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# A Note on a Theorem of W. Gaschütz and N. Itô

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**Abstract.** Given a finite group $G$ and $p$ an odd prime number, we conclude that $\mathbf{O}^p(G) \cap G'$ is $p$-nilpotent when for every subgroup $H$ of $G$ of order $p$ there exists a subgroup $K$ of $G$ such that $G = HK$ and $H$ permutes with every subgroup of $K$.

## Introduction

In Su [**6**] (also see Wang [**7**]), the concept of seminormality of a subgroup is introduced. Equivalently to [**6**] and [**7**] we may define: A subgroup $H$ of a finite grupo $G$ is said to be *semi-normal* in $G$ if there exists a subgroup $K$ of $G$ such that $HK = G$ and $H$ permutes with every subgroup of $K$. Clearly, every normal subgroup of $G$ is semi-normal.

Also every subgroup of prime index is semi-normal. Our main attention we direct to finite groups in which every minimal subgroup of odd order is semi-normal.

Gaschütz and Itô ([**3**], Kap. IV, Satz 5.7) have shown that if every minimal subgroup of odd order is normal in $G$, then $G'$ is $p$-nilpotent for each prime number $p > 2$, that is, $G'$ has a normal Sylow-2-subgroup with nilpotent factor group.

The purpose of this note is the presentation of the following:

**Theorem.** *Let $G$ be a group and $p$ a prime number. Suppose all subgroups of order $p$ (all cyclic subgroups of order 2 and 4 if $p = 2$) are semi-normal in $G$. Then $\mathbb{O}^p(G) \cap G'$ is $p$-nilpotent. In particular, $G$ is $p$-solvable of $p$-length at most one. Here $\mathbb{O}^p(G)$ denotes the smallest*

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*normal subgroup of G with p-quotient.*

## 1. Preliminary Results

We prepare the proof of the Theorem.

**Lemma 1.** *Let $H$ be a semi-normal subgroup of a group $G$. If $H \leq L \leq G$, then $H$ is semi-normal in $L$.*

For a proof see (Wang, [**7**]).

**Lemma 2.** *Let $G$ be a group.*

(a) *Let $p$ be an odd prime number. If every subgroup of order $p$ of $G$ is in the center of $G$, then $G$ is $p$-nilpotent.*

(b) *If all elements of order 2 and 4 are in the center of $G$, then $G$ is 2-nilpotent.*

For a proof see ([**3**], Kap. IV, Satz 5.5).

**Lemma 3.** *Let $p$ be a prime number and $H \leq G$ a quasinormal $p$-subgroup of $G$. Then the group of automorphisms induced by $G$ on $H^G/H_G$ is a $p$-group.*

For a proof see (Maier-Schmid [**5**]).

If $H$ and $K$ are subgroup of a group $G$ such that every subgroup of $H$ is permutable with every subgroup of $K$ we say that $H$ and $K$ are *totally permutable.*

**Lemma 4.** *Let $\mathcal{F}$ be a saturated formation which contains the class of supersolvable groups. Let $G = HK$ be a group such that $H$ and $K$ are totally permutable subgroups. If $H$ and $K$ lie in $\mathcal{F}$, then $G$ is an $\mathcal{F}$-group.*

For a proof see (Maier, [**4**]).

A generalization for an arbitrary number of factors of Maier's result is given in Carocca, [**1**], also see [**2**].

**Remark.** Let $p$ be a prime number. The class of all groups $G$ such that $\mathbb{O}^p(G) \cap G'$ is $p$-nilpotent, is a saturated formation which contains all supersolvable groups. See ([**3**], p. 696, p. 689, Satz 6.3 and p. 716, Satz 9.1(b)).





## Proof of the theorem

**Theorem.** *Let $G$ be a group and $p$ a prime number. Suppose all subgroups of order $p$ (all cyclic subgroups of order 2 and 4 if $p = 2$) are semi-normal in $G$. Then $\mathbb{O}^p(G) \cap G'$ is $p$-nilpotent. In particular, $G$ is $p$-solvable of $p$-length at most one.*

**Proof.** Let $G$ be a group of smallest order in which the theorem is not true. Clearly the hypothesis is inherited by subgroups.

Let $H$ denote any one of the subgroups of order $p$ (cyclic subgroups of order 2 or 4 if $p = 2$).

For any such $H$ we have some subgroup $K \leq G$ such that $G = HK$ and $H$ permutes with every subgroup of $K$.

**Case I.** $K = G$ for all $H$.

In this case all $H$ are quasinormal $p$-subgroups of $G$. By Lemma 3, $G/\mathbb{C}_G(H^G/H_G)$ is a $p$-group.

**Sub-Case (i).** Let $|H| = p$. If $H_G = 1$, then $\mathbb{O}^p(G) \leq \mathbb{C}_G(H^G) \leq \mathbb{C}_G(H)$. If $H \trianglelefteq G$, then $G' \leq \mathbb{C}_G(H^G) = \mathbb{C}_G(H)$.

**Sub-Case (ii).** Let $|H| = 4$. Let $x \in G$ be of odd order. Since $H$ is quasinormal in $G$, one has $\langle H, x\rangle = H \times \langle x\rangle$, so $x$ centralizes $H$.

Since $\mathbb{O}^2(G)$ is generated by the elements of odd order of $G$, also in this case $\mathbb{O}^2(G) \leq \mathbb{C}_G(H)$.

In any case, those of our subgroups $H$ which are in $G' \cap \mathbb{O}^p(G)$ are central in this subgroup. So $G' \cap \mathbb{O}^p(G)$ is $p$-nilpotent, by Lemma 2.

**Case II.** For some $H$ we have $K < G$.

Let $\mathcal{F}_p$ denote the formation of all groups $G$ which have $G' \cap \mathbb{O}^p(G)$ $p$-nilpotent.

By the minimality of $|G|$, we have $K \in \mathcal{F}_p$. Also $H \in \mathcal{F}_p$.

If $|H| = p$, then $K$ and $H$ are totally permutable. By Lemma 4, we have $G \in \mathcal{F}_p$.

Let $|H| = 4$. If $H$ and $K$ are not totally permutable, then $Y$, the subgroup of order 2 of $H$ is not quasinormal in $G$. Since $Y$ is semi-normal in $G$, there exists a subgroup $L < G$ such that $G = LY$ and $L$ and $Y$ are totally permutable. Since $Y$, $L \in \mathcal{F}_p$, again $G \in \mathcal{F}_p$.

That $G$ is $p$-solvable of $p$-length at most one follows now easily from the fact that $G/G' \cap \mathbb{O}^p(G)$ is nilpotent.

**Corollary.** *Let $G$ be a group.*

(a) *If all minimal subgroups of $G$ of odd order are semi-normal in $G$, then $G^*$ has a normal Sylow-2-subgroup with nilpotent factor group. ($G^*$ denotes the smallest normal subgroup of $G$ with nilpotent factor group).*

(b) *If all minimal subgroups and all cyclic subgroups of order 4 of $G$ are semi-normal, then $G$ is a solvable group of Fitting-length at most two.*

For the proof of Corollary, we mention that $G^* = \bigcap_p \mathbb{O}^p(G) \cap G'$.

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