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Dicritical Singularities in Foliations with Algebraic Limit Sets

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Abstract. Given a holomorphic foliation of the 2-dimensional projective space, sufficient conditions are given in order to be the pull back of a linear foliation by a rational map.

Introduction

We address here the problem of characterizing the analytic foliations of the complex projective plane $\mathbb{C}P(2)$ which have simple limit sets. The most trivial example is given by a hyperbolic linear foliation; pull backs by rational transformations of $\mathbb{C}P(2)$ of this foliation will generate more complicated examples. They exhibit an algebraic curve as limit set whose holonomy group contains a hyperbolic attractor. It has been proved (see [**1**]) that, under some mild condition imposed on the singularities of the foliation, the existence of such an algebraic limit set implies the pullback property. The condition is that the singularities are nondicritical ones (or equivalently, admit only a finite number of separatrices) and no saddle-nodes appear along their desingularization process. We call them first-order singularities.

Here we extend the result stated beforehand by allowing dicritical singularities, but not many in fact. The main point is to construct a closed meromorphic 1-form which represents the foliation in a neighborhood of the limit set; then by a Levi type of argument the form extends to all $\mathbb{C}P(2)$. In order to carry out the construction one has to go to surfaces that appear after blowing ups of the singularities. The pres-

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ence of dicritical singularities may then obstruct the extension of 1-forms defined in neighborhoods of the limit set. More precisely we need the complement of the limit set to be a Stein surface; this is automatic when all singularities are first-order ones but no longer necessarily true when dicritical singularities are involved. We propose here some conditions in order to overcome such a difficulty.

A word should be said about the presence of singularities of saddle-node type. We think that they change radically the analysis of the problem, as it is suggested by the following theorem of Malmquist ([**3**]): suppose that some leaf of a foliation of $\mathbb{C}P(1) \times \mathbb{C}P(1)$ (this is "almost" $\mathbb{C}P(2)$) has as its limit set a finite number of vertical lines, that is, a set like

$$\bigcup_{i=1}^{n} \{p_i\} \times \mathbb{C}P(1).$$

Then the foliation is pullback by a rational map of a Riccati equation. It is very easy to construct examples where the hypothesis holds, with singularities of saddle-node type. We still avoid in this paper the presence of these singularities.

In Section 1 we state our theorem and give some examples; in Section 2 we develop the proof until the point where we guarantee the Stein property. From that place on the same steps taken in [**1**] are followed in order to complete the proof. This reference contains all the background material which is needed here.

Finally I want to acknowledge the interesting conversations I had with M. Carnicer; they were helpful in achieving the statement of the theorem.

1. Statement of the Theorem

Let $\mathcal{F}$ be an analytic foliation of $\mathbb{C}P(2)$ having some (transcendental) leaf whose limit set is an algebraic curve C (which is of course invariant for $\mathcal{F}$). Assume that:

(i) some component of C has an attractor in its holonomy group.

(ii) The singularities p_j, $i \le j \le m$ of $\mathcal{F}$ in C fall in two different

classes: $p_{m+1}, \dots, p_{m+n}$ are first-order singularities and $p_1, \dots, p_m$ are dicritical ones which under one blowing up give rise only to first-order new singularities (if any).

(iii) if $d_j = \text{mult}_{p_j}(C)$, then $\sum_{j=1}^{m} d_j <$ degree (C).

Theorem. *An analytic foliation $\mathcal{F}$ of $\mathbb{C}P(2)$ that satisfies conditions (i), (ii), (iii) is the pullback of a linear foliation of $\mathbb{C}P(2)$ by a rational map.*

The conditions (ii), (iii) mean in some sense that there are "few" dicritical singularities of $\mathcal{F}$ along C. We remark that radial singularities (locally given by the equation $xdy - ydx = 0$) are allowed after (ii), since their blowing ups produce no new singularities. Also, whenever all singularities are first-order ones the theorem holds, but it should be noticed a small improvement, compared to [**1**], of the present version: we only demand some (transcendental) leaf to have an algebraic curve as its limit set, instead of requiring all leaves to satisfy this condition.

Let us now look to some examples.

Example 1. Consider the pullback of

$$\mathcal{L}\colon \frac{dv}{v} + \lambda \frac{du}{u} = 0,\ \lambda \notin \mathbb{R}\ ,$$

by the rational map

$$(x, y) \overset{\Phi}{\to} \left(x + y, \frac{x}{y}\right).$$

The limit set will be the union of the lines $L_1 = c\ell\{x = 0\}$, $L_2 = c\ell\{x + y = 0\}$, $L_3 = c\ell\{y = 0\}$ and L_∞, the line at infinity. The singularities $L_1 \cap L_\infty$, $L_3 \cap L_\infty$, $L_1 \cap L_3$ are all first-order singularities, but $L_2 \cap L_\infty$ is dicritical (locally given by $\eta d\xi - \xi d\eta = 0$). It is so because the line $v = -1$ is collapsed to the point $L_2 \cap L_\infty$ by ϕ^{-1}. The singularity $(0, 0) = L_1 \cap L_3$ needs one blowup to be desingularized.

Now we will apply another pullback to $\phi^*\mathcal{L}$. First of all we take an automorphism ψ of $\mathbb{C}P(2)$ such that $\psi((1, -1)) = (0, 0)$ and the line $c\ell\{y = -1\}$ is not invariant far $\psi^*(\phi^*\mathcal{L})$. Finally we consider the foliation $\phi^*(\psi^*\phi^*\mathcal{L})$; then ϕ^{-1} collapses $c\ell\{y = -1\}$ into a dicritical singularity whose resolution needs more than one blowup (since it involves a singularity like $L_1 \cap L_3$ in the foliation $\phi^*\mathcal{L}$).

In both cases conditions (i), (ii), (iii) are easily verified.

Example 2. Consider the foliation $\mathcal{F}$ defined by a Riccati equation $\omega = (y^2 - \lambda y)dx - R(x)dy = 0$, where

$$\lambda \neq 0, \quad R(x) = \frac{P(x)}{Q(x)}$$

is a rational function with degree $(P) \geq$ degree $(Q) + 2$, and P has only simple roots $x_1, \ldots, x_p$, $p \geq 2$. One sees easily that the "vertical" lines $L_j = c\ell\{x = x_j\}$, $j = 1, \ldots, p$, the line L_∞ at infinity and $L_0 = c\ell\{y = 0\}$, $L_\lambda = c\ell\{y = \lambda\}$ are all invariant for $\mathcal{F}$. We may also assume that the singularities at the intersections $L_j \cap L_0$ and $L_j \cap L_\lambda$ for $j = 1, 2$, are in the Poincaré domain and for $j > 2$ are in Siegel domain (with local holomorphic first integrals); this depends only upon a convenient choice of the polynomials P and Q. At L_∞ there are three singularities: $L_0 \cap L_\lambda \cap L_\infty = \{p_1\}$, the point $p_2 \in L_\infty$ "at infinity" (both radial singularities) and there is another $p_3 \in L_\infty$ whose local expression is given by the equation $udx + xdu = 0$ (that is, with local holomorphic first integral).

The limit set for $\mathcal{F}$ is

$$C = L_0 \cup L_\lambda \cup \bigcup_{j=1}^{n} L_j,$$

an algebraic curve of degree $p + 2$, and there are only two dicritical singularities. Nevertheless we have $\text{mult}_{p_1}(C) + \text{mult}_{p_2}(C) = 2 + p =$ degree(C), violating condition (iii). On the other hand $\mathcal{F}$ is not the pullback of a linear foliation of $\mathbb{C}P(2)$, since one of the separatrices of p_3 is not an algebraic curve.

This example shows that pullbacks of linear equations are too restrictive as a class if we search far a characterization of "simple" limit sets in general.

2. Proof of the Theorem

Let us denote by C the algebraic curve which is the limit set of a (transcendental) leaf F. The blowup of $\mathbb{C}P(2)$ at $p_1, \ldots, p_m$ produces a new surface $\tilde{M}$ and a holomorphic map $\pi\colon \tilde{M} \to \mathbb{C}P(2)$ such that

each $\pi^{-1}(p_j)$, $1 \leq j \leq m$ is a projective line L_j and π restricted to $\tilde{M} \setminus \pi^{-1}(\{p_1, \ldots, p_m\})$ is a diffeomorphism onto $\mathbb{C}P(2) \setminus \{p_1, \ldots, p_m\}$. There are also the strict transform $\tilde{C}$ of C, that is, $\tilde{C} = c\ell(\pi^{-1}(C) \setminus \pi^{-1}(\{p_1, \ldots, p_m\}))$ and the pullback $\pi^*\mathcal{F} = \mathcal{F}^*$ of the foliation $\mathcal{F}$. Since $p_1, \ldots, p_m$ are dicritical singularities satisfying condition (ii), the lines $L_1, \ldots, L_m$ are not leaves of $\mathcal{F}^*$.

Lemma 1. *$\tilde{M} \setminus \tilde{C}$ is a Stein surface.*

Proof. We proceed to verify that $\tilde{M} \setminus \tilde{C}$ has the following two properties (see [**2**], Chapter V).

(a) there exists no analytic compact curve $C_1 \subset \tilde{M} \setminus \tilde{C}$.

(b) given a discrete infinite subset of points $\{q_n\}_{n\in\mathbb{N}}$ in $\tilde{M} \setminus \tilde{C}$, there exists a holomorphic function $\varphi \in \mathcal{O}(\tilde{M} \setminus \tilde{C})$ such that $\overline{\lim_{n\to\infty}} |\varphi(q_n)| = \infty$.

We verify (a) first. Assume by contradiction that there exists $C_1 \subset \tilde{M} \setminus \tilde{C}$, an analytic compact curve; this means that $\pi(C_1)$, which is an algebraic curve in $\mathbb{C}P(2)$, intersects C only at dicritical singularities $p_{\ell_1}, \ldots, p_{\ell_k}$ with distinct tangents. It follows from Bezout's theorem that

$$\sum_{j=1}^{k} \text{mult}_{p_{\ell_j}}(C) \cdot \text{mult}_{p_{\ell_j}}(C_1) = \text{degree } (C) \cdot \text{degree } (C_1). \qquad (*)$$

But

$$\sum_{j=1}^{k} \text{mult}_{p_{\ell_j}}(C) < \text{ degree } (C)$$

and

$$\max_{1\leq j\leq k} \text{mult}_{p_j}(C_1) \leq \text{degree } (C_1),$$

contradicting (*). Therefore there is no curve such as C_1.

Remarks.

(i) an analogous argument shows that $\tilde{C}$ is a connected curve.

(ii) if we look back at Example 2, we see that blowups at the points p_1 and p_2 will separate the strict transform $\tilde{C}$ of C and the strict transform C_1 of L_∞.

Let us now proceed to verify (b). Assume first that $\{q_n\}$ has an accumulation point $\tilde{q} \in \tilde{C} \setminus L_1 \cup \cdots \cup L_m$. We choose affine coordinates in $\mathbb{C}P(2)$ such as to have $q_j = [1: a_j: b_j]$, $0 \le j \le m$, and the vertical direction different from all the straight lines joining q_j to $q = \pi(\tilde{q})$. Finally we may assume also that $[0:0:1] \in C$.

Let $f(x,y) = 0$ be the (reduced) polynomial equation that defines C, and consider the function

$$\varphi(x,y) = f(x,y)^{-1}[(x-a_1)^{d_1}\dots(x-a_m)^{d_m}].$$

Claim. $\varphi \circ \pi \in \mathcal{O}(\tilde{M} \setminus \tilde{C})$ and $\overline{\lim_{n\to\infty}} |\varphi \circ \pi(q_n)| = \infty$. There are three different situations to be analyzed.

(i) $\varphi \circ \pi$ is holomorphic at $\tilde{r} = \pi^{-1}(r)$ where $r \in L_\infty \setminus C$. Putting $u = x^{-1}, v = yx^{-1}$ and $d =$ degree (C) we get:

$$\begin{aligned}\varphi(x,y) &= \frac{[(x-a_1)^{d_1}\dots(x-a_m)^{d_m}]}{[f_0(x,y)+\cdots+f_d(x,y)]}\\ &= \frac{[(x-a_1)^{d_1}\dots(x-a_m)^{d_m}]}{[f_0(u^{-1},u^{-1}v)+\cdots+f_d(u^{-1},u^{-1}v)]}\\ &u^{-(d_1+\cdots+d_m)}\frac{[(1-a_1u)^{d_1}\dots(1-a_mu)^{d_m}]}{[f_0(1,v)+\cdots+u^{-d}f_d(1,v)]}\\ &u^{d-(d_1+\cdots+d_m)}\frac{[(1-a_1u)^{d_1}\dots(1-a_mu)^{d_m}]}{[u^d f_0(1,v)+\cdots+f_d(1,v)]},\end{aligned}$$

where $f_0,\dots,f_d$ are homogeneous polynomials. This last expression is clearly holomorphic at any $r \in L_\infty \setminus C$ (notice that $r \neq [0:0:1]$).

(ii) $\varphi \circ \pi$ is holomorphic at $\pi^{-1}(r)$ when $r \in \mathbb{C}^2 \setminus C$: this is obvious.

(iii) $\varphi \circ \pi$ is holomorphic along any $L_j \setminus \tilde{C}$, $1 \le j \le m$.

For simplicity let us work the proof at L_1, with $p_1 = (0,0)$. Take coordinates $(x, t = x^{-1}y)$ in a neighborhood of L_1. Then:

$$\begin{aligned}\varphi(x,t) &= \frac{x^{d_1}(x-a_2)^{d_2}\dots(x-a_m)^{d_m}}{x^{d_1}[f_{d_1}(1,t)+xf_{d_1+1}(1,t)+\dots]}\\ &= \frac{(x-a_2)^{d_2}\dots(x-a_m)^{d_m}}{[f_{d_1}(1,t)+xf_{d_1}(1,t)+\dots]},\end{aligned}$$

where

$$f(x,y) = \sum_{j=d_1}^{d} f_j(x,y)$$

with all f_j being homogeneous polynomials.

Again this expression is holomorphic around points where $x = 0$ and $f_{d_1}(1,t) \neq 0$ (that is, points in $L_1 \setminus \tilde{C}$).

We must still compute φ in coordinates $(u = y^{-1}x, y)$:

$$\begin{aligned}\varphi(u,y) &= \frac{u^{d_1}y^{d_1}(uy-a_2)^{d_2}\dots(uy-a_m)^{d_m}}{y^{d_1}[f_{d_1}(u,1)+yf_{d_1+1}(u,1)+\dots]}\\ &= \frac{u^{d_1}(uy-a_2)^{d_2}\dots(uy-a_m)^{d_m}}{[f_{d_1}(u,1)+yf_{d_1+1}(u,1)+\dots]}\end{aligned} \qquad (**)$$

and since $f_{d_1}(u,1) \neq 0$ outside $\tilde{C}$, we conclude that φ is holomorphic at $y = 0$, $f_{d_1}(u,1) \neq 0$.

So far we have verified that $\varphi \circ \pi \in \mathcal{O}(\tilde{M} \setminus \tilde{C})$.

In order to check that $\overline{\lim_{n\to\infty}} |\varphi(q_n)| = \infty$, we simply notice that

$$x^{d_1}(x-a_1)^{d_2}\dots(x-a_n)^{d_m} \neq 0$$

at $\tilde{q}$, by the choice of the affine coordinates.

We are now left with the case where the sequence $\{q_n\}$ accumulates only at points of $\tilde{C} \cap (L_1 \cup \cdots \cup L_m)$, say $\tilde{q} \in L_1$. The proof is analogous, but at the very beginning we have to choose the vertical direction not contained in the set of tangent cones of C at the points $p_1,\dots,p_m$ otherwise the expression (**) wouldn't yield $\overline{\lim_{n\to\infty}} |\varphi(q_n)| = \infty$ if $\tilde{q}$ were not supposed to satisfy $f_{d_1}(0,1) \neq 0$. □

Lemma 2. *Let f, g be local holomorphic diffeomorphisms at $0 \in \mathbb{D}$, with $f(0) = g(0) = 0$ and $|f'(0)| < 1$, $g'(0) = 1$. Assume that there exists $x_0 \neq 0$ such that the orbit of x_0 by the action of the pseudo group generated by f and g accumulates only at $0 \in \mathbb{D}$. Then $g \equiv id$.*

Proof. We may assume that $f(z) = \lambda z$, for $|\lambda| < 1$. Let $A \subset \mathbb{D}$ be a fundamental domain for f. If $g \neq id$, we may suppose $g^m(x_0) \to 0$ as $m \to \infty$ (the argument is analogous if $g^{-m}(x_0) \to 0$ as $m \to \infty$). Put $\ell - 1 = \#A \cap \mathcal{O}(x_0)$ and $D_j = \{g^{(j-1)\ell}(x_0),\dots,g^{j\ell-1}(x_0)\}$, $j \in \mathbb{N}$. As

$\#D_j = \ell$, there exist integers $n_j > \overline{n_j}$ in $[(j-1)\ell, j\ell - 1]$ and $m_j \geq \overline{m_j}$ such that

$$\lambda^{-m_j} g^{n_j}(x_0) = \lambda^{-\overline{m_j}} g^{\overline{n_j}}(x_0) \in A.$$

Let $y_j = g^{\overline{n_j}}(x_0)$, it follows that

$$\lambda^{\overline{m_j} - m_j} \; g^{n_j - \overline{n_j}} \; (y_j) = y_j \qquad (***)$$

Since $g'(0) = 1$ and $|f'(0)| < 1$ we have that $\overline{m_j} - m_j$ is bounded. Therefore, from (***) we conclude that there exist integers M, N such that the equality

$$\lambda^M g^N(y_j) = y_j$$

holds for infinitely many $y_j \to 0$, what is absurd. So that $g \equiv id$. □

Lemmas 1 and 2 are now used to go on with the proof of the Theorem. Essentially the steps are the following:

1) the resolution of the singularities of $\tilde{C}$ introduces another surface $\tilde{M}$ with an exceptional divisor E which is the union of projective lines. Any irreducible component of $\tilde{C} \cup E$ will have an attractor in its (virtual) holonomy group, since we may transfer the attractor of that group of one of the components of $\tilde{C}$ to all components "going through" corners of $\tilde{C} \cup E$. Then Lemma 2 guarantees that all these groups are abelian and then can be made linear in convenient coordinate systems. This yields the construction of closed meromorphic 1-forms which represent the foliation in neighborhoods of the components of $\tilde{C} \cup E$ (the polar sets are the components).
2) the next step is to glue together all there 1-forms and, simultaneously, get an extension to $\tilde{M}$. This is possible because of Lemma 1.
3) once we have a meromorphic 1-form which represents the foliation $\mathcal{F}$ outside C (its polar set with order one), it is easy to prove the pullback property.

All the details can be found in [1], pg.431-440.

The simplest situation where the Theorem applies is the following.

Corollary. *Let $\mathcal{F}$ be an analytic foliation of $\mathbb{C}P(2)$ which has some transcendental leaf whose limit set is an algebraic curve C. If $\mathcal{F}$ satisfies*

(a) *the singularities of $\mathcal{F}$ along C are first-order singularities (that is, in their desingularizations no saddle-nodes or eventually dicritical singularities appear) or of radial type (that is, in local coordinates given as $ydx - xdy = 0$).*
(b) *if $r \in \mathbb{N}$ denotes the number of singularities of $\mathcal{F}$ along C of radial type, then $r < \deg(C)$.*
(c) *some irreducible component of C has a hyperbolic attractor in its holonomy group.*

Then it is the pull-back of a linear foliation of $\mathbb{C}P(2)$ by a rational map.

3. A remark about the solvable case

A natural question is: what happens when the condition (iv) of the Theorem is dropped? It is no longer true that the holonomy group of each irreducible component of $\tilde{C} \cup E$ is abelian, since we can not use Lemma 3. Anyway, we may apply a similar argument when a pseudo group has two elements which are tangent to the identity

$$(g_i(z) = z + \sum_{j=2}^{\infty} a_i^{(i)} z^j)$$

but do not commute to prove that no orbit is discrete; since we have a discrete orbit (which correspond to the embedded transcendental leaf) for the holonomy group of each component of $\tilde{C} \cup E$, we conclude that their holonomy groups are solvable ones (as an alternative, we may apply Nakai's Separatrix Theorem [4]). We notice that the transcendental leaf accumulates along all points of C^* (so that we have the conclusion about the holonomy groups of all components) since the singularities of C^* are locally of the form

$$xdy - \lambda ydx + h \cdot o \cdot t = 0, \text{ where } \lambda \in \mathbb{R} \setminus (\mathbb{Q}_+ \cup \{0\}).$$

Now the non-abelian solvable groups are, with a few exceptions, conjugate to ramifications of subgroups of Moebius transformations; this case appears in example 2 (with a minor modification). It is a interesting problem to classify the foliations in this situation.

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