overline{x},\overline{y}). \tag{2.1}$$

Take a vector (v_1,v_2) in the cone $\mathcal{C}_t(x,y)$. By definition,

$$(\theta_t)_*(v_1,v_2) = (\overline{v}_1,\overline{v}_2) = (a_t(x,y)\cdot v_1 + c_t(x,y)\cdot v_2, b_t(x,y)\cdot v_1 + d_t(x,y)\cdot v_2).$$

Since (v_1,v_2) belongs to $\mathcal{C}_t(x,y)$ one has

$$|v_1| < \alpha \cdot x^2 \cdot |v_2|. \tag{2.2}$$

Now, from (2.1–2),

$$\begin{aligned} |\overline{v}_2| &= |b_t(x,y)\cdot v_1 + d_t(x,y)\cdot v_2| \\ &\geq |d_t(x,y)| \cdot |v_2| - |b_t(x,y)| \cdot |v_1| \geq \\ &\geq (|d_t(x,y)| - |b_t(x,y)| \cdot \alpha \cdot x^2) \cdot |v_2| \geq (\gamma \cdot t + \beta' \cdot \overline{y}^2) \cdot |v_2|. \end{aligned} \tag{2.3}$$

On the other hand, from (2.2) there is $\beta'' > 0$ with

$$|\overline{v}_1| = |a_t(x,y)\cdot v_1 + b_t(x,y)\cdot v_2| \leq \beta'' \cdot |v_2| \quad \forall\, (v_1,v_2) \in \mathcal{C}_t(x,y). \tag{2.4}$$

Now, by taking

$$\beta = \max\{\frac{\beta''}{\beta'}, \frac{\beta''}{\gamma}\}$$

from (2.3–4) it follows that $(\theta_t)_*(\mathcal{C}_t) \subset \overline{\mathcal{C}}_t$.

To get the lemma it is enough to increase β'' and to shrink α, β' and γ to guarantee (2.1) and (2.4) for every $\tilde{\theta}$ in some neighbourhood $\mathcal{D}$ of θ. $\square$

Associated with the arc of diffeomorphisms θ_t above we define the two-parameter family of diffeomorphisms $\{\theta_{\mu,t}\}_{\{\mu>0, t\in[0,1]\}}$ by

$$\theta_{\mu,t}\colon [-\mu,\mu]^2 \to [-\mu,\mu]^2, \quad \theta_{\mu,t}(x,y) = \mu \cdot \theta_t(\frac{x}{\mu}, \frac{y}{\mu}).$$

Lemma 2.2. *For every $\tilde{\theta}$ in $\mathcal{D}$ take the family of diffeomorphisms $\{\tilde{\theta}_{\mu,t}\}_{\mu,t}$ above and the conefields $\{\mathcal{C}_{\mu,t}\}_{\mu,t}$ and $\{\overline{\mathcal{C}}_{\mu,t}\}_{\mu,t}$, $\mu > 0$ and $t \in [0,1]$, defined on $[-\mu,\mu]^2$ by*

$$\mathcal{C}_{\mu,t}(x,y) = \{(v_1,v_2)\colon |v_1| < \frac{\alpha}{\mu} \cdot x^2 \cdot |v_2|\},\ (x,y) \in [-\mu,\mu]^2.\},$$

$$\overline{\mathcal{C}}_{\mu,t}(x,y) = \{(v_1,v_2)\colon |v_1| < \frac{\mu^2 \cdot \beta}{\mu^2 \cdot t + |y|^2} \cdot |v_2|\},\ (x,y) \in [-\mu,\mu]^2,\ y \neq 0\},$$

where α and β are as in Lemma 2.1. Then

$$(\tilde{\theta}_{\mu,t})_*(\mathcal{C}_{\mu,t}(x,y)) \subset \overline{\mathcal{C}}_{\mu,t}(\tilde{\theta}_{\mu,t}(x,y)) \text{ for all } \mu \in]0,1[\text{ and } t \in [0,1].$$

Proof. Given a point z and $\varepsilon > 0$ define the cone of size ε at z, $\mathcal{C}(z,\varepsilon)$, by

$$\mathcal{C}(z,\varepsilon) = \{(v_1,v_2)\colon |v_1| < \varepsilon \cdot |v_2|\}.$$

Let h_κ be the homotethy of ratio κ on $[-1,1]^2$, i.e. $h_\kappa(x,y) = (\kappa \cdot x, \kappa \cdot y)$. Take $z = (x(z), y(z))$ and observe that

$$\begin{aligned}(h_{\frac{1}{\mu}})_*(\mathcal{C}(z, \frac{\alpha}{\mu} \cdot x(z)^2)) &= \mathcal{C}(\frac{z}{\mu}, \frac{\alpha}{\mu} \cdot x(z)^2) = \\ &= \mathcal{C}(z', \mu \cdot \alpha \cdot x(z')^2),\end{aligned} \qquad z' = \frac{z}{\mu} = (\frac{x(z)}{\mu}, \frac{y(z)}{\mu}). \tag{2.5}$$

Write $\overline{z} = \tilde{\theta}(z')$. Since μ is less than 1, by Lemma 2.1

$$(\tilde{\theta}_t)_*(\mathcal{C}(z', \mu \cdot \alpha \cdot x(z')^2)) \subset (\tilde{\theta}_t)_*(\mathcal{C}(z', \alpha \cdot x(z')^2)) \subset \mathcal{C}(\overline{z}, \frac{\beta}{t + y(\overline{z})^2}). \tag{2.6}$$

Arguing as in (2.5) one has

$$(h_\mu)_*(\mathcal{C}(\overline{z}, \frac{\beta}{t + y(\overline{z})^2})) \subset \mathcal{C}(\tilde{z}, \frac{\mu^2 \cdot \beta}{\mu^2 \cdot t + y(\tilde{z})^2}), \qquad \tilde{z} = \overline{z} \cdot \mu. \tag{2.7}$$

From (2.5–7) and the definition of $\tilde{\theta}_{\mu,t}$ one has

$$(\tilde{\theta}_{\mu,t})_*(\mathcal{C}_{\mu,t}(z)) \subset \overline{\mathcal{C}}_{\mu,t}(\tilde{\theta}_{\mu,t}(z)).$$

This completes the proof of the lemma. □

We close this section by making the following remark whose proof is immediate:

Remark 2.3. In the square $[-1,1]^2$ consider the symmetry $s(x,y) = (y,x)$. Then $s \circ \theta^{-1} \circ s \in \Theta$.

In what follows the main step of the proof is the construction of the invariant stable and unstable conefields for the diffeomorphisms we consider. The simmetry between the roles of θ and θ^{-1} in the remark above will allow us to restrict our attention only to the construction of the unstable conefield.

3. Heteroclinic cubic tangencies

Let f be a $\mathcal{C}^\infty$-diffeomorphism defined on a compact boundaryless surface M having a nontrivial basic set Λ_f. For any pair of points x and y in Λ_f one has $W^s(x,f) \pitchfork W^u(y,f) \neq \varnothing$. Take any $q \in \Lambda_f$ so that $q \in W^s(P_1,f) \pitchfork W^u(P_2,f)$, where P_1 and P_2 are two periodic saddles of Λ_f with disjoint orbits. In this section we construct a two-parameter family of diffeomorphisms $\{f_{\mu,t}\}_{\{\mu>0, t\in[0,1]\}}$ isotopic to f and such that $f_{\mu,t}$ coincides with f outside a neighbourhood of size μ of q (thus P_1 and P_2 are hyperbolic periodic points of $f_{\mu,t}$). Moreover, for small μ the invariant manifolds $W^s(P_1, f_{\mu,0})$ and $W^u(P_2, f_{\mu,0})$ have a cubic tangency throughout the orbit of q and $f_{\mu,t}$ has a basic set conjugate to Λ_f for every $t \in]0,1]$. In other words, each $f_{\mu,0}$ is a diffeomorphism with a heteroclinic cubic tangency which is the first bifurcation of the arc $\{f_{\mu,t}\}$.

3.1. Semi-local properties

In this paragraph we aim to prove some semi-local properties of the diffeomorphisms we consider (see Lemma 3.1-2 below). The perturbations $f_{\mu,t}$ of the diffeomorphism f we consider are far (in the $\mathcal{C}^1$-topology) from f. So it is not clear *a priori* that the locally maximal set of $f_{\mu,t}$ in $\mathcal{W}$ is compact (neither if such a set is hyperbolic). However, the region were the perturbations are far from f is contained in a very small neighbourhood of Λ_f. This enables us to prove that the new locally maximal set in $\mathcal{W}$ remains compact. Moreover, when such a set is hyperbolic it is conjugate to Λ_f. These preliminary considerations hold trivially when the diffeomorphism f is Anosov (i.e. $\Lambda_f = M = T^2$).

Let f be as above and $\mathcal{W}$ an isolating compact neighbourhood of the basic set Λ_f, i.e. $\Lambda_f = \cap_{i\in\mathbb{Z}} f^i(\mathcal{W})$ and $\Lambda_g = \cap_{i\in\mathbb{Z}} g^i(\mathcal{W})$ is a basic set conjugate to Λ_f for every diffeomorphism g $\mathcal{C}^1$-nearby f.

We can choose $\mathcal{W}$ and a neighbourhood $\mathcal{V}$ of f in the $\mathcal{C}^3$-topology such that there are a Riemannian metric $\|\cdot\|$, a constant $\rho > 1$ and continuous conefields $\mathcal{C}^u$ and $\mathcal{C}^s$ such that for every $g \in \mathcal{V}$ one has

$$\begin{aligned} &\text{for all } z \in \mathcal{W} \cap g^{-1}(\mathcal{W}) \text{ and } v \in \mathcal{C}^u(z) \quad \begin{cases} g_*(v) \in \mathcal{C}^u(g(z)), \\ \|(g)_*(v)\| \geq \rho \cdot \|v\|, \end{cases} \\ &\text{for all } z \in \mathcal{W} \cap g(\mathcal{W}) \text{ and } v \in \mathcal{C}^s(z)) \quad \begin{cases} (g^{-1})_*(v) \in \mathcal{C}^s(g^{-1}(z)), \\ \|(g^{-1})_*(v)\| \geq \rho \cdot \|v\|. \end{cases} \end{aligned} \tag{3.1}$$

Lemma 3.1. *There are a compact neighbourhood $\mathcal{W}_0$ of Λ_f contained in the interior of $\mathcal{W}$ and a C^0-neighbourhood $\mathcal{O}$ of the restriction $f|_{(\mathcal{W}\setminus\mathcal{W}_0)}$ such that for every diffeomorphism g such that $g|_{(\mathcal{W}\setminus\mathcal{W}_0)}$ belongs to $\mathcal{O}$ the compact set $\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(\mathcal{W})$ is contained in the interior of $\mathcal{W}$.*

Proof. Take k so that

$$\mathcal{W}_1 = \bigcap_{i=-k}^{i=k} f^i(\mathcal{W})$$

is in the interior of $\mathcal{W}$. Observe that if z belongs to the closure of $(\mathcal{W} \setminus \mathcal{W}_1)$ there is $i(z) = i$, $|i| \leq k+1$, with $f^i(z) \notin \mathcal{W}$ and $f^j(z) \in (\mathcal{W} \setminus \mathcal{W}_1)$ for all $0 \leq j < i$ if $i > 0$, or for all $0 \geq j > i$ if $i < 0$. Now, choose a neighbourhood $\mathcal{W}_0$ of Λ_f in the interior of $\mathcal{W}_1$.

By compactness of the closure of $(\mathcal{W} \setminus \mathcal{W}_1)$ there is a neighbourhood $\mathcal{O}$ of f in $\mathrm{Diff}^0(\mathcal{W} \setminus \mathcal{W}_0)$ such that for each $z \in (\mathcal{W} \setminus \mathcal{W}_1)$ and every $0 \leq j < i$ (or $0 \geq j > i$) $g^j(z) \notin \mathcal{W}_0$. Hence either $g^j(z) \notin \mathcal{W}$ (and then $z \notin \Lambda_g$), or $g^j(z) \in (\mathcal{W} \setminus \mathcal{W}_0)$ (and then $g^{j+1}(z)$ remains close to $f^{j+1}(z)$), hence $g^i(z) \notin \mathcal{W}$. Now the maximal invariant set of g in $\mathcal{W}$ is contained in $\mathcal{W}_1$. $\square$

From Lemma 3.1 we get:

Lemma 3.2. *There is a neighbourhood $\mathcal{V}$ of f in* $\mathrm{Diff}^3(M)$ *such that for every $g_1 \in \mathcal{V}$ and every isotopy $\{g_t\}_{\{t\in[0,1]\}}$ from $g = g_1$ to g_0 satisfying*
(1) *$g_t \in \mathcal{O}$,*
(2) *$\Lambda_t = \cap_{i\in\mathbb{Z}} g_t^i(\mathcal{W})$ is hyperbolic,*

for all t, one has that the restriction $g_t|_{\Lambda_{g_t}}$ is conjugate to $f|_{\Lambda_f}$.

Proof. For every t the set Λ_{g_t} is hyperbolic and locally maximal in $\mathcal{W}$. Thus there is $\varepsilon_t > 0$ such that Λ_{g_t} is conjugate to Λ_{g_s} for all $s \in]t-\varepsilon_t, t+\varepsilon_t[$. So Λ_{g_t} is conjugate to Λ_{g_1}. Now, if $\mathcal{V}$ is small enough, Λ_{g_1} is conjugate to Λ_f. Thus, Λ_{g_t} is conjugate to Λ_f. □

3.2. Choice of coordinates

Consider a diffeomorphism f as above. For simplicity we suppose that the points P_1 and P_2 are both fixed. To construct the diffeomorphisms $f_{\mu,t}$ we begin by taking appropriate coordinates around the points P_1, P_2, q and $\hat{q} = f^{-1}(q)$. We assume that f is linearizable at P_1 and P_2 and that the linearizing coordinates can be taken depending continuously on a small neighbourhood $\mathcal{V}$ of f in the C^3 topology (for that it suffices to demand a finite number of nonresonance properties on the eigenvalues of P_1 and P_2, see [**St**]) . We denote by (x_i, y_i) such coordinates defined on a neighbourhood $\mathcal{U}_i$ of P_i $(i = 1, 2)$ independent of the diffeomorphism. Here for notational simplicity we omit the dependence of the coordinates on the diffeomorphism.

Given $z \in \mathcal{U}_i$ denote $(x_i(z), y_i(z))$ its coordinates. Then $x_i(z) = 0$ (resp. $y_i(z) = 0$) implies $z \in W^s_{\text{loc}}(P_i)$ (resp. $z \in W^u_{\text{loc}}(P_i)$). Without loss of generality we can assume that q and $\hat{q}$ are in $\mathcal{U}_2 \cap \mathcal{U}_1$. So in some fixed neighbourhoods of q and $\hat{q}$ the functions $(x_2(z), y_1(z))$ are coordinates. To avoid misunderstandings let (x_s, y_s) and (x_r, y_r) be the coordinates at $\hat{q}$ and q, respectively. The expression of any diffeomorphisms $g \in \mathcal{V}$ in these coordinates is linear:

$$x_r(g(z)) = \sigma_2 \cdot x_s(z) \quad \text{and} \quad y_r(g(z)) = \lambda_1 \cdot y_s(z), \tag{3.2}$$

where λ_i and σ_i denote the eigenvalues of g_* at P_i , $i = 1, 2$. Note that $|\lambda_i| > 1 > |\sigma_i|$. Again, we omit the dependence of the eigenvalues on the diffeomorphism.

For small τ, $\delta > 0$ define the sets

$$\begin{aligned} R_1^{\pm} &= \{(x_1, y_1) \in R_1 \colon |x_1| < \delta \text{ and } y_1 \in [\pm\tau, (\pm\lambda_1 \cdot \tau))\} \subset \mathcal{W}, \\ R_2^{\pm} &= \{(x_2, y_2) \in R_2 \colon x_2 \in [\pm\tau, (\pm\sigma_2 \cdot \tau)) \text{ and } |y_2| < \delta\} \subset \mathcal{W}. \end{aligned} \tag{3.3}$$

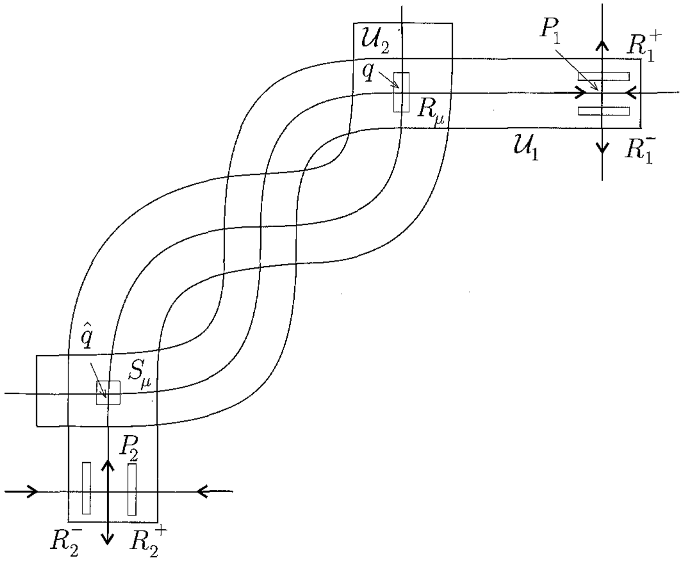

Figure 3

Remark 3.3. *There is $\delta > 0$ so that for every $g \in \mathcal{V}$*

$$\frac{\partial}{\partial y_1} \subset \mathcal{C}^u(z), \quad \frac{\partial}{\partial x_2} \subset \mathcal{C}^s(z) \quad \textit{for every } z \in R_i^{\pm} \cap \mathcal{W},\ i = 1,\ 2.$$

Proof. Observe that the tangent space of the local unstable manifold of P_1 (spanned by $\frac{\partial}{\partial y_1}$) is in the unstable conefield. The first part of the remark follows from the continuity of $\mathcal{C}^u$. The proof of the second part is similar, so we omit it. □

For every diffeomorphism g in $\mathcal{V}$ and sufficiently small $\mu > 0$ consider the square S_μ centered at $\hat{q}$ given in the coordinates (x_s, y_s) by $S_\mu = [-\mu, \mu]^2$. Notice that the segments $[-\mu, \mu] \times \{0\}$ and $\{0\} \times [-\mu, \mu]$ are in the stable manifold of P_1 and the unstable manifold of P_2, respectively. Let $R_\mu = g(S_\mu)$. In the coordinates (x_r, y_r) we have from (3.2),

$$R_\mu = [-\sigma_2 \cdot \mu, \sigma_2 \cdot \mu] \times [-\lambda_1 \cdot \mu, \lambda_1 \cdot \mu] \subset \mathcal{U}_1 \cap \mathcal{U}_2.$$

Now we are ready to define the arcs $\{g_{\mu,t}\}_{\{\mu>0,\, t\in[0,1]\}}$ for every $g \in \mathcal{V}$

as mentioned above.

3.3. The family of perturbations $\{g_{\mu,t}\}$

Notice that for each pair (μ, t) the two-parameter family of diffeomorphisms $\{\theta_{\mu,t}\}$ in Section 2 defines a family of diffeomorphisms from S_μ into itself which is the identity on the boundary of S_μ. So the formula below defines a two-parameter family of diffeomorphisms $\{g_{\mu,t}\}_{\{\mu>0,\, t\in[0,1]\}}$

$$g_{\mu,t}(x) = \begin{cases} g(x) \text{ if } x \notin S_\mu, \\ (g \circ \theta_{\mu,t})(x) \text{ if } x \in S_\mu. \end{cases}$$

Clearly, $g_{\mu,0}(\hat{q}) = q$ and the image $g_{\mu,t}(S_\mu)$ does not depend on t. So $R_\mu = g_{\mu,0}(S_\mu) = g_{\mu,t}(S_\mu)$.

Lemma 3.4. *By shrinking the neighbourhood $\mathcal{V}$ of f in* $\mathrm{Diff}^3(M)$ *if necessary one gets $\mu_0 > 0$ such that for every $g \in \mathcal{V}$ and $\mu \in]0, \mu_0[$ the invariant manifolds $W^s(P_1(g), g_{\mu,0})$ and $W^u(P_2(g), g_{\mu,0})$ have a cubic tangency at $\hat{q}(g)$.*

Proof. Take $\mu_0 > 0$ small so that for every $\mu \in]0, \mu_0[$ all the negative (resp. positive) iterates of the segment of $W^u(P_2, f) \cap S_\mu$ (resp. $W^s(P_1, f) \cap S_\mu$) through $\hat{q}$ are disjoint from S_μ. If $\mathcal{V}$ is small enough this holds for every $g \in \mathcal{V}$. In the (x_s, y_s)-coordinates one has

$$\{0\} \times [-\mu, \mu] \subset W^u(P_2, g_{\mu,0}), \quad \theta_{\mu,0}^{-1}([-\mu, \mu] \times \{0\}) \subset W^s(P_1, g_{\mu,0}).$$

Now the lemma follows from the definition of $\theta_{\mu,t}$. $\square$

We are now ready to state the key result about the arcs $\{g_{\mu,t}\}$.

Proposition 3.5. *Let f be a C^∞-diffeomorphism of a compact boundaryless surface M having a nontrivial basic set Λ_f. Suppose that there are fixed points P_1 and P_2 of Λ_f, $P_1 \neq P_2$, such that f is C^3-linearizable at P_1 and P_2 and the eigenvalues of f_* at P_i, denoted by λ_i and σ_i, $|\lambda_i| > 1 > |\sigma_i|$, $(i = 1, 2)$, satisfy $|\lambda_1^2 \cdot \sigma_1| < 1 < |\lambda_2 \cdot \sigma_2^2|$.*

Then there are a neighbourhood $\mathcal{V}$ of f in $\mathrm{Diff}^3(M)$ *and $\mu_0 > 0$ such that for every $g \in \mathcal{V}$ the two-parameter family of diffeomorphisms $\{g_{\mu,t}\}$ defined as above satisfies*

(1) *for every* $0 < \mu \leq \mu_0$ *the set*

$$\tilde{\Lambda}_{g_{\mu,0}} = \bigcap_{i \in \mathbb{Z}} (g_{\mu,0})^i (\mathcal{W} \setminus \{q(g)\})$$

is nonuniformly hyperbolic and $g_{\mu,0}$ *has a cubic tangency throughout the orbit of* $q(g)$ *(the continuation of* q*),*

(2) *for every* $0 < \mu \leq \mu_0$ *and* $t > 0$ *the set*

$$\Lambda_{g_{\mu,t}} = \bigcap_{i \in \mathbb{Z}} (g_{\mu,t})^i (\mathcal{W})$$

is hyperbolic and conjugate to Λ_g.

To prove the proposition for $\mu > 0$ and $t \in [0, 1]$ we analize the first return map of $g_{\mu,t}$ in R_μ and we show that for small $\mu > 0$ this map is hyperbolic (uniformly if $t > 0$ and nonuniformly if $t = 0$). From now on we fix g and so we omit the dependence on g.

More precisely: For $\mu > 0$ consider the subset $R^+_{\mu,t}$ of R_μ defined by $R^+_{\mu,t} = \{z \in R_\mu$ such that $g^i_{\mu,t}(z) \in R_\mu$ and $g^j_{\mu,t}(z) \in \mathcal{W}$ for every $0 \leq j \leq i\}$.

We remark that the set $R^+_{\mu,t}$ is nonempty: the set Λ_f is transitive and $q \in \Lambda_f$, thus there are points of Λ_f in R_μ. In particular, there are points of Λ_f in R_μ whose forward orbit is contained in $\mathcal{W}$ that return to R_μ. That implies that $R^+_{\mu,t} \neq \varnothing$.

Given $z \in R^+_{\mu,t}$ the *return time of* z *to* R_μ, $n_{\mu,t}(z)$, is

$$n_{\mu,t}(z) = \min\{i > 0 \text{ such that } g^i_{\mu,t}(z) \in R_\mu\}.$$

Note that by construction $g_{\mu,t} = g$ outside S_μ, so that $g^i_{\mu,t}(z) = g^i(z)$ for all $0 \leq i \leq n_{\mu,t}(z) - 1$. Therefore $n_{\mu,t}(z) = n_\mu(z)$ does not depend on t. Finally, the relation

$$T_{g_{\mu,t}} \colon R^+_{\mu,t} \to R_\mu, \quad z \mapsto g^{n_\mu(z)}_{\mu,t}(z)$$

defines the *Poincaré return map* $T_{g_{\mu,t}}$ *associated with* $g_{\mu,t}$ *and* R_μ.

Proposition 3.5. follows from the lemma below which we prove in the next section. Let

$$\Lambda_{\mu,t} = \bigcap_{i \in \mathbb{Z}} (T^i_{g_{\mu,t}}(R_\mu)).$$

Lemma 3.6. *Under the hypotheses of the Proposition 3.5 there are a neighbourhood $\mathcal{V}_0$ of f and $\mu_0 > 0$ such that for every $g \in \mathcal{V}_0$ one has*
(1) *for every $\mu \in]0, \mu_0[$ and $t \in]0, 1]$ the set $\Lambda_{\mu,t}$ is hyperbolic,*
(2) *for every $\mu \in]0, \mu_0[$ the set $\Lambda_{\mu,0}$ is nonuniformly hyperbolic.*

3.4. Hyperbolicity of the Poincaré map $T_{g_{\mu,t}}$: proof of Lemma 3.6

For the sake of clarity we first prove the lemma for the diffeomorphism f. To prove the hyperbolicity of the Poincaré return map $T_{f_{\mu,t}} = T_{\mu,t}$ we shall write $T_{\mu,t}$ as composition of several functions of the type $f^i_{\mu,t}$. We use that $f^i_{\mu,t}(z) = f^i(z)$ for all $0 \le i < n_\mu(z)$ and the fact that the orbit of any $z \in R^+_{\mu,t}$ must visit some fixed regions we list below before coming back to R_μ. Recalling the definitions of R^+_i and R^-_i (see (3.3)) one has immediatly:

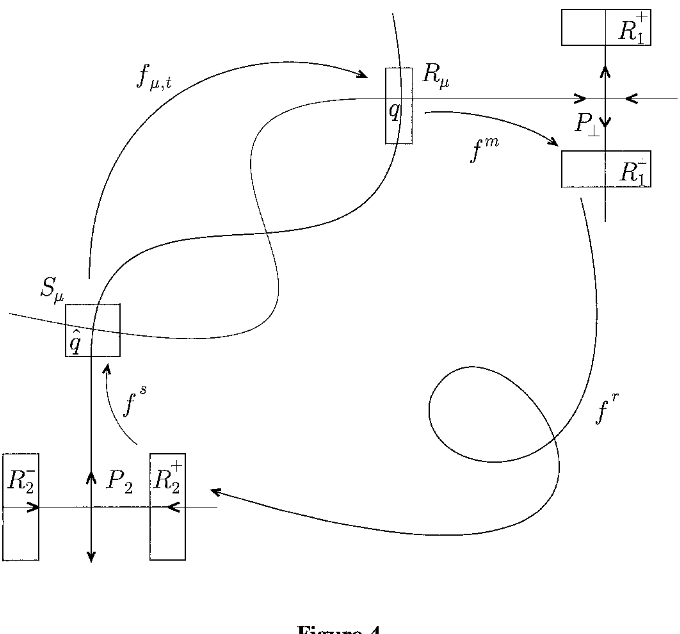

Figure 4

Remark 3.7. *There is $\mu_1 > 0$ such that for every $z \in R^+_{\mu,t}$ and $\mu \in]0, \mu_1]$ there are positive integers $m = m(z)$ and $r = r(z)$ for which the forward* *orbit* $\{z, \cdots, f_{\mu,t}^{n_\mu(z)}(z)\}$ *may be split as follows:*

$$f^m(z) \in R_1^{\pm} \text{ and } (\cup_{i=0}^{m-1} f^i(z)) \subset \mathcal{U}_1,$$
$$f^{m+r}(z) \in R_2^{\pm}, \text{ and}$$
$$f^i(z) \in \mathcal{U}_2 \text{ for every } m+r \leq i < n_\mu(z).$$

Hence

$$T_{\mu,t}(z) = f_{\mu,t} \circ f^{s(z)} \circ f^{r(z)} \circ f^{m(z)}(z) \text{ where } s(z) = n_\mu(z) - 1 - r(z) - m(z).$$

To prove the hyperbolicity of $T_{\mu,t}$ we exhibit a $(T_{\mu,t})_*$-invariant expanding conefield in $R_{\mu,t}^+$ (the unstable conefield) and a $(T_{\mu,t}^{-1})_*$-invariant expanding conefield (the stable conefield) as above, see (3.1). Here we construct the unstable conefield. By Remark 2.3 this will imply the existence of the stable one. Our proof deals with the four different coördinates systems defined in Section 3.2 and six conefields. So, for simplicity, in the sequel $\mathcal{C}(\omega)$ means that the conefield $\mathcal{C}$ is expressed in the ω-coordinates. Before going into the details let us give the scheme of the construction of the $(T_{\mu,t})_*$-invariant expanding conefield:

(1) We begin with a conefield $\mathcal{C}_1(x_r, y_r)$ on R_μ coinciding with $\overline{\mathcal{C}}_{\mu,t}(x,y)$ as defined in Lemma 2.2. We exhibit a conefield $\mathcal{C}_2(x_1, y_1)$ on $R_\mu \subset \mathcal{U}_1$ containing $\mathcal{C}_1$.

(2) By using the linear expression of $f_{\mu,t}$ in $\mathcal{U}_1$ we construct a conefield $\mathcal{C}_3(x_1, y_1)$ on $R_1^{\pm}$ so that the derivative of the transition $f^{m(z)}$ from $R_{\mu,t}^+$ to $R_1^{\pm}$ maps $\mathcal{C}_2$ into $\mathcal{C}_3$.

(3) From the hyperbolicity of f we get $\mathcal{C}_4(x_2, y_2)$ on $R_2^{\pm}$ so that the derivative of $f^{r(z)}$ from $R_1^{\pm}$ to $R_2^{\pm}$ maps $\mathcal{C}_3$ into $\mathcal{C}_4$.

(4) Having in mind that (in the (x_2, y_2)-coordinates) $f_{\mu,t} = f$ is linear, we prove that the derivative of $f^{s(z)}$ maps $\mathcal{C}_4$ into a conefield $\mathcal{C}_5(x_2, y_2)$. This conefield satisfies:

(5) The conefield $\mathcal{C}_5(x_2, y_2)$ on S_μ is contained in the conefield $\mathcal{C}_6(x_s, y_s)$ defined exactly as $\mathcal{C}_{\mu,t}$ in Lemma 2.2. So Lemma 2.2 implies that $(f_{\mu,t})_*(\mathcal{C}_6) \subset \mathcal{C}_1$.

Clearly, (1)–(5) give the $(T_{\mu,t})_*$-invariance of the conefield $\mathcal{C}_1$. Now we go into the details of our construction.

Remark. Throughout the proof of the lemma all the constants k_i and c_i we will introduce do not depend on $\mu > 0$.

First step: conefields on R_μ.
Let $\mathcal{C}_1(x_r, y_r)$ be the conefield on R_μ which is (in those coordinates) the image by the linear map $(x, y) \mapsto (\sigma_2 \cdot x, \lambda_1 \cdot y)$ of the conefield $\overline{\mathcal{C}}_{\mu,t}$ in Lemma 2.2:

$$\mathcal{C}_1(x_r, y_r) = \{(v_1, v_2) \colon |v_1| < \sigma_2^{-1} \cdot \lambda_1 \cdot \frac{\mu^2 \cdot \beta}{\mu^2 \cdot t + y_r^2} \cdot |v_2|\}.$$

Claim 1. *There exists a constant $k_1 > 0$ such that the conefield $\mathcal{C}_2$ defined on R_μ by*

$$\mathcal{C}_2(x_1, y_1) = \{(v_1, v_2) \colon |v_1| \leq \frac{k_1 \cdot \mu^2}{\mu^2 \cdot t + y_1^2} \cdot |v_2|\}$$

contains $\mathcal{C}_1$.

Proof. The coordinates (x_r, y_r) on S_μ satisfy $x_r(z) = x_2(z)$ and $y_r(z) = y_1(z)$. Hence $\partial/\partial x_r$ and $\partial/\partial x_1$ are colinear. Now the claim follows from an easy calculation. □

Second step: transition from R_μ to $R_1^\pm$.
Using the coordinates (x_1, y_1) on $R_1^\pm$ let

$$\mathcal{C}_3(x_1, y_1) = \{(v_1, v_2) \colon |v_1| \leq k_2 \cdot \mu^2 \cdot |v_2|\}.$$

Claim 2. *There is a constant $k_2 > 0$ such that for every $z \in R_{\mu,t}^+$*

$$(f^{m(z)})_*(\mathcal{C}_2(z)) \subset \mathcal{C}_3(f^{m(z)}(z)).$$

Proof. Given a vector $v = (v_1, v_2)$ in $\mathcal{C}_2(x_1(z), y_1(z))$ let

$$(w_1, w_2) = (f_{\mu,t}^m)_*(v_1, v_2) = (\sigma_1^m \cdot v_1, \lambda_1^m \cdot v_2).$$

Thus

$$|w_1| = |\sigma_1^m \cdot v_1| \leq \frac{|\sigma_1^m|}{|\lambda_1^m|} \cdot \frac{k_1 \cdot \mu^2}{t \cdot \mu^2 + y_1(z)^2} \cdot |w_2| \leq \frac{|\sigma_1^m|}{|\lambda_1^m|} \cdot \frac{k_1 \cdot \mu^2}{y_1(z)^2} \cdot |w_2|. \quad (3.4)$$

The definitions of $m(z)$ and τ (see 3.3) give

$$\tau \leq |\lambda_1^m \cdot y_1(z)| = |y_1(f^m(z))| \leq |\lambda_1| \cdot \tau. \quad (3.5)$$

Now, by (3.4-5) and $|\sigma_1 \cdot \lambda_1| < 1$ one gets k_2 such that

$$|w_1| \leq k_2 \cdot |\sigma_1^m| \cdot |\lambda_1^m| \cdot \mu^2 \cdot |w_2| \leq k_2 \cdot \mu^2 \cdot |w_2|.$$

This ends the proof of the claim. □

Third step: the transition from $R_1^{\pm}$ to $R_2^{\pm}$
Define the conefield $\mathcal{C}_4$ on $R_2^{\pm}$ by

$$\mathcal{C}_4(x_2, y_2) = \{(v_1, v_2) \colon |v_1| \leq k_3 \cdot |v_2|\}.$$

Notice we can assume that $\mathcal{C}_4$ contains the unstable conefield $\mathcal{C}^u(z, \varepsilon)$ in (3.1). Now, for each point $z \in R_1^{\pm}$ let

$$r(z) = \sup\{i \in \mathbb{N} \colon f^i(z) \in R_2^{\pm} \text{ and } f^j(z) \notin S_\mu \text{ for all } 0 < j < i\}.$$

If the forward orbit of z does not meet $R_2^{\pm}$ we let $r(z) = \infty$. Having in mind that the unstable foliation is transverse to the horizontal lines $\{y_i(z) = k\}$ (Remark 3.3), it follows that

Claim 3. *There exists a constant $k_3 > 0$ such that for every $z \in R_1^{\pm}$ with $r(z) < \infty$*

$$(f^{r(z)})_*(\mathcal{C}_3(z)) \subset \mathcal{C}_4(f^{r(z)}(z)).$$

Proof. For every point $z \in R_1^{\pm}$ the vertical vector field $\partial/\partial y_2$ is contained in the interior of the conefield $\mathcal{C}^u$ defined on $\mathcal{W}$ (see Remark 3.3). Thus there exists $\mu_0 > 0$ such that the cone $\mathcal{C}_3(z) = \{(v_1, v_2) \colon |v_1| \leq k_2 \cdot \mu^2 \cdot |v_2|\}$ is contained in $\mathcal{C}^u(z)$ for all $\mu \in]0, \mu_0]$ and any $z \in R_1^{\pm}$. On the other hand, for $z \in R_2^{\pm}$ the horizontal vector field $\partial/\partial x_2$ is in the interior of $\mathcal{C}^s$ in (3.1) (see Remark 3.3). So it is transverse to the unstable conefield $\mathcal{C}^u$. Thus there is a constant $k_3 > 0$ such that for each $z \in R_2^{\pm}$ the cone $\mathcal{C}^u(z)$ is contained in $\mathcal{C}_4(z)$. Now using the $(f)_*$-invariance of $\mathcal{C}^u$ one has:

$$(f^r)_*(\mathcal{C}_3(z)) \subset (f^r)_*(\mathcal{C}^u(z)) \subset \mathcal{C}^u(f^r(z)) \subset \mathcal{C}_4(f^r(z)).$$

This completes the proof of the claim. □

Fourth step: transition from $R_2^{\pm}$ to S_μ.
In S_μ define the conefield $\mathcal{C}_5$ by

$$\mathcal{C}_5(x_2, y_2) = \{(v_1, v_2) \colon |v_1| \leq k_4 \cdot x_2^2 \cdot |v_2|\}.$$

Given $z \in R_2^{\pm}$ let

$$s(z) = \inf\{i \in \mathbb{N}\colon f^i(z) \in S_\mu \text{ and } f^j(z) \in \mathcal{U}_2 \text{ for all } 0 \le j \le i\}.$$

As above we let $s(z) = \infty$ if the forward orbit of z does not intersect S_μ.

Claim 4. *There exists $k_4 > 0$ such that for each $z \in R_2^{\pm}$ with $s(z) < \infty$ one has that*

$$(f^{s(z)})_*(\mathcal{C}_4(z)) \subset \mathcal{C}_5(f^{s(z)}(z)).$$

Proof. Take $z \in R_2^{\pm}$ with $s = s(z) < \infty$. Take $v = (v_1, v_2) \in \mathcal{C}_4(x_2(z), y_2(z))$. Let

$$(w_1, w_2) = (f^s)_*(v_1, v_2) = (\sigma_2^s \cdot v_1, \lambda_2^s \cdot v_2).$$

Thus, since $|\sigma_2 \cdot \lambda_2| > 1$,

$$|w_1| = |\sigma_2^s \cdot v_1| \le \frac{|\sigma_2^s|}{|\lambda_2^s|} \cdot k_3 \cdot |w_2| \le (|\sigma_2^s|)^2 \cdot k_3 \cdot |w_2|. \tag{3.6}$$

By definition of $R_2^{\pm}$,

$$|\sigma_2| \cdot \tau \le |x_2(z)| \le \tau,$$

therefore

$$|x_2(f^{s(z)}(z))| = |\sigma_2^s \cdot x_2(z)| \ge |\sigma_2^s \cdot (\sigma_2 \cdot \tau)|. \tag{3.7}$$

So from (3.6–7) one has that

$$|w_1| \le k_4 \cdot |x_2(f^s(z))|^2 \cdot |w_2|, \text{ where } k_4 = \frac{k_3}{(|\sigma_2| \cdot \tau)^2}.$$

This implies the claim. □

Fifth step: transition from S_μ to R_μ.

To end the proof of Lemma 3.3 it remains to show that $(f_{\mu,t})_*(\mathcal{C}_5) \subset \mathcal{C}_1$. For that consider the auxiliary conefield $\mathcal{C}_6$ defined in S_μ by

$$\mathcal{C}_6(x_s, y_s) = \{(v_1, v_2)\colon |v_1| < \frac{\alpha}{\mu} \cdot x_s^2 \cdot |v_2|\}.$$

Claim 5. *There exists $\mu_0 > 0$ such that $\mathcal{C}_5(z) \subset \mathcal{C}_6(z)$ for all $\mu \in]0, \mu_0]$ and z in S_μ.*

Proof. The coordinates (x_s, y_s) satisfy $x_s(z) = x_2(z)$ and $y_s(z) = y_1(z)$. Thus the vector fields $\partial/\partial y_s$ and $\partial/\partial y_2$ are colinear. Now the claim is clear. □

Finally the $(T_{\mu,t})_*$-invariance of $\mathcal{C}_1$ follows from the next claim.

Claim 6. *For every $\mu \in]0, \mu_0[$ the conefield $\mathcal{C}_1$ defined on R_μ is $(T_{\mu,t})_*$-invariant. In other words,*

$$(T_{\mu,t})_*(\mathcal{C}_1(z)) = (f_{\mu,t}^{n_\mu(z)})_*(\mathcal{C}_1(z)) \subset \mathcal{C}_1(f_{\mu,t}^{n_\mu(z)}(z)) = \mathcal{C}_1(T_{\mu,t}(z))$$

for all $\mu \in]0, \mu_0]$ and every $z \in R_{\mu,t}^+$.

Proof. The Claims 1-5 imply $(f^{n_\mu(z)-1})_*(\mathcal{C}_1(z)) \subset \mathcal{C}_6(f^{n_\mu(z)-1}(z))$. Moreover, the conefield $\mathcal{C}_6$ coincides (in the (x_s, y_s)-coordinates) with $\mathcal{C}_{\mu,t}$ in Lemma 2.2. The expression of $f_{\mu,t} = f \circ \theta_{\mu,t}: S_\mu \to R_\mu$ in the coordinates (x_s, y_s) and (x_r, y_r) is given by

$$x_r(f(z)) = \sigma_2 \cdot x_s(z) \quad \text{and} \quad y_r(f(z)) = \lambda_1 \cdot y_s(z).$$

Lemma 2.2 claims that $(f_{\mu,t})_*(\mathcal{C}_6)$ is in the conefield that (in the (x_r, y_r)-coordinates) is the image by the linear map $(x, y) \mapsto (\sigma_2 \cdot x, \lambda_1 \cdot y)$ of the conefield $\overline{\mathcal{C}}_{\mu,t}$ in Lemma 2.2. This means that

$$(f_{\mu,t})_*(\mathcal{C}_6(z)) \subset \mathcal{C}_1(f_{\mu,t}(z)).$$

This ends the proof of the claim. □

The proof of the $(T_{\mu,t})_*$-invariance of $\mathcal{C}_1$ is now complete. To finish the proof of Lemma 3.6 (for f) we need to show that the vectors in $\mathcal{C}_1(z)$ are expanded by $(T_{\mu,t})_*$ for every $z \in \cap_{i \in \mathbb{N}} T_{\mu,t}^i(R_\mu)$.

Sixth step:. $(T_{\mu,t})_*$ expands every vector in $\mathcal{C}_1$.

First we prove that:

Claim 7. *There is $\mu_2 \in]0, \mu_1]$ such that for every $\mu \in]0, \mu_2]$ and $z \in R_{\mu,t}^+$ one has*

$$|v_r((T_{\mu,t})_*(u))| \geq \frac{1}{\mu} \cdot |v_r(u)| \text{ for all } u \in \mathcal{C}_1(z)$$

where $v_r(u)$ is the vertical component of the vector u in the (x_r, y_r) coordinates.

Proof. For simplicity take $u = (h_r, v_r)$ any vector in $\mathcal{C}_1(z)$ with $|v_r| = 1$. Since $(x_r, y_r) = (x_2, y_1)$ the coordinates $(h_1(u), v_1(u))$ of u satisfy $|v_1(u)| = |v_r(u)| = 1$. Thus there is a constant c_0 with

$$||(f^m)_*(u)|| \geq c_0 \cdot |v_1((f^m)_*(u))| = c_0 \cdot |\lambda_1|^m. \tag{3.8}$$

Now, from (3.8) and the fact that $(f)_*$ expands the vectors in the conefield $\mathcal{C}^u$ (see (3.1)), $f_*^{m+r}(u)$ is in the conefield $\mathcal{C}_4$ and its norm is greater than $c_0 \cdot |\lambda_1^m|$. Hence, there exists a constant c_1 such that

$$|v_2((f^{m+r})_*(u))| \geq c_1 \cdot |\lambda_1|^m. \tag{3.9}$$

Let $\bar{z} = f_{\mu,t}^{m+r+s}(z) = f_{\mu,t}^{n_\mu(z)-1}(z) = f_{\mu,t}^{-1} \circ T_{\mu t}(z)$, and $\bar{u} = (f^{m+r+s})_*(u) = (f_{\mu,t}^{-1} \circ T_{\mu,t})_*(u)$. From (3.9), one has that

$$|v_2(\bar{u})| \geq c_2 \cdot |\lambda_1^m \cdot \lambda_2^s|$$

for some constant c_2. The change of coordinates gives that $v_s(\bar{u})$ is a function of h_2 and v_2. Now, one gets c_3 such that

$$|v_s(\bar{u})| \geq c_3 \cdot |\lambda_1^m \cdot \lambda_2^s|. \tag{3.10}$$

The coordinate $v_r(T_{\mu,t}(u)) = v_r((f_{\mu,t})_*(\bar{u}))$ is given by the following formula:

$$\begin{aligned} v_r((f_{\mu,t})_*(\bar{u})) = \lambda_2 \cdot \mu \cdot [b_t(\frac{x_s(\bar{z})}{\mu}, \frac{y_s(\bar{z})}{\mu}) \cdot h_s(\bar{u}) + \\ + d_t(\frac{x_s(\bar{z})}{\mu}, \frac{y_s(\bar{z})}{\mu}) \cdot v_s(\bar{u})], \end{aligned} \tag{3.11}$$

see Section 2 for the definitions of b_t and d_t. Since $\bar{u}$ is in the conefield $\mathcal{C}_6$ one has

$$|h_s(\overline{u})| \leq \frac{\alpha \cdot x_r(\overline{z})^2}{\mu} \cdot |v_s(\overline{u})|.$$

Hence, from (3.11)

$$|v_r((f_{\mu,t})_*(\bar{u}))| \geq |\lambda_2| \cdot \mu \cdot (|d_t(\frac{\overline{z}}{\mu})| - \frac{\alpha \cdot x_r(\overline{z})^2}{\mu} \cdot |b_t(\frac{\overline{z}}{\mu})|) \cdot |v_s(\overline{u})|. \tag{3.12}$$

From properties (H3–H4) of d_t and (2.1) in Lemma 2.1, one gets positive constants c_4 and c_5 with

$$d_t(x,y) - c_4 \cdot x^2 \cdot |b_t(x,y)| \geq c_5 \cdot x^2.$$

In our coordinates this inequality is read as follows

$$d_t(\frac{\overline{z}}{\mu}) - c_4 \cdot \frac{x_s(\overline{z})^2}{\mu^2} \cdot |b_t(\frac{\overline{z}}{\mu})| \geq c_5 \cdot \frac{x_s(\overline{z})^2}{\mu^2}. \tag{3.13}$$

Take a small μ_2 such that

$$\frac{\alpha}{\mu} < \frac{c_4}{\mu^2} \quad \text{for every } \mu \in]0, \mu_2]. \tag{3.14}$$

From (3.12–14), there is $c_6 > 0$ with

$$|v_r((f_{\mu,t})_*(\bar{u}))| \geq c_6 \cdot \frac{x_s(\bar{z})^2}{\mu^2} \cdot |v_s(\bar{u})|. \tag{3.15}$$

From (3.15), (3.7), the definition of $\overline{z}$ and (3.10) one gets constants c_7–c_8 with

$$\begin{aligned} |v_r((f_{\mu,t})_*(\bar{u}))| &\geq |c_6 \cdot \frac{x_s(\bar{z})^2}{\mu^2}| \cdot |v_s(\bar{u})| \geq |c_7 \cdot \frac{\sigma_2^{2s}}{\mu^2}| \cdot |v_s(\bar{u})| \geq \\ &\geq \frac{|c_8 \cdot \lambda_1^m \cdot \lambda_2^s \cdot \sigma_2^{2s}|}{\mu^2}. \end{aligned} \tag{3.16}$$

By hypothesis, $\lambda_2 > \sigma_2^{-2}$. Thus, by (3.16) one has that

$$|v_r((f_{\mu,t})_*(\bar{u}))| \geq |c_8 \cdot \frac{\lambda_1^m}{\mu^2}|. \tag{3.17}$$

Notice that $\frac{1}{|y_r|} \sim |\lambda_1^m|$ (recall the definition of m, see also (3.5)). From $z \in R_\mu$ one has $|y_r| \leq c_9 \cdot \mu$ for some constant c_9. Finally, from (3.17) we get c_{10} and μ_2 with:

$$|v_r((f_{\mu,t})_*(\bar{u}))| \geq |c_8 \cdot \frac{\lambda_1^m}{\mu^2}| \geq |\frac{c_{10}}{\mu^2}| \geq \frac{1}{\mu} \quad \text{for every } \mu \in]0, \mu_2[.$$

The proof of the claim is now complete. □

Claim 8. *Let $\mu_2 > 0$ be as in Claim 7. There is a metric $|||\cdot|||$ such that for every $\mu \in]0, \mu_2[$ one has*

(1) *for every $t > 0$ there is N_t with*

$$|||(T_{\mu,t}^{N_t})_*(v)||| > 2 \cdot |||v||| \textit{ for all } v \in \mathcal{C}_1(z) \textit{ and } z \in \Lambda_{\mu,t},$$

(2) *for each $z \in \Lambda_{\mu,0}$ there is $N(z)$, $N(z) \to \infty$ as $z \to q$, such that*

$$|||(T_{\mu,t}^{N(z)})_*(v)||| > 2 \cdot |||v||| \textit{ for all } v \in \mathcal{C}_1(z).$$

Proof. Define the metric $|||\cdot|||$ by

$$|||u||| = \max\{|h_r(u)|, |v_r(u)|\}.$$

For every $u \in \mathcal{C}_1$ one has

$$|||u||| \leq \max\{\frac{|\sigma_2^{-1} \cdot \lambda_1| \cdot \mu^2 \cdot \beta}{\mu^2 \cdot t + y_r^2} \cdot |v_r(u)|, |v_r(u)|\}. \tag{3.18}$$

Let us first suppose that $t > 0$. Since $\mu < \mu_2 < 1$ there is N_t with

$$\left(\frac{1}{\mu}\right)^{N_t} \geq 2 \cdot (\max\{\frac{\beta}{t}, 1\}) \geq 2 \cdot (\max\{\frac{\mu^2 \cdot \beta}{\mu^2 \cdot t + y_r^2}, 1\}). \tag{3.19}$$

Claim 7 and (3.18) imply that

$$\begin{aligned} |||(T_{\mu,t}^{N_t})_*(u)||| &\geq \left(\frac{1}{\mu}\right)^{N_t} \cdot |v_r(u)| \geq \\ &\geq 2 \cdot (\max\{\frac{|\sigma_2^{-1} \cdot \lambda_1| \cdot \mu^2 \cdot \beta}{\mu^2 \cdot t + y_r^2} \cdot |v_r(u)|, |v_r(u)|\}) \geq \\ &\geq 2 \cdot |||v|||. \end{aligned}$$

This completes the proof of the claim when $t > 0$.

If $t = 0$ and $z = (x_r(z), y_r(z)) \in \Lambda_{\mu,0}$ take $N(z)$ with

$$\left(\frac{1}{\mu}\right)^{N(z)} \geq 2 \cdot \left(\max\{\frac{|\sigma_2^{-1} \cdot \lambda_1 \cdot \beta \cdot \mu^2|}{y_r(z)^2}, 1\}\right). \tag{3.20}$$

The definition of $\mathcal{C}_1$ gives

$$|||u||| \leq \max\{\frac{|\sigma_2^{-1} \cdot \lambda_1 \cdot \beta \cdot \mu^2|}{y_r(z)^2} \cdot |v_r(u)|, |v_r(u)|\}. \tag{3.21}$$

From Claim 7 and (3.20–21) one has

$$|||(T_{\mu,0}^{N(z)})_*(u)||| \geq \left(\frac{1}{\mu}\right)^{N(z)} \cdot |v_r(u)| \geq 2 \cdot |||v|||.$$

This ends the proof of the claim. □

The proof of Lemma 3.6 is now complete for arcs of the form $f \circ \theta_{\mu,t}$. For arcs of the form $g_{\mu,t} = g \circ \theta_{\mu,t}$, $g \in \mathcal{V}$, it is enough to observe that the constants k_i and c_i above can be taken uniformly on $g \in \mathcal{V}$ if $\mathcal{V}$ is small. This ends the proof of Lemma 3.6. □

4. Proof of Theorem A

Let f be a C^∞-diffeomorphism with a nontrivial basic set Λ_f as in Proposition 3.5. Consider a diffeomorphism f_0 derived from f by the deformation in Section 2, i.e. $f_0 = f \circ \theta_{\mu,0}$ where $\theta \in \mathcal{D}$. For simplicity let us assume that $\mu = 1$. Thus $f_0 = f \circ \theta_0$. So we write R and S in the place of R_μ and S_μ.

For any g_0 in a small C^3-neighbourhood $\mathcal{E}$ of f_0 denote by P_{i,g_0} the continuation of the hyperbolic fixed point P_i of f_0, by λ_{i,g_0} and σ_{i,g_0} the eigenvalues of g_0 at P_{i,g_0}, and by (x_{i,g_0}, y_{i,g_0}) the linearizing coordinates of g_0 in the neighbourhood $\mathcal{U}_i$ of P_i, $i = 1, 2$. By the choice of f_0 we can take, and we do, these coordinates depending C^3 on g_0.

Denote by R_g the connected component of $\{z \in \mathcal{U}_1 \cap \mathcal{U}_2, (x_{2,g}(z), y_{1,g}(z)) \in [-\sigma_{2,g}, \sigma_{2,g}] \times [-\lambda_{1,g}, \lambda_{1,g}]\}$ containing q. Similarly, S_g is the component of $\{z \in \mathcal{U}_1 \cap \mathcal{U}_2, (x_{1,g}(z), y_{2,g}(z)) \in [-1,1]^2\}$ containing $\hat{q}$.

Notice that the pair of functions (x_{2,g_0}, y_{1,g_0}) does not define a coordinate system on S_g and R_g. But they define coordinates out of a small neighbourhood of $\hat{q}$. In R_g the vector field $\partial/\partial y_{2,g_0}$ is given by

$$\frac{\partial}{\partial y_{2,g_0}}(z) = \chi_{g_0}(z)\frac{\partial}{\partial x_{1,g_0}} + \delta_{g_0}(z)\frac{\partial}{\partial y_{1,g_0}}. \tag{4.1}$$

Recall that the coordinates (x_{i,g_0}, y_{i,g_0}) depend C^3 on g_0 and δ_{f_0} is a Morse function at $(0,0)$. Therefore, δ_{g_0} depends C^2 on g_0. Thus δ_{g_0} has a unique minimum at the point z_{g_0} which depends differentially on g_0. Moreover, such a minimum is also of Morse type. These remarks imply that

$$\phi\colon \mathcal{E} \to \mathbb{R}^3, \quad \phi(g) = (\delta_g(z_g), x_{2,g}(z_g), y_{1,g}(z_g)),$$

is a $\mathcal{C}^2$ function. Consider the set

$$\Gamma = \{\phi^{-1}(0,0,0)\} \cap \mathcal{E}.$$

Lemma 4.1. *The map ϕ is a submersion at f_0. In particular, Γ is a submanifold of codimension 3 of* $\mathrm{Diff}^3(M)$.

Proof. Take isotopies h_τ and G_τ defined on the square $S = [-1,1]^2$, and H_τ defined on $R = [-\sigma_2, \sigma_2] \times [-\lambda_1, \lambda_1]$ such that:

- h_τ is the identity on a neighbourhood of the boundary of $[-1,1]^2$ and in a vicinity of $(0,0)$ is given by $(x,y) \mapsto (x+\tau, y)$,
- G_τ is the identity on a neighbourhood of the boundary of S and its derivative at $(0,0)$ is a rotation of angle τ,

- H_τ is the identity on a neighbourhood of the boundary of R and in a vicinity of $(0,0)$ is given by $(x,y) \mapsto (x, y+\tau)$.

Consider the map $\Psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\Psi(\tau_1,\tau_2,\tau_3) = \phi(H_{\tau_1} \circ f_0 \circ G_{\tau_2} \circ h_{\tau_3})$. Now the lemma follows from the fact that Ψ is a submersion at $(0,0,0)$. $\square$

Proposition 4.2. *There is a neighbourhood $\mathcal{V}_0$ of f_0 in $\mathrm{Diff}^3(M)$ such that every arc $\{h_t\}_{t\geq 0}$ with $h_0 \in \Gamma \cap \mathcal{V}_0$ and $\frac{d}{dt}(\delta_{h_t}(z_{h_t})) > 0$ satisfies the following properties*

(1) *there is $t_0 > 0$ such that the set $\Lambda_{h_t} = \bigcap_{i\in\mathbb{Z}} h_t^i(\mathcal{W})$ is hyperbolic and conjugate to Λ_f for every $t \in]0, t_0[$,*

(2) *the diffeomorphism h_0 has a heteroclinic cubic tangency associated to the continuations of P_1 and P_2.*

Proof. To prove the proposition we will see that $h_t = g_t \circ \tilde{\theta}_t$ for some $g_t \in \mathcal{V}$ and $\tilde{\theta}_t \in \mathcal{D}$. From Proposition 3.5 this will imply the proposition.

For $i = 1, 2$ denote by $(\hat{x}_i, \hat{y}_i)$ the linearizing coordinates of f_0 in $\mathcal{U}_i$. Remark that $(\hat{x}_1, \hat{y}_1)$ (resp. $(\hat{x}_2, \hat{y}_2)$) coincides with (x_1, y_1) (resp. (x_2, y_2)) in the complementary of a small neighbourhood of $\hat{q}$ (contained in S) in $\mathcal{U}_1$ (resp. in the complementary of a small neighbourhood of q (contained in R) in $\mathcal{U}_2$).

Lemma 4.3. *For every $t \geq 0$ there is a function $\tilde{x}_{2,t}$ (resp. $\tilde{y}_{1,t}$) C^3-close to $\hat{x}_2$ in $\mathcal{U}_2$ (resp. $\hat{y}_1$ in $\mathcal{U}_1$) which coincides with x_{2,h_t} (resp. y_{1,h_t}) on a neighbourhood of $(\mathcal{U}_2 \setminus R)$ (resp. of $(\mathcal{U}_1 \setminus S)$) and such that $\tilde{x}_{2,t}(z_{h_t}) = 0$ (resp. $\tilde{y}_{1,t}(h_t^{-1}(z_{h_t})) = 0$).*

Proof. Just notice that x_{2,h_t} is C^3-close to $\hat{x}_2$ and that $\hat{x}_2(z_{h_t})$ is close to 0. $\square$

Remark 4.4. *The pair of functions $(\tilde{x}_{2,t}, \tilde{y}_{1,t})$ defines coordinates on R and S. Furthermore, there is a neighbourhood of ∂S such that in such coordinates h_t is given by*

$$\tilde{x}_{2,t}(h_t(z)) = \sigma_{2,h_t} \cdot \tilde{x}_{2,t}(z), \quad \tilde{y}_{1,t}(h_t(z)) = \lambda_{1,h_t} \cdot \tilde{y}_{1,t}(z).$$

Proof. The first assertion follows from the proximity between $\tilde{x}_{2,t}$ and

$\hat{x}_2$ and $\tilde{y}_{1,t}$ and $\hat{y}_1$. The second one follows by recalling that $\tilde{x}_{2,t}$ and $\tilde{y}_{1,t}$ are linearizing coordinates outside a neighbourhood of $(0,0)$. □

Lemma 4.5. *Consider g_t with $g_t(z) = h_t(z)$ for every $z \notin S$ and such that in the $(\tilde{x}_{2,t}, \tilde{y}_{1,t})$-coordinates g_t is given by:*

$$\tilde{x}_{2,t}(g_t(z)) = \sigma_{2,h_t} \cdot \tilde{x}_{2,t}(z), \quad \tilde{y}_{1,t}(g_t(z)) = \lambda_{1,h_t} \cdot \tilde{y}_{1,t}(z).$$

Then for small $t \geq 0$ one has $g_t \in \mathcal{V}$. Moreover, $\tilde{\theta}_t = g_t^{-1} \circ h_t \in \mathcal{D}$.

Proof. To prove the first part of the lemma observe that h_t is C^3-close to f_0 and $f(x) = f_0(x)$ for every $x \notin S$. Moreover, the eigenvalues of f_* at P_i and $(h_t)_*$ at P_{i,h_t} are close, f (resp. g_t) is linear on S in the $(\hat{x}_2, \hat{y}_1)$-coordinates (resp. $(\tilde{x}_{2,t}, \tilde{y}_{1,t})$), and $(\hat{x}_2, \hat{y}_1)$ and $(\tilde{x}_{2,t}, \tilde{y}_{1,t})$ are C^3-close. Finally, since $g_t(x) = h_t(x)$ if $x \notin S$ one gets that g_t is C^3-close to f on M and $g_t \in \mathcal{V}$.

The proximity between θ_0 and $\tilde{\theta}_t$ follows from $\theta_0 = f^{-1} \circ f_0$ and $\tilde{\theta}_t = g_t^{-1} \circ h_t$ as well as from the proximity between f and g_t, and f_0 and h_t.

By construction, $\tilde{\theta}_t$ is the identity in a neighbourhood of the boundary of $S = [-1,1]^2$ and $\tilde{\theta}_t(0,0) = (0,0)$ (hypotheses (H1–H2)). Write

$$(\tilde{\theta}_t)_*(\frac{\partial}{\partial \tilde{y}_{1,t}}) = \tilde{c}_t(\tilde{x}_{2,t}, \tilde{y}_{1,t}) \cdot \frac{\partial}{\partial \tilde{x}_{2,t}} + \tilde{d}_t(\tilde{x}_{2,t}, \tilde{y}_{1,t}) \cdot \frac{\partial}{\partial \tilde{y}_{1,t}}. \tag{4.2}$$

It remains to see that $\tilde{d}_t$ satisfies the inequality (2.1) (that defines the set $\mathcal{D}$, see the end of the Lemma 2.1). In particular, $\tilde{d}_0$ is Morse at $(0,0)$. By construction in a vicinity of $(0,0)$ one has $h_t = g_t \circ \tilde{\theta}_t$, where g_t is linear in the $(\tilde{x}_{2,t}, \tilde{y}_{1,t})$-coordinates. On one hand, from (4.2) one has

$$(h_t)_*(\frac{\partial}{\partial \tilde{y}_{1,t}}(z)) = \tilde{k}_t(z) \cdot \frac{\partial}{\partial \tilde{x}_{2,t}}(h_t(z)) + \lambda_{1,h_t} \cdot \tilde{d}_t(z) \cdot \frac{\partial}{\partial \tilde{y}_{1,t}}(h_t(z)), \tag{4.3}$$

for some function $\tilde{k}_t$. On the other hand, in the (x_{2,h_t}, y_{2,h_t})-coordinates (that from now on we denote by $(x_{2,t}, y_{2,t})$) h_t is linear, i.e.

$$(h_t)_*(\frac{\partial}{\partial y_{2,t}}(z)) = \lambda_{2,t} \cdot \frac{\partial}{\partial y_{2,t}}(h_t(z)).$$

Having in mind (4.1) this means that

$$\begin{aligned}(h_t)_*(\frac{\partial}{\partial y_{2,t}}(z)) &= \lambda_{2,t}\cdot\frac{\partial}{\partial y_{2,t}}(h_t(z)) = \\ &= \lambda_{2,t}\cdot(\chi_{h_t}(z)\cdot\frac{\partial}{\partial\tilde{x}_{1,t}}(h_t(z))+\delta_{h_t}(z)\cdot\frac{\partial}{\partial\tilde{y}_{1,t}}(h_t(z))).\end{aligned}\tag{4.4}$$

The vector fields $\dfrac{\partial}{\partial y_{2,t}}(z)$ and $\dfrac{\partial}{\partial\tilde{y}_{1,t}}(z)$ are colinear. So there exists a function $\rho_t(z)$ which is uniformly bounded from below with $\rho_t(z)\neq 0$ such that

$$\frac{\partial}{\partial\tilde{y}_{1,t}}(z)=\rho_t(z)\cdot\frac{\partial}{\partial y_{2,t}}(z).$$

In particular, from (4.3)

$$\begin{aligned}(h_t)_*(\frac{\partial}{\partial y_{2,t}}(z)) &= \frac{1}{\rho_t(z)}\cdot(h_t)_*(\frac{\partial}{\partial\tilde{y}_{1,t}}(z)) = \\ = \frac{1}{\rho_t(z)}\cdot(\tilde{k}_t(z)&\cdot\frac{\partial}{\partial\tilde{x}_{2,t}}(h_t(z))+\lambda_{1,h_t}\cdot\tilde{d}_t(z)\cdot\frac{\partial}{\partial\tilde{y}_{1,t}}(h_t(z))).\end{aligned}\tag{4.5}$$

From (4.4–5)

$$\tilde{d}_t(z)=\frac{\rho_t(z)\cdot\lambda_{2,t}}{\lambda_{1,t}}\cdot\delta_{h_t}(z).\tag{4.6}$$

The functions ρ_t, $\lambda_{2,t}$ and $\lambda_{1,t}$ satisfy

$$0<C_1<|\rho_t|,|\lambda_{2,t}|,|\lambda_{1,t}|<C_2.$$

Finally from $\dfrac{d}{dt}\delta_{h_t}(z_{h_t})>0$ it follows $\dfrac{d}{dt}d_t(z_{h_t})>0$ for small $t\geq 0$. This implies the proposition. □

End of the Proof ot Theorem A

Take any diffeomorphism F with a basic set $\Lambda_F=\bigcap_{\mathbb{Z}}F^i(\mathcal{W})$ satisfying the hypotheses of the theorem. Then there is a diffeomorphism f satisfying the hypotheses of Proposition 3.5 and an arc $\{F_t\}_{t\in[0,1]}$ with $F_0=f$ and $F_1=F$ such that the set $\Lambda_t=\bigcap_{\mathbb{Z}}F_t^i(\mathcal{W})$ is conjugate to $\Lambda=\bigcap_{\mathbb{Z}}F^i(\mathcal{W})$ for every t. The theorem follows from Propositions 3.5 and 4.2. □

5. Cubic tangencies at lateral points

5.1. Introduction.

In this section we will prove Theorem B. This result will follow essentially from the arguments in the proof of Theorem A. So we will omit some technical details and focus our attention on the main differences between their proofs.

Let f be a surface diffeomorphism and Λ_f be a nontrivial basic set of f different from the whole manifold. Assume that Λ_f is either of saddle type or an attractor (otherwise we replace f by f^{-1}). Then there is a lateral periodic point $P \in \Lambda_f$ such that the set $(\mathcal{U} \backslash W^u_{\text{loc}}(P, f))$ has a connected component that does not intersect Λ_f for every small neighbourhood $\mathcal{U}$ of P.

From now on we suppose that P is a fixed point of f. By deforming f by an isotopy we can assume that f is $\mathcal{C}^3$-linearizable in a neighbourhood $\mathcal{U}$ of P and that the linearizing coordinates (x, y) at P depend continuously in the $\mathcal{C}^3$-topology. Finally, we take a homoclinic point q of P such that q and $f^{-1}(q) = \hat{q}$ are in $\mathcal{U}$.

Take an open set $\mathcal{W}$ in which Λ_f is locally maximal. Since P is a lateral point of Λ_f we can take a neighbourhood $\mathcal{U}$ of P and coordinates (x, y) such that the backward orbit of any point $z \in \{x > 0\}$ leaves $\mathcal{W}$ straightforwardly, meaning the following: Let $j = \inf\{k \in \mathbb{N}$ such that $f^{-k}(z) \notin \mathcal{U}\}$. Then there is i such that $f^{-i}(z) \notin \mathcal{W}$ and $f^{-k}(z) \notin \mathcal{U}$ for every $j \leq k \leq i$. Notice that this assertion holds for every diffeomorphism nearby f.

In what follows, to get a diffeomorphism with a cubic tangency at q which is a first bifurcation, we will perturb f in a square S_μ of size μ centered at $\hat{q}$. As in the nonlateral case we will consider diffeomorphisms θ_μ from S_μ into itself and local perturbations $f_\mu = f \circ \theta_\mu$ of f.

The property about lateral points above plays a key role to check the hyperbolicity of the diffeomorphisms before the bifurcation. It will allow us to focus our atention on the points (x, y) with $x \leq 0$. More precisely, and bearing in mind the proof of the nonlateral case, it will be enough to consider the first return map from $S_\mu \cap \{x \leq 0\}$ into itself. That will allow us to consider generic perturbations θ_μ in the following sense. The image by f_μ of the vertical foliation in S_μ has a nondegenerate cubic tangency at q with the horizontal foliation defined on the square

$f(S_\mu) = R_\mu$. Such a nondegenerate cubic tangency is accumulated by quadratic tangencies between the image by f_μ of the vertical foliation and the horizontal foliation. However, all the tangencies correspond to leaves $f_\mu(\{x_0\} \times [-1,1])$ with $x_0 > 0$, see the figure below.

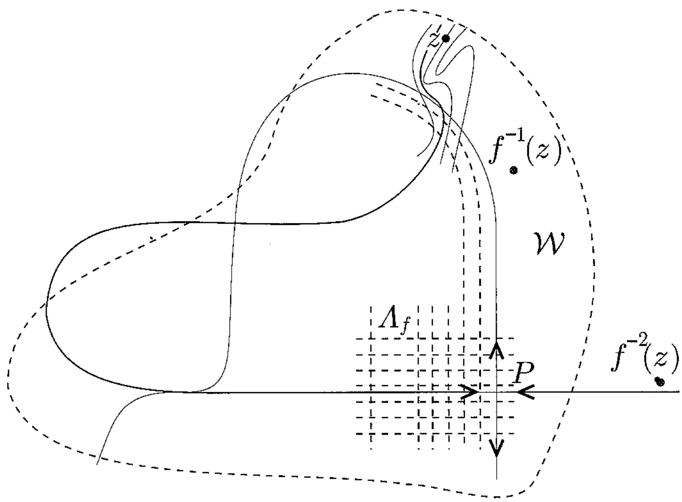

Figure 5

The possibility of using generic perturbations θ_μ with a nondegenerate cubic tangency justifies why in the lateral case the submanifold $\mathcal{C}_\ell$ of diffeomorphisms with cubic tangencies has codimension 2 instead of 3. In the proof we see that using such a perturbation one does not need to use the condition $|\lambda_1^2 \cdot \sigma_1| < |\lambda_2 \cdot \sigma_2^2|$ on the eigenvalues, such a possibility allows us to consider also homoclinic tangencies.

5.2. Families of local perturbations

As in Section 2 we consider diffeomorphisms θ defined on the cube $[-1,1]^2 \times [0,1]$, $\theta(x,y,t) = (\theta_t(x,y),t)$, that satisfy hypotheses H0–H2 in Section 2 and

H3b: $d_0(0,0) = 0$, $d(x,y,t)$ is strictly positive on $([-1,0]^2 \times [0,1]) \setminus \{(0,0,0)\})$, $\dfrac{\partial d_t}{\partial t}(0,0) \geq k_0 > 0$, $\dfrac{\partial d_t}{\partial x}(0,0) = d_{tx} < 0$, $\dfrac{\partial d_t}{\partial y}(0,0) = d_{ty} = 0$, and $\dfrac{\partial^2 d_t}{\partial^2 y}(0,0) = d_{tyy} > 0$.

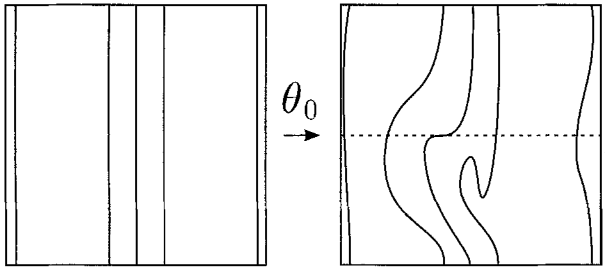

Figure 6

We denote by Θ_ℓ the set of C^3-diffeomorphisms of $[-1,1]^2 \times [0,1]$ satisfying the hypotheses (H0–H2) and (H3b).

Lemma 5.2. *For every $\theta \in \Theta_\ell$ there are a C^3-neighbourhood $\mathcal{D}$ of θ in Θ_ℓ and strictly positive constants α, β and γ such that for every $\tilde{\theta} \in \mathcal{D}$ the conefields*

$$\begin{aligned} \mathcal{C}_t(x,y) &= \{(v_1,v_2)\colon |v_1| < \alpha \cdot |x| \cdot |v_2|\}, \\ \overline{\mathcal{C}}_t(x,y) &= \{(v_1,v_2)\colon |v_1| < \frac{\gamma}{t+|y|} \cdot |v_2| \ ,y \neq 0\} \end{aligned}$$

satisfy

$$(\tilde{\theta}_t)_*(\mathcal{C}_t(x,y)) \subset \overline{\mathcal{C}}_t(\tilde{\theta}_t(x,y)) \text{ for every } x \leq 0 \text{ and } t \geq 0.$$

Proof. Write $\theta_t(x,y) = (\overline{x}_t, \overline{y}_t)$. The proof of the lemma follows from the next result:

Lemma 5.3. *There is a constant K such that*

$$d_t(x,y) \geq K\cdot(t+|\overline{y}_t|+|x|) \text{ for all } t \geq 0 \text{ and } (\overline{x}_t,\overline{y}_t) = \theta_t(x,y) \text{ with } x \leq 0.$$

Proof. Write

$$\begin{aligned} &\overline{y}_t = \overline{y}_t(x,y) = \overline{y}_t(0,y) + b_t(0,0)\cdot x + H_t(x,y)\cdot x, \\ &H_t(x,y) \to 0 \text{ as } y \to 0. \end{aligned} \tag{5.1}$$

Having in mind (H3b) in the definition of Θ_ℓ we have

$$\overline{y}_t(0,y) = \frac{d_{tyy}}{6}\cdot y^3 + y^3 \cdot G_t(y), \quad G_t(y) \to 0 \text{ as } y \to 0. \tag{5.2}$$

From (5.1–2) there is a constant $k_1 > 0$ such that for every (x,y) nearby $(0,0)$ one has

$$|b_t(x,y) + H_t(x,y)| \cdot |x| = |\overline{y}_t - (\frac{d_{tyy}}{6} + G_t(y)) \cdot y^3| \geq |\overline{y}_t| - k_1 \cdot |y^3|.$$

Notice that $b_0(0,0) \neq 0$ (recall that $d_0(0,0) = 0$ and that θ_0 is a diffeomorphism). Thus

$$|x| \geq k_2 \cdot |\overline{y}_t| - k_3 \cdot |y^3|, \tag{5.3}$$

for some strictly positive constants k_2 and k_3. Finally, from (5.3), $d_{tyy} > 0$, $d_{tx} < 0$, $x \leq 0$, $t \geq 0$, and $\dfrac{\partial d_t}{\partial t}(0,0) > 0$ (see (H3b)) we get positive constants k_4–k_6 such that

$$\begin{aligned} d_t(x,y) &\geq k_4 \cdot t + \frac{d_{tyy}}{3} \cdot y^2 + 2 \cdot \frac{|d_{tx}|}{3} \cdot |x| \geq \\ &\geq k_4 \cdot t + \frac{d_{tyy}}{3} \cdot y^2 + \frac{|d_{tx}|}{3} \cdot |x| + \\ &\quad + \frac{|d_{tx}| \cdot k_2}{3} \cdot |\overline{y}_t| - \frac{|d_{tx}| \cdot k_3}{3} \cdot |y|^3 \geq \\ &\geq k_4 \cdot t + k_5 \cdot |x| + k_6 \cdot |\overline{y}_t|, \end{aligned} \tag{5.4}$$

for every small y and $|x|$. The map d_t is strictly positive outside a neighbourhood of $(0,0)$ in $\{x \leq 0\}$. Therefore by shrinking k_4–k_6 the inequality (5.4) holds for all (x,y) with $x \leq 0$ and $t \geq 0$. □

As in the nonlateral case for each $\mu > 0$ we consider the two-parameter family of diffeomorphisms $\theta_{\mu,t}$ defined on $[-\mu,\mu]^2$ by

$$\theta_{\mu,t}(x,y) = \mu \cdot \theta_t(x/\mu, y/\mu).$$

Arguing as in Lemma 2.2 one gets the following result:

Lemma 5.3. *For every $\tilde{\theta}$ in $\mathcal{D}$ define the two-parameter family of diffeomorphisms $\{\tilde{\theta}_{\mu,t}\}$ as above. For each $\mu > 0$ consider the conefields $\mathcal{C}_{\mu,t}$ and $\overline{\mathcal{C}}_{\mu,t}$ defined on $[-\mu,\mu]^2$ by*

$$\mathcal{C}_{\mu,t}(x,y) = \{(v_1,v_2)\colon |v_1| < \alpha \cdot \frac{|x|}{\mu} \cdot |v_2|\},$$

$$\overline{\mathcal{C}}_{\mu,t}(x,y) = \{(v_1,v_2)\colon |v_1| < \frac{|\mu \cdot \beta|}{t + |y|} \cdot |v_2|, \text{ for all } y \neq 0\}.$$

Then

$$(\tilde{\theta}_{\mu,t})_*(\mathcal{C}_{\mu,t}(x,y)) \subset \overline{\mathcal{C}}_{\mu,t}(\tilde{\theta}_{\mu,t}(x,y)),$$

for every sufficiently small $\mu > 0$ *and* $x \leq 0$.

5.3. Choice of coordinates

Consider a nontrivial basic set Λ_f of a diffeomorphism f having a lateral fixed point P. To construct the diffeomorphisms $f_{\mu,t}$ we begin by taking coordinates at P, q and $\hat{q} = f^{-1}(q)$. Recall that f is linearizable at P and that the linearizing coordinates depend continuously on a small neighbourhood $\mathcal{V}$ of f in the $\mathcal{C}^3$-topology. Moreover, there are neighbourhoods $\mathcal{U}_1$ and $\mathcal{U}_2$ of P where f is linearizable so that $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) contains the segment $[P,\hat{q}]^s$ in $W^s(P)$ (resp. the segment $[P,q]^u$ in $W^u(P)$), see figure.

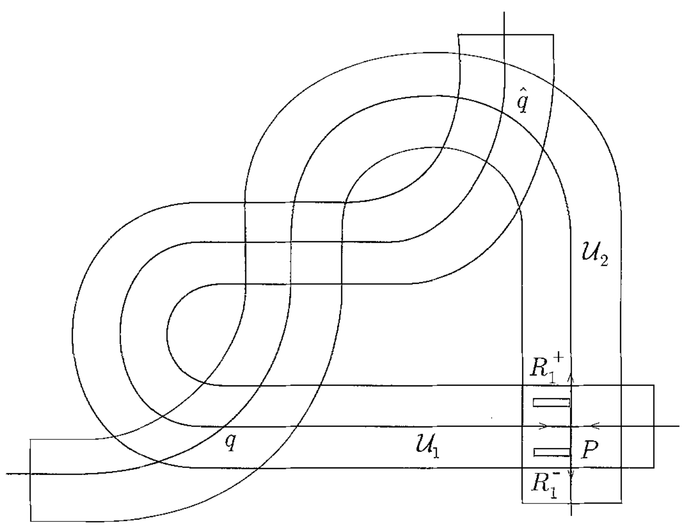

Figure 7

Denote by (x_i, y_i) the linearizing coordinates in $\mathcal{U}_i$ $(i = 1, 2)$. Pick neighbourhoods of q and $\hat{q}$ where the pair $(x_2(z), y_1(z))$ defines coordinates. As in the nonlateral case we call such coordinates (x_r, y_r) and (x_s, y_s), respectively. In these coordinates any $g \in \mathcal{V}$ is linear:

$$x_r(g(z)) = \sigma \cdot x_s(z) \quad \text{and} \quad y_r(g(z)) = \lambda \cdot y_s(z), \tag{5.5}$$

where λ and σ ($|\lambda| > 1 > |\sigma|$) are the eigenvalues of g_* at the continuation P_g of P.

In what follows given any $z \in \mathcal{U}_i$ we will analize the returns of the forward orbit of z to $\mathcal{U}_1$. For that as in the nonlateral case we will consider the exit and entry boxes $R_1^{\pm}$ below. Recall that we are only interested on the points such that their first coordinate is negative. For small τ and $\delta > 0$ define the sets

$$R_1^{\pm} = \{(x_1, y_1) \in \mathcal{U}_1\colon -\delta < x_1 \leq 0 \text{ and } y_1 \in [\pm\tau, (\pm\lambda \cdot \tau))\} \subset \mathcal{W}. \quad (5.6)$$

For every g in $\mathcal{V}$ and small $\mu > 0$ consider the square S_μ centered at $\hat{q}(g)$ (the continuation of $\hat{q}$) that in the coordinates (x_s, y_s) is given by $S_\mu = [-\mu, \mu]^2$. The segments $[-\mu, \mu[\times \{0\}$ and $\{0\} \times [-\mu, \mu]$ are in the stable and the unstable manifolds of P_g, respectively. Let $R_\mu = g(S_\mu)$. From (5.5) in the coordinates (x_r, y_r) we have

$$R_\mu = [-\sigma \cdot \mu, \sigma \cdot \mu] \times [-\lambda \cdot \mu, \lambda \cdot \mu] \subset \mathcal{U}_1 \cap \mathcal{U}_2.$$

Now for each $g \in \mathcal{V}$ we define the two-parameter family of diffeomorphisms $\{g_{\mu,t}\}_{\{\mu>0,\, t\in[0,1]\}}$ by

$$g_{\mu,t}(x) = \begin{cases} g(x) \text{ if } x \notin S_\mu, \\ (g \circ \theta_{\mu,t})(x) \text{ if } x \in S_\mu. \end{cases}$$

Clearly, $g_{\mu,0}(\hat{q}(g)) = q(g)$ (the continuation of q). Moreover, by construction $g_{\mu,t}(S_\mu)$ does not depend on t. So we let $R_\mu = g_{\mu,0}(S_\mu) = g_{\mu,t}(S_\mu)$. Finally, if the neighbourhood $\mathcal{V}$ of f is small there is $\mu_0 > 0$ such that for every $g \in \mathcal{V}$ and $\mu \in]0, \mu_0[$ the invariant manifolds $W^s(P_1(g), g_{\mu,0})$ and $W^u(P_2(g), g_{\mu,0})$ have a nondegenerate cubic tangency at $\hat{q}(g)$.

Proposition 5.4. *Let f be a C^∞-diffeomorphism, Λ_f a nontrivial basic set of f, and P a lateral fixed point of Λ_f as above.*

There are a neighbourhood $\mathcal{V}$ of f in $\mathrm{Diff}^3(M)$ *and $\mu_0 > 0$ such that for every $g \in \mathcal{V}$ the two-parameter family of diffeomorphisms $\{g_{\mu,t}\}$ satisfies*

(1) *the set*

$$\tilde{\Lambda}_{g_{\mu,0}} = \bigcap_{i\in\mathbb{Z}} (g_{\mu,0})^i(\mathcal{W} \setminus \{q(g)\})$$

is nonuniformly hyperbolic for every $0 < \mu \leq \mu_0$ *and* $g_{\mu,0}$ *has a nondegenerate cubic tangency at* $q(g)$,

(2) *the set*

$$\Lambda_{g_{\mu,t}} = \bigcap_{i\in\mathbb{Z}} (g_{\mu,t})^i(\mathcal{W})$$

is hyperbolic and conjugate to Λ_g *for every* $0 < \mu \leq \mu_0$ *and* $t > 0$.

As in the nonlateral case to prove Proposition 5.4 we will study the first return map of $g_{\mu,t}$ defined on R_μ.

5.4. The Poincaré return map in R_μ

Notice that here we are not concerned with the returns of those points whose forward orbit meet $\{x_1 > 0\}$ before intersecting R_μ: by construction such points do not belong to the locally maximal set of $g_{\mu,t}$ in $\mathcal{W}$.

For $g_{\mu,t}$, we consider the subset $R^+_{\mu,t}$ of R_μ of points such that their forward orbits return to R_μ before leaving $\mathcal{W}$. More precisely,

$$R^+_{\mu,t} = \left\{ z \in R_\mu \text{ such that there is } i \text{ with } \begin{array}{l} g^i_{\mu,t}(z) \in R_\mu, \text{ and} \\ g^j_{\mu,t}(z) \in \mathcal{W} \text{ for every } 0 \leq j \leq i. \end{array} \right\}$$

Given $z \in R^+_{\mu,t}$ the *return time of* z *to* R_μ, $n_\mu(z)$, is defined by

$$n_\mu(z) = \min\{i > 0 \text{ so that } g^i_{\mu,t}(z) \in R_\mu\}.$$

The *Poincaré return map* $T_{g_{\mu,t}}$ *associated with* $g_{\mu,t}$ *and* R_μ is given by

$$T_{g_{\mu,t}} \colon R^+_{\mu,t} \to R_\mu, \quad z \mapsto g^{n_\mu(z)}_{\mu,t}(z).$$

By the comments above the image $T_{g_{\mu,t}}(R^+_{\mu,t})$ is in $g_{\mu,t}(\{x_s \leq 0\})$.

Given any $g_{\mu,t}$, with g C^3-close to f and $\mu > 0$, consider the set

$$\Lambda_{\mu,t} = \bigcap_{i\in\mathbb{Z}} T^i_{g_{\mu,t}}(R^+_{\mu,t}).$$

Proposition 5.4 follows from the next lemma. This lemma plays the role of Lemma 3.6 in the nonlateral case.

Lemma 5.5. *Under the hypotheses of Proposition 5.4, there are a neighbourhood* $\mathcal{V}_0$ *of* f *and* $\mu_0 > 0$ *such that for every* $g \in \mathcal{V}_0$,

(1) $\Lambda_{\mu,t}$ *is hyperbolic for every* $\mu \in (0,\mu_0)$ *and* $t \in]0,1]$,

(2) $\Lambda_{\mu,0}$ *is nonuniformly hyperbolic for every* $\mu \in (0,\mu_0)$.

From now on we fix a diffeomorphism $g \in \mathcal{V}$ and we write $T_{\mu,t}$ in the place of $T_{g_{\mu,t}}$. Since the orbit of any point $z \in R^+_{\mu,t}$ visit some fixed sets before returning to R_μ, we write $T_{\mu,t}$ as the composition of some transition functions we will define below:

Remark. *Take* $z \in R^+_{\mu,t}$ *and* μ *small. Define* $m = m(z) < n_\mu(z)$ *by*

$$g^m_{\mu,t}(z) \in R^\pm_1 \text{ and } \bigcup_{i=0}^{m-1} g^i_{\mu,t}(z) \subset \mathcal{U}_1,$$

where $g^m_{\mu,t}(z) \in R^+_1$ *if* $y_r(z) > 0$ *and* $g^m_{\mu,t}(z) \in R^-_1$ *if* $y_r(z) < 0$.

Let $r = r(z)$ *be the biggest integer less than* $n_\mu(z)$ *such that* $g^r_{\mu,t}(z) \in R^+_1$. *By definition,* $g^i_{\mu,t}(z) \in \mathcal{U}_2$ *for all* $r \le i \le n_\mu(z)$. *Notice that there are points with* $y_r(z) > 0$ *and* $m = r$.

For every $z \in R^+_{\mu,t}$ *the forward orbit* $\{z, \ldots, g^{n_\mu(z)}_{\mu,t}(z)\}$ *splits as follows:*

$$T_{\mu,t}(z) = g_{\mu,t} \circ g^{s(z)} \circ g^{k(z)} \circ g^{m(z)}(z),$$
$$s(z) = n_\mu(z) - r - 1, \quad \text{and}$$
$$k = k(z) = r - m,$$

see the figure below.

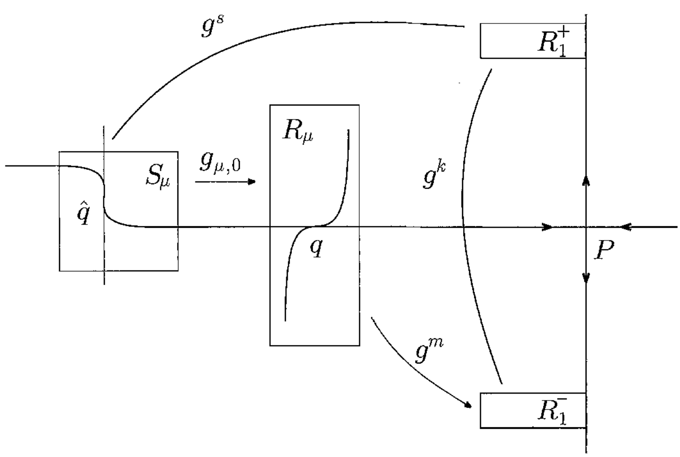

Figure 8

The proof of the lemma follows essentialy as in the nonlateral case. We will exhibit a $(T_{\mu,t})_*$-invariant expanding conefield in $R^+_{\mu,t}$ (the unstable conefield) and a $(T^{-1}_{\mu,t})_*$-invariant expanding conefield (the stable conefield). The proof will involve 6 conefields which are expressed in different coordinates. So for clarity, $\mathcal{C}(\omega)$ will mean that the conefield $\mathcal{C}$ is given in the ω-coordinates. Now, let us sketch the construction of the $(T_{\mu,t})_*$-invariant unstable conefield.

(1) We will begin with a conefield $\mathcal{C}_1(x_r, y_r)$ in R_μ coinciding with $\overline{\mathcal{C}}_\mu(x, y)$ in Lemma 5.3. We will verify that $\mathcal{C}_1$ is contained in a conefield $\mathcal{C}_2(x_1, y_1)$ with equivalent size.
(2) Using the linearity of $g_{\mu,t}$ in $\mathcal{U}_1$ in the (x_1, y_1)-coordinates, we will construct a conefield $\mathcal{C}_3(x_1, y_1)$ in $R^\pm_1$ such that the derivative of the transition $g^{m(z)}$ from $R^+_{\mu,t}$ to $R^\pm_1$ maps $\mathcal{C}_2$ into $\mathcal{C}_3$.
(3) By the hyperbolic behaviour of g in $\mathcal{W}$, the derivative of the transition $g^{k(z)}$ from $R^\pm_1$ to R^+_1 maps $\mathcal{C}_3$ into $\mathcal{C}_5(x_1, y_1) = \{|v_1| \leq |x_1|\cdot|v_2|\}$.
(4) From the linearity of g in $\mathcal{U}_2$, after a change of coordinates, we will get that $(g_\mu^{s(z)})_*(\mathcal{C}_5)$ is in the conefield $\mathcal{C}_6(x_s, y_s)$ in S_μ defined as $\mathcal{C}_\mu$ in Lemma 5.3.
(5) Finally, Lemma 5.3 will imply that $(g_{\mu,t})_*(\mathcal{C}_6) \subset \mathcal{C}_1$. That will give the $(T_{\mu,t})_*$-invariance of $\mathcal{C}_1$.

Next, we fill the details in the outline of the proof of the lemma above. This proof will follow along the ideas of the proof of Lemma 3.6. So we will just emphasize the differences between the lateral and the nonlateral cases.

First step: conefields in R_μ.

Let $\mathcal{C}_1(x_r, y_r)$ be the conefield on R_μ which is the image by the linear map $(x, y) \mapsto (\sigma \cdot x, \lambda \cdot y)$ of $\overline{\mathcal{C}}_{\mu,t}$ in Lemma 5.3:

$$\mathcal{C}_1(x_r, y_r) = \{(v_1, v_2) \colon |v_1| < \frac{|\sigma^{-1} \cdot \lambda \cdot \mu \cdot \beta|}{t + |y_r|} \cdot |v_2|, \quad y_r \neq 0\}.$$

Since $\dfrac{\partial}{\partial x_r}$ and $\dfrac{\partial}{\partial x_1}$ are colinear, there is a constant k_1 independent of

μ with $\mathcal{C}_1(z) \subset \mathcal{C}_2(z)$, where

$$\mathcal{C}_2(x_1, y_1) = \{|v_1| \leq \frac{k_1 \cdot \mu}{t + |y_2|} \cdot |v_2|,\ |y_2| \neq 0\}.$$

Second step: transition from R_μ to $R_1^\pm$.
In R_1^+ consider the conefield $\mathcal{C}_3$

$$\mathcal{C}_3(x_1, y_1) = \{|v_1| \leq k_2 \cdot \mu \cdot |x_1| \cdot |v_2|\},$$

where $k_2 > 0$ is some fixed constant we define below.

Claim 1. *There is $k_2 > 0$ so that $(g^{m(z)})_*(\mathcal{C}_2(z)) \subset \mathcal{C}_3(g^{m(z)}(z))$ for every $z \in R_{\mu,t}^+$.*

Proof. Let $m = m(z)$ and $g_\mu^m(z) = (x_1(g_\mu^m(z)), y_1(g_\mu^m(z))) = (\tilde{x}_1, \tilde{y}_1)$. From the definition of m and the linearity of f one gets constants such that

$$k_3 \cdot |\sigma_1^m| \leq |\tilde{x}_1| \leq k_4 \cdot |\sigma_1^m| \text{ and } k_5 \leq |\lambda_1^m| \cdot |y_2(z)| \leq k_6. \tag{5.7}$$

Take a vector $(w_1, w_2) = (g^m)_*(v_1, v_2)$, $(v_1, v_2) \in \mathcal{C}_2(z)$. From (5.7)

$$|w_1| \leq \frac{k_1 \cdot \mu \cdot \sigma^m}{|\lambda^m \cdot y_2(z)|} \cdot |w_2| \leq k_2 \cdot \mu \cdot |\tilde{x}_1| \cdot |w_2|,$$

for some constant $k_2 > 0$. Now the proof of the claim is complete. □

Third step: transition from $R_1^\pm$ to R_1^+.
If $r = r(z) \neq m(z) = m$, we split the orbit of z as follows.

Consider the rectangle D that is equal to $[\sigma \cdot x_0, x_0] \times [-1, 1]$ for some x_0 in the (x_1, y_1)-coordinates. Take $j = j(z)$ maximum so that $g_{\mu,t}^{m+j}(z) \in D$ and $m+j < n_\mu(z)$. Pick a conefield $\mathcal{C}_4$ of constant size, say k_6, in D. Due to the hyperbolicity of g on $\mathcal{W}$, one has $(g^j)_*(\mathcal{C}_3(g^m(z))) \subset \mathcal{C}_4(g^{m+j}(z))$ for small μ. See the figure.

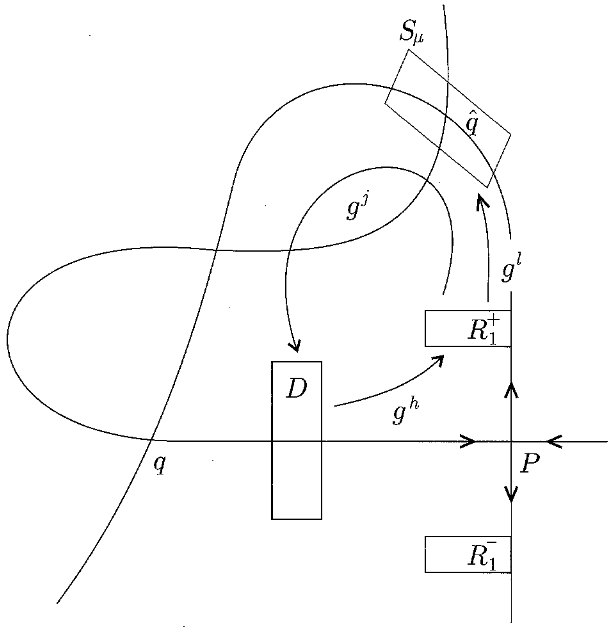

Figure 9

Consider the conefield $\mathcal{C}_5(x_1, y_1) = \{|v_1| \leq |x_1| \cdot |v_2|\}$. Clearly, if μ is small $\mathcal{C}_5(z)$ contains $\mathcal{C}_3(z)$. In particular, $\mathcal{C}_3(g^r(z)) \subset \mathcal{C}_5(g^r(z))$ if $r = m$.

Claim 2. *Let $h = h(z) = r(z) - m(z) - j(z)$, where $j(z)$ is defined as above. There is $\mu_0 > 0$ such that*

$$(g^h)_*(\mathcal{C}_4(g^{m+j}(z)) \subset \mathcal{C}_5(g^r(z)),$$

for every $0 < \mu \leq \mu_0$.

Proof. Let

$$(\hat{x}_1, \hat{y}_1) = (x_1(g^{m+j+h}(z)), y_1(g^{m+j+h}(z))).$$

Define $\nu > 0$ by $|\sigma| = |\lambda^{-\nu}|$. Given $v = (v_1, v_2) \in \mathcal{C}_4(g^{m+j}(z))$ let $w = (w_1, w_2) = (g^h)_*(v) = (\sigma^h v_1, \lambda^h v_2)$. Notice that $n_\mu(z) - m(z) - j(z)$ is uniformly bounded. Thus $|\hat{x}_1| \leq k_7 \cdot \mu$ for some $k_7 > 0$. As in the claim above one gets positive constants k_8 and k_9 with

$$\frac{|w_1|}{|w_2|} \leq k_6 \cdot \frac{|\sigma^h|}{|\lambda^h|} \leq k_8 \cdot |\hat{x}_1| \cdot (\sigma^h)^\nu \leq k_9 \cdot |\hat{x}_1| \cdot \mu^\nu.$$

Now the claim is obvious if μ is small. □

Fourth step: transition from R_1^+ to S_μ.
Define the conefield $\mathcal{C}_6$ in S_μ coinciding with $\mathcal{C}_\mu$ in Lemma 5.3, i.e. $\mathcal{C}_6(x_s, y_s) = \{|v_1| \leq \frac{\alpha \cdot |x_2|}{\mu} \cdot |v_2|\}$. Since $1 << \frac{1}{\mu}$ if μ is small and $l = n_\mu(z) - r - 1$ is uniformly bounded,

$$(g^l)_*(\mathcal{C}_5(g^r(z)) \subset (\mathcal{C}_6(g^{n_\mu(z)-1}(z))). \qquad \square$$

Final step: transition from S_μ to R_μ.
By the definitions of $\mathcal{C}_1$ and $\mathcal{C}_6$ and Lemma 5.3, one has

$$(g_{\mu,t})_*(\mathcal{C}_6(g^{n_\mu(z)-1}(z))) \subset \mathcal{C}_1(g_{\mu,t}^{n_\mu(z)}(z)).$$

Then, the $(T_{\mu,t})_*$-invariance of $\mathcal{C}_1$ follows from Claims 1–2. $\square$

In the sequel we fix $\mu > 0$ such that Claims 1–2 hold. Now, we prove the expansion of the vectors in $\mathcal{C}_1$ by $(T_{\mu,t})_*$. Notice that it is equivalent to see that the return map in S_μ, say $\tilde{T}_{\mu,t} = g_{\mu,t}^{-1} \circ T_{\mu,t} \circ g_{\mu,t}$, expands the vectors of the $(\tilde{T}_{\mu,t})_*$-invariant conefield $\mathcal{C}_6$. Since the calculations for the transition from S_μ into itself are easily understood than the transition from R_μ into itself, let us consider $\tilde{T}_{\mu,t}$ instead of $T_{\mu,t}$. As in the nonlateral case, the result follows from the expansion of the vertical component of the vectors in $\mathcal{C}_6$ by $(\tilde{T}_{\mu,t})_*$, see Claims 7 and 8 in Section 3.4.

Claim 3. *There is $\mu_2 \in]0, \mu_1]$ such that for every $\mu \in]0, \mu_2]$ and every $z \in S_\mu$ such that $\tilde{T}_{\mu,t}$ is defined one has:*

$$|v_s((\tilde{T}_{\mu,t})_*(u))| \geq 2 \cdot |v_s(u)| \quad \textit{for every } u \in \mathcal{C}_6(z),$$

where $v_s(u)$ is the vertical component of u in the (x_s, y_s)-coordinates.

Sketch of the proof. Take $u = (h_s(u), v_s(u)) \in \mathcal{C}_6(z)$ with $|v_s(u)| = 1$. Since $g_{\mu,t}$ is a diffeomorphism and $w = (g_{\mu,t})_*(u) \in \mathcal{C}_2(g_{\mu,t}(z))$, there is $k > 0$ (independent of μ and z) such that $|v_1(w)| \geq k \cdot \frac{|\tilde{y}_1|}{\mu}$, where $\tilde{y}_1 = y_1(g_\mu(z))$. Now, $g_{\mu,t}^{m+1}(z)$ is in $R_1^\pm$. Thus, $|\lambda^m \cdot \tilde{y}_1| \geq k'$ for some constant $k' > 0$ independent of μ and z. Then, the vertical component

$v_1(\hat{u})$ of $\hat{u} = (g_{\mu,t}^{m+1})_*(u)$ satisfies

$$|v_1(\hat{u})| = \frac{|\lambda^m \cdot k \cdot \tilde{y}_1|}{\mu} \geq \frac{k \cdot k'}{\mu}.$$

To end the proof of the claim, just remark that if μ is small, then the vertical component $v_1(\hat{u})$ is arbitrarily large. Hence, the derivative of the transition from $R_1^{\pm}$ to S_μ expands such a vertical component. □

The proof of the existence of a $(T_{\mu,t})_*$-invariant expanding unstable conefield is now complete.

Now, we are left to construct the stable bundle. For this we consider the return map T_μ^{-1} of f_μ^{-1} in R_μ. We take the stable conefield $\mathcal{C}_1'$ as the complementary of $\mathcal{C}_1$. The $(T_\mu^{-1})_*$-invariance of $\mathcal{C}_1'$ follows from the $(T_\mu)_*$-invariance of $\mathcal{C}_1$. So it remains to show that $(T_\mu^{-1})_*$ expands the vectors in $\mathcal{C}_1'$. For this just observe that the complementary of $\mathcal{C}_{\mu,t}$ in Lemma 5.3 is a conefield $\overline{\mathcal{C}}_{\mu,t}'$ with size equal to the size of $\overline{\mathcal{C}}_{\mu,t}$, where the coordinate x is in the place of y. Conversely, the complementary of $\overline{\mathcal{C}}_{\mu,t}$ is a conefield $\mathcal{C}_{\mu,t}'$ with size equal to the size of $\mathcal{C}_{\mu,t}$. Now the expansion follows as in the unstable case.

The proof of Lemma 5.5 is now complete. □

5.5. End of the proof of Theorem B

The theorem follows arguing as in Theorem A. We recall that in the lateral case the codimension of the submanifold $\mathcal{C}_\ell$ is 2 instead of 3. Now let us outline the construction of $\mathcal{C}_\ell$.

Take a $\mathcal{C}^\infty$-diffeomorphism f with a nontrivial basic set Λ_f as in Proposition 5.4. Consider a diffeomorphism f_0 derived from f by the deformation Section 5.1, i.e. $f_0 = f \circ \theta_{\mu,0}$, where $\theta \in \mathcal{D}$. Since from now on μ is fixed, let us omit it.

Pick any g in a small C^3-neighbourhood $\mathcal{E}$ of f_0. Denote by P_g the continuation of the fixed point P of f, by λ_g and σ_g the eigenvalues of g at P_g, and by $(x_{i,g}, y_{i,g})$ the linearizing coordinates of g in the neighbourhood $\mathcal{U}_i$ of P, $i = 1, 2$. Recall that these coordinates depend C^3 on g.

Denote by R_g the connected component of

$$\{z \in \mathcal{U}_1 \cap \mathcal{U}_2, (x_{2,g}(z), y_{1,g}(z)) \in [-1,1]^2]\}$$

containing $q(g)$. Similarly, S_g is the component of $\hat{q}(g)$ in $\{z \in \mathcal{U}_1 \cap \mathcal{U}_2, (x_{1,g}(z), y_{2,g}(z)) \in [-1,1]^2\}$. In R_g the vector field $\dfrac{\partial}{\partial y_{2,g}}$ is given by

$$\frac{\partial}{\partial y_{2,g}}(z) = \chi_g(z)\frac{\partial}{\partial x_{1,g}} + \delta_g(z)\frac{\partial}{\partial y_{1,g}}.$$

For each $x_0 \in [-1,1]$ consider the function $\Delta^g_{x_0}(y) = \delta_g(x_0, y)$. If $\mathcal{E}$ is small there is a unique x_g nearby 0 so that Δ^g_{xg} has a unique zero at, say, y_g, see the figure.

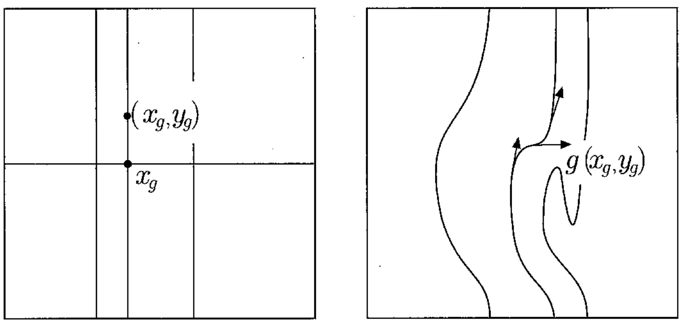

Figure 10

Define functions υ, $\varrho: \mathcal{E} \to [-1,1]$ by $\upsilon(g) = x_g$ and $\varrho(g) = y_g$. Since the coordinates $(x_{i,g}, y_{i,g})$ depend $\mathcal{C}^3$ on g, the functions υ and ϱ are $\mathcal{C}^2$. Hence

$$\psi: \mathcal{E} \to \mathbb{R}^2, \quad \psi(g) = (\upsilon(g), \varrho(g))$$

is also $\mathcal{C}^2$. Define

$$\Gamma_\ell = \{\psi^{-1}(0,0)\} \cap \mathcal{E}.$$

By construction, every $g \in \Gamma_\ell$ has a (nondegenerate) homoclinic cubic tangency related to P_g.

Arguing as in Lemma 4.1 one has

Lemma 5.6. *The map ψ is a submersion at f_0. In particular, Γ_ℓ is submanifold of codimension* 2 *of* $\mathrm{Diff}^3(M)$.

Now Theorem B follows from the arguments in Section 4 (see Lemmas 4.1–5).

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