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Strange Attractors in Higher Dimensions

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Abstract. We consider generic one-parameter families of diffeomorphisms on a manifold of arbitrary dimension, unfolding a homoclinic tangency associated to a sectionally dissipative saddle point (the product of any pair of eigenvalues has norm less than 1). We prove that such families exhibit strange attractors in a persistent way: for a positive Lebesgue measure set of parameter values. In the two-dimensional case this had been obtained in a joint work with L. Mora, based on and extending the results of Benedicks-Carleson on the quadratic family in the plane.

1. Introduction

It is a well established fact that, notwithstanding a simple and deterministic formulation, natural systems very often exhibit complicated and apparently erratic dynamical behaviour. The mathematical study of such *chaotic* behaviour gained a renewed impetus in recent years, stimulated by the discovery of a number of notable dynamical phenomena such as the Lorenz-like attractors, the Hénon-like attractors or Feigenbaum and Coullet-Tresser's cascades of bifurcations. These discoveries, obtained mostly in the numerical study of systems modeling natural phenomena, showed that unstable (nonhyperbolic) dynamics is a much more common feature than it was thought at a certain stage. On the other hand, it was Lorenz fundamental contribution to have identified the sensitive dependence of orbits on their initial conditions, exhibited by many relevant systems, as a main source of the unpredictability of their dynamical behaviour. Notions such as this one are central to the way we now try to build-up a mathematical structure to understand complicated dynamics.

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In the one-dimensional context, a substantial comprehension of the mechanisms of chaotic behaviour (namely, existence of absolutely continuous invariant measures with positive Liapounov exponent) was provided by the works of Jakobson [**Ja**], Collet-Eckmann [**CE**], Rees [**Re**] and Benedicks-Carleson [**BC1**], among others. These works showed that such behaviour has a remarkable *persistence*, which must be formulated in *measure-theoretical* terms - positive Lebesgue measure in parameter space - rather than topological ones.

The extension of this study to higher-dimensional systems presents considerable additional difficulties. In [**BC2**] Benedicks-Carleson were able to overcome many of these difficulties and prove that the occurrence of strange attractors is, in the same measure-theoretical sense, a persistent phenomenon in the Hénon (or quadratic) family of diffeomorphisms on the plane. A construction of SRB-invariant measures for these attractors was recently given by Benedicks-Young [**BY**].

Another application of Benedicks-Carleson's methods was made in [**MV**], where their results were generalized to the setting of homoclinic bifurcations on surfaces. More precisely, it was shown that the unfolding of a homoclinic tangency by a generic one-parameter family of surface diffeomorphisms always includes the presence, for a positive measure set of parameter values, of strange attractors or repellers of Hénon type. Homoclinic bifurcations are a main way for the development of complicated dynamics and they are observed in most relevant dynamical systems. Their unfolding is accompanied by a great variety of complex dynamical phenomena including, among others, cascades of period-doubling bifurcations [**YA**], coexistence of infinitely many periodic attractors [**Ne**], creation of saddle-node cycles and, as stated before, persistence of Hénon-like attractors or repellers. We refer the reader to [**PT**] for a detailed exposition on these and related topics.

It is this result of [**MV**] that we now extend to the full generality of homoclinic bifurcations on manifolds of arbitrary dimension. Let us state this in a precise form. We take $f_\mu: M \to M$, $\mu \in \mathbb{R}$, to be a smooth one-parameter family of diffeomorphisms on an $m-$dimensional manifold M, $m \geq 2$, exhibiting a homoclinic tangency associated to

a hyperbolic fixed (or periodic) point p of f_0. We assume that f_0 is *sectionally dissipative* at p, meaning that the product of any pair of eigenvalues of $Df_0(p)$ is less than 1 in absolute value. Then,

Theorem A. *For generic one-parameter families $(f_\mu)_\mu$ as above there is $S \subset \mathbb{R}$ such that*

- *$S \cap (-\varepsilon, \varepsilon)$ has positive Lebesgue measure for every $\varepsilon > 0$;*
- *for all $\mu \in S$, f_μ exhibits nonhyperbolic strange attractors in a* (const $|\mu|$)*$-$neighbourhood of the orbit of tangency.*

As in [**MV**], we define an *attractor* of a transformation f to be a compact, $f-$invariant and transitive set Λ whose basin $W^s(\Lambda) = \{z \in M: \text{dist}(f^n(z), \Lambda) \to 0 \text{ as } n \to +\infty\}$ has nonempty interior. We call the attractor *strange* if it contains a dense orbit $\{f^n(z_1): n \geq 0\}$ displaying exponential growth of the derivative:

$$\|Df^n(z_1)\| \geq e^{cn} \text{ for all } n \geq 0 \text{ and some } c > 0.$$

When proving the theorem we take the point z_1 to be *critical* in the sense that there exists a direction in the tangent space to M at z_1 which is exponentially contracted by both positive and negative iterates of Df_μ. Clearly, the presence of such a point is an obstruction to (uniform) hyperbolicity of the attractor.

For the proof of the theorem we assume the homoclinic tangency to be quadratic and to be generically unfolded by the family $(f_\mu)_\mu$. A few other mild (open and dense) conditions of a somewhat technical nature are also used in Sections 2-3 and are stated there. By *smooth* above we mean that $\Phi: \mathbb{R} \times M \to \mathbb{R} \times M$, $\Phi(\mu, x) = (\mu, f_\mu(x))$, is a C^∞ map; however, a much weaker requirement of differentiability should be sufficient for the conclusion of the theorem. It is also very likely that the statement will remain true if we demand in the definition of attractor that $W^s(\Lambda)$ be a full neighbourhood of Λ. We note, in addition, that our arguments and conclusions are valid even if M is $\infty-$dimensional (a manifold over a Hilbert space).

We point out that the assumption of sectional dissipativeness is in fact a necessary condition for the dynamics of the f_μ near the tangency to contain attractors: otherwise only a nowhere dense set of points remain in the neighbourhood of the orbit of tangency for all positive times. This is also related to the fact that the strange attractors one encounters in this context of homoclinic bifurcations are always topologically one-dimensional. Persistent strange attractors displaying higher-dimensional non-uniform expansion do not seem to have been exhibited yet. It is an interesting problem to give examples of such attractors and to describe mechanisms (bifurcations) through which they can be created.

Although the global structure of the proof of Theorem A follows closely the arguments of [**BC2**] and [**MV**], some difficulties arise when extending these arguments to the present higher-dimensional setting. A main conceptual difference lies in the control of the geometry of the unstable manifold required for the construction of the *critical points*. These are always defined in nearly-straight segments of the unstable manifold. The two-dimensional argument is based on the simple, and yet crucial remark that two such segments which are nearby, must also be nearly parallel, in order to avoid intersecting each other. Clearly, this can not be expected to hold in dimensions greater that two. Instead, we derive the necessary geometric information directly from the binding construction, in Section 7. This also requires that our definition of critical point be somewhat more restrictive than in [**MV**], see Section 6. Another point worth attention concerns the topological characterization of the attractor, namely the fact that its basin has nonempty interior. In the two-dimensional situation this follows in a simple way from the area-dissipativeness (and the Jordan curve theorem). Here we have to combine this with the existence of invariant strong-contracting foliations and with some bound on the geometry of iterates of sections transverse to such foliations, see Section 3. The control of the distributions of contracting hyperplanes, in Section 4, has been improved with respect to [**MV**]; on the other hand, in dimension greater than two these distributions are, in general, not integrable, see Section 5.

This paper is organized as follows. In Section 2 we establish a renormalization procedure which reduces the proof of Theorem A, to proving

it for a special class of families of (nearly one-dimensional) maps, which we call *quadratic-like families*. In Section 3 the attractor is exhibited and characterized as the closure of the unstable manifold of a hyperbolic saddle-point. In Sections 4-5 we develop some main tools (contractive hyperplanes, critical points algorithms) for proving that the attractor contains a dense orbit exhibiting exponential growth of the derivative. The global structure of this proof is described in Section 6. It consists of an induction argument and the content of the induction hypothesis is stated there. Sections 7-8 are devoted to the inductive step of the argument: showing that the properties in the induction hypothesis can be recovered at the next stage, as long as some parameter values are excluded. In Section 9 it is shown that a positive measure set of values of the parameter remain after all the exclusions. Several of the facts in Sections 2-9 can be proved in the same way as in two dimensions and in this case we just refer the reader to the corresponding results in [**MV**] or [**BC2**]. Apart from that, we present here the complete argument to prove Theorem A.

This work corresponds to my doctoral thesis. I am grateful to Prof. J. Palis for his friendship and advise and to my colleagues at IMPA for many pleasant discussions. I am also grateful to the hospitality of the Royal Institute of Technology of Stockholm, where part of this work was done. Finally, I acknowledge partial financial support from CNPq and Fundação Calouste Gulbenkian.

2. Renormalization. Quadratic-like families

First we describe a higher-dimensional version of the renormalization scheme in [**MV, Sec. 2**]. Our argument is a natural extension of the two-dimensional one ([**TY**], [**PT, Ch.3**]) and so we only sketch its main points, leaving the details to the reader. Our definition of *quadratic-(or Hénon-)like family* is somewhat more general than in [**MV**]: here we only require closeness to the family of quadratic endomorphisms, cf. (QL) below.

Let $(f_\mu)_{\mu\in\mathbb{R}}$ be as in the statement of Theorem A. The assumption of sectionally dissipativenes implies that $Df_0(p)$ has a unique expanding eigenvalue which we denote by σ_0; for our purposes it is no restriction to assume that σ_0 is positive and we do so from now on. Recall also that we denote $\Phi\colon \mathbb{R}\times M \to \mathbb{R}\times M$, $\Phi(\mu,x)=(\mu,f_\mu(x))$. In this section we assume that the homoclinic tangency is quadratic and that it is generically unfolded by the family $(f_\mu)_\mu$. For simplicity we assume also that, for μ small, f_μ is C^k–linearizable in a neighbourhood of the point p. Here $k\geq 3$ is a fixed integer and we also consider a constant $A\geq 3$.

Theorem 2.1. *There are $l\geq 1$ and a sequence $\Theta_n\colon [1/A,A]\times[-A,A]^m \to \mathbb{R}\times M$ of C^k diffeomorphisms such that as $n\to+\infty$ the sequence $\varphi_n=\Theta_n^{-1}\circ\Phi^{n+l}\circ\Theta_n$ converges to the map*

$$\phi(a,x,y_1,\dots,y_{m-1})=(a,1-ax^2,0,\dots,0)$$

in the C^k topology.

We describe the construction of Θ_n. Let $(\xi,H)=(\xi,\eta_1,\dots,\eta_{m-1})$ be C^k μ–dependent coordinates on a neighbourhood U of p linearizing f_μ: $f_\mu(\xi,H)=(\sigma_\mu\xi,\Lambda_\mu H)$, with $\sigma_\mu\in\mathbb{R}$, $\Lambda_\mu\in\mathcal{L}(\mathbb{R}^{m-1})$ and $|\sigma_\mu|<1<\lambda_\mu=\|\Lambda_\mu\|$. The assumption of sectional dissipativeness means that, up to a convenient choice of the metric, we have $0<\lambda_\mu\sigma_\mu<1$ for μ close to zero. Clearly, we may take $\{(\xi,H)\colon \|(\xi,H)\|\leq 2\}$ to be contained in U and $q=(1,0^{m-1})=(1,0,\dots,0)$ to be a point in the orbit of tangency. We fix $l\geq 1$ such that $f_0^l(q)=(0,H_0)\in U$ and then we write

$$\begin{aligned} f_\mu^l(\xi,H)=&(\alpha(\xi-1)^2+\beta(\xi-1)\mu+\gamma\mu^2+g\cdot H+v\mu+r,\\ &H_0+F(\xi-1)+G\cdot H+V\mu+R)\end{aligned}$$

where $r=r(\mu,\xi,H)$ and $R=R(\mu,\xi,H)$ are such that

$$r, R, Dr, DR, \partial_{\xi\xi}r, \partial_{\mu\xi}r \quad\text{and}\quad \partial_{\mu\mu}R \quad\text{vanish at}\quad (0,1,0^{m-1}). \tag{1}$$

The hypotheses of nondegeneracy and generic unfolding of the tangency amount to having $\alpha\neq 0$ and $v\neq 0$ and, up to reparametrizing $(f_\mu)_\mu$, we may suppose $v=1$. For the definition of Θ_n we first consider the n–dependent reparametrization

$$\nu=\nu_n(\mu)=\sigma_\mu^{2n}\mu+g\sigma_\mu^{2n}\Lambda_\mu^n H_0-\sigma_\mu^n. \tag{2}$$

It is easy to check that, given any constant A_1 (we will use $A_1\gg A$), for n sufficiently large ν_n maps a small interval I_n close to $\mu=0$ diffeomorphically onto $[-A_1,A_1]$. We let $\mu_n=(\nu_n|I_n)^{-1}$. Then we introduce (n,μ)–dependent coordinates $(\tilde{x},\tilde{Y})$ given by

$$(\xi,H)=\tilde{\theta}_{n,\mu}(\tilde{x},\tilde{Y})=(1+\sigma_\mu^{-n}\tilde{x},\Lambda_\mu^n H_0+\rho_\mu^{-n}\tilde{Y}),\ \text{where}\ \rho_\mu=\sigma_\mu\sqrt{\sigma_\mu/\lambda_\mu}. \tag{3}$$

Now we define $\tilde{\Theta}_n\colon[-A_1,A_1]\times[-A_1,A_1]^m\to\mathbb{R}\times M$ by $\tilde{\Theta}_n(\nu,\tilde{x},\tilde{Y})=(\mu,\xi,H)$ with $\mu=\mu_n(\nu)$ and $(\xi,H)=\tilde{\theta}_{n,\mu}(\tilde{x},\tilde{Y})$. A straightforward calculation gives for $\tilde{\varphi}_n=\tilde{\Theta}_n^{-1}\circ\Phi^{n+l}\circ\tilde{\Theta}_n$

$$\begin{aligned}\tilde{\varphi}_n(\nu,\tilde{x},\tilde{Y})=&(\nu,\alpha\tilde{x}^2+\beta\tilde{x}(\sigma_\mu^n\mu)+\gamma(\sigma_\mu^n\mu)^2+\nu+g\sigma_\mu^{2n}\rho_\mu^{-n}\tilde{Y}+\sigma_\mu^{2n}r,\\ &\rho_\mu^n\sigma_\mu^{-n}\Lambda_\mu^nF\tilde{x}+\rho_\mu^n\Lambda_\mu^nG\Lambda_\mu^nH_0+\Lambda_\mu^nG\tilde{Y}+\rho_\mu^n\Lambda_\mu^nV\mu+\rho_\mu^n\Lambda_\mu^nR)\end{aligned} \tag{4}$$

where r and R are calculated at $(\mu,\xi,H)=\tilde{\Theta}_n(\nu,\tilde{x},\tilde{Y})$. Note that $\sigma_\mu^n\mu=(1+\sigma_\mu^{-n}\nu-g\sigma_\mu^n\Lambda_\mu^nH_0)\to 1$ and $\|\rho_\mu^n\sigma_\mu^{-n}\Lambda_\mu^n\|\leq(\sqrt{\lambda_\mu\sigma_\mu})^n=\sigma_\mu^{2n}\rho_\mu^{-n}\to 0$ as $n\to\infty$. It is also clear that $\|\rho_\mu^n\Lambda_\mu^nG\Lambda_\mu^nH_0\|$, $\|\Lambda_\mu^nG\tilde{Y}\|$ and $\|\rho_\mu^n\Lambda_\mu^nV\mu\|$ converge to zero as $n\to\infty$. Finally, the same holds for $|\sigma_\mu^{2n}r|$ and $\|\rho_\mu^n\Lambda_\mu^nR\|$, as a consequence of (1) and the fact that, recall (2), (3), $|\mu|,|\xi-1|\leq \text{const}\ \sigma_\mu^{-n}\leq\text{const}\ \sigma_0^{-n}$ and $\|H\|\leq\text{const}\ \lambda_\mu^n\leq\text{const}\ \lambda_0^n$ (throughout this section const denotes a positive constant depending only on the family $(f_\mu)_\mu$). This proves that

$$\tilde{\varphi}_n(\nu,\tilde{x},\tilde{Y})\to\tilde{\phi}(\nu,\tilde{x},\tilde{Y})=(\nu,\alpha\tilde{x}^2+\beta\tilde{x}+\gamma+\nu,0^{m-1})$$

as $n\to\infty$ (uniformly on $[-A_1,A_1]\times[-A_1,A_1]^m$). Moreover, the same kind of estimates apply to all derivatives up to order k, proving that this convergence holds in the C^k topology. On the other hand, $\tilde{\phi}$ is conjugated to ϕ by

$$h\colon(a,x,Y)\mapsto\left(-\frac{a}{\alpha}-\frac{\beta}{2\alpha}+\frac{\beta^2}{4\alpha}-\gamma,-\frac{a}{\alpha}x-\frac{\beta}{2\alpha},\tilde{Y}\right). \tag{5}$$

Thus, in order to complete our construction it is now sufficient to take $\Theta_n=\tilde{\Theta}_n\circ h$ (clearly, the domain of definition of Θ_n contains $[1/A,A]\times[-A,A]^m$, as long as A_1 is large enough).

Remark 2.1. Let us remark that the choice of the coefficient $\rho_\mu=$

$\tau_\mu\sqrt{\sigma_\mu/\lambda_\mu}$ in (3) is somewhat arbitrary: the argument works, essentially without change, for any ρ_μ strictly in between σ_μ^2 and σ_μ/λ_μ. Moreover, taking $\rho_\mu = \sigma_\mu^2$ also leads to a convergent sequence $\tilde\varphi_n$, although in this case the limit map is $(\nu, \tilde x, \tilde Y) \mapsto (\nu, \alpha\tilde x^2 + \beta\tilde x + \gamma + \nu + g\cdot\tilde Y, 0^{m-1})$. Let us also describe the results obtained by taking $\rho_\mu = \sigma_\mu/\lambda_\mu$, which are of interest for Section 3. Generically, $Df_0(p)$ has either (I) a unique, real, least contracting eigenvalue, or (II) a pair of complex conjugate least contracting eigenvalues. In the first case we may write

$$\Lambda_\mu = \begin{pmatrix} \tilde\lambda_\mu & 0 \\ 0 & \tilde\Lambda_\mu \end{pmatrix} \quad \text{with } \tilde\lambda_\mu \in \mathbb{R},\ \tilde\Lambda_\mu \in \mathcal{L}(\mathbb{R}^{m-2})$$

and $\|\tilde\Lambda_\mu\| < \lambda_\mu = |\tilde\lambda_\mu|$. Then the sequence $\tilde\varphi_n$ converges to

$$\bar\phi\colon (\nu, \tilde x, \bar y_1, \bar y_2, \dots, \bar y_{m-1}) \mapsto (\nu, \alpha\tilde x^2 + \beta\tilde x + \gamma + \nu, f_1\tilde x + v_1, 0^{m-2}),$$

where $F = (f_1, \dots, f_{m-1})$ and $V = (v_1, \dots, v_{m-1})$. In case (II) we write

$$\Lambda_\mu = \begin{pmatrix} \lambda_\mu\cos\tau & \lambda_\mu\sin\tau & 0 \\ -\lambda_\mu\sin\tau & \lambda_\mu\cos\tau & 0 \\ 0 & 0 & \tilde\Lambda_\mu \end{pmatrix} \quad \text{with } \tau = \tau_\mu \in [0, 2\pi),\ \tilde\Lambda_\mu \in \mathcal{L}(\mathbb{R}^{m-3})$$

and $\|\tilde\Lambda_\mu\| < \lambda_\mu = \|\Lambda_\mu\|$. The same kind of calculations as before show that in this case the C^k–norm of $(\tilde\varphi_n - R_\tau^n \circ \bar\phi)$ converges to zero as $n \to \infty$, where R_τ is the rotation of angle τ in the $(\bar y_1\bar y_2)$–plane and $\bar\phi(\nu, \tilde x, \bar y_1, \bar y_2, \bar y_3, \dots, \bar y_{m-1}) = (\nu, \alpha\tilde x^2 + \beta\tilde x + \gamma + \nu, f_1\tilde x + v_1, f_2\tilde x + v_2, 0^{m-3})$.

Having in mind introducing a convenient notation to be used in the forthcoming sections, we restate the convergence of φ_n in the following way: there is $K > 0$ and for each $b > 0$ there exists $n_0 \geq 1$ such that every $\varphi = \varphi_n$, $n \geq n_0$, satisfies

$$\|\varphi - \phi\|_{C^k(a,x,Y)} \leq K\sqrt{b}. \qquad (QL)$$

Let $\varphi_a = \varphi_{n,a}$ be defined by $\varphi_n(a, x, Y) = (a, \varphi_{n,a}(x, Y))$ and let $D\varphi_{n,a}$ denote its derivative. As an immediate consequence of (QL), the maps φ_a are (strongly) sectionally dissipative. We state this explicitly. Let $\det|(u, v) = \|(u\cdot u)v - u(u\cdot v)\|/\|u\|$ denote the area of the parallelogram generated by vectors $u, v \in \mathbb{R}^m$. Then, for $n \geq n_0$, $a \in [1/A, A]$ and $u, v \in \mathbb{R}^m$,

$$|\det|(D\varphi_a u, D\varphi_a v) \leq Kb|\det|(u, v). \qquad (SD)$$

The previous arguments reduce the proof of Theorem A to proving it for *quadratic-like families*, i.e. for families of diffeomorphisms $\varphi = (\varphi_a)_a$ satisfying (QL) (and (SD)) for some sufficiently small $b > 0$. This is done in Sections 4-9. Before that, in Section 3, we give a topological characterization of the attractor, which requires revisiting the present construction. The value of K is fixed from now on; we take $K \geq 10$ so that, in particular, $\|\varphi\|_{C^k} \leq K$. Moreover, we fix $1/2 < c < c_1 < c_0 < \log 2$: c is to be a lower bound for the rate of exponential growth of the derivative. Our construction also involves small constants $1 \gg \beta \gg \alpha \gg \delta$. (A minor simplification with respect to [**MV**] is that we avoid the use of an extra constant ε: here we always take $\varepsilon = \alpha$). Besides, we fix an interval $\Omega \in (1, 2)$ in parameter space and a large integer N related to it (N is the first return time for the φ_a, $a \in \Omega$); Ω is taken close enough to $a = 2$ so that $N \gg 1/\delta$. Finally, b is always assumed to be small with respect to any of these constants.

3. The attractor

We keep the notations of the previous section. Let $\Omega \subset (1, 2)$ be a compact interval and $R = \{(x, Y)\colon -1 - \varepsilon_1 \leq x \leq 1 + \varepsilon_1 \ and \ \|Y\| \leq 1\}$. Here we suppose $\varepsilon_1 > 0$ small with respect to $2 - a_0$, $a_0 = \sup\Omega$, so that $\phi_a(R) \subset (-1 - \varepsilon_1, 1 + \varepsilon_1) \times \{0^{m-1}\}$ for all $a \in \Omega$. It follows that $\varphi_{n,a}(R) \subset \text{interior}(R)$ for all $a \in \Omega$, as long as n is sufficiently large. We also note that $\varphi_{n,a}$ has a hyperbolic fixed point $P_{n,a}$ in the interior of R, which is just the continuation of the fixed point $((\sqrt{1+4a} - 1)/2a, 0^{m-1})$ of ϕ_a. We let $\Lambda_{n,a} = \text{closure}(W^u(P_{n,a}))$ and denote $W^s(\Lambda_{n,a}) = \{z \in \mathbb{R}^m\colon \text{dist}(\varphi_{n,a}^j(z), \Lambda_{n,a}) \to 0 \ as \ j \to +\infty\}$. The main result in this section states that, as long as n is large enough, $W^s(\Lambda_{n,a})$ contains a nontrivial open set.

Theorem 3.1. *$W^s(\Lambda_{n,a})$ has nonempty interior for all $a \in \Omega$.*

The proof of this theorem requires two more generic conditions on the family $(f_\mu)_\mu$, which we state below; the remaining sections are independent of these conditions. Let $1 \leq w \leq m - 1$ be such that the contractive eigenvalues of $Df_0(p)$ satisfy $|\lambda_1| = \cdots = |\lambda_w| > |\lambda_{w+1}| \geq$

$\cdots \geq |\lambda_{m-1}|$. We identify the neighbourhood U of p with an open subset of $\mathbb{R}^m$ via the linearizing coordinates (ξ, H) and we consider the splitting $T_U\mathbb{R}^m = E^u \oplus E^w \oplus E^{ss}, E^u(z) = \mathbb{R} \times \{0^{m-1}\}, E^w(z) = \{0\} \times \mathbb{R}^w \times \{0^{m-1-w}\}, E^{ss}(z) = \{0^{1+w}\} \times \mathbb{R}^{m-1-w}$, induced by this identification. Clearly, we may take $H = (\eta_1, \ldots, \eta_{m-1})$ in such a way that the expression of $Df_\mu(p)$ with respect to this splitting is

$$Df_\mu(p) = \begin{pmatrix} \sigma_\mu & 0 & 0 \\ 0 & \Lambda_\mu^w & 0 \\ 0 & 0 & \Lambda_\mu^s \end{pmatrix} \text{ with } \Lambda_\mu^w \in \mathcal{L}(E^w), \Lambda_\mu^s \in \mathcal{L}(E^{ss})$$

and, up to a convenient choice of the metric on $\mathbb{R}^m$,

$$\|\Lambda_\mu^s\| < \|(\Lambda_\mu^w)^{-1}\|^{-1} \leq \|\Lambda_\mu^w\|.$$

For the proof of the next result we need the following open and dense condition (where $l \geq 1$ and $q = (1, 0^{m-1})$ are as in the previous section):

$$Df_0^l(q) \cdot (E^u \oplus E^w) \text{ is transversal to } E^{ss} \tag{1}$$

or, equivalently,

$$Df_0^{-l}(f_0^l(q)) \cdot E^{ss} \text{ is transversal to } E^u \oplus E^w \tag{2}$$

In what follows we denote $Q = [-A, A]^m$.

Proposition 3.2. *There is $0 < \tau < 1$ and for each $a \in (1,2)$ and $n \geq 1$ sufficiently large there exists a continuous splitting $T_Q\mathbb{R}^m = E_{n,a}^{uw} \oplus E_{n,a}^{ss}$ satisfying*

(a) $\dim E_{n,a}^{uw}(z) = 1 + w$ *and* $\dim E_{n,a}^{ss}(z) = m - 1 - w$ *for* $z \in Q$;

(b) $D\varphi_{n,a}(z) \cdot E_{n,a}^*(z) = E_{n,a}^*(\varphi_{n,a}(z)), * = uw$ *or* ss, *for* $z \in Q \cap \varphi_{n,a}^{-1}(Q)$;

(c) $\|D\varphi_{n,a}(z)|E_{n,a}^{ss}(z)\| \leq \tau^n$ *for* $z \in Q$;

(d) $\|D\varphi_{n,a}(z)|E_{n,a}^{ss}(z)\| \cdot \|(D\varphi_{n,a}(z)|E_{n,a}^{uw}(z))^{-1}\| \leq \tau^n$ *for* $z \in Q \cap \varphi_{n,a}^{-1}(Q)$.

Moreover, $E_{n,a}^{ss}$ admits an integral foliation $\mathcal{F}_{n,a}^{ss}$

Proof. We describe the construction of $E_{n,a}^{ss}$. This follows from a standard fixed-point argument and we start with a preliminary remark. By (2) we may write for μ small $Df_\mu^{-l}(f_\mu^l(q)) \cdot E^{ss} = \text{graph}(\hat{u}_\mu, \hat{W}_\mu)$, with $(\hat{u}_\mu, \hat{W}_\mu) \in \mathcal{L}(E^{ss}, E^u \oplus E^w)$. We denote by $\hat{E}_\mu$ the parallel subbundle of $T_Q\mathbb{R}^m$ given by $\hat{E}_\mu(z) = \text{graph}(0, \hat{W}_\mu)$ and fix $\varepsilon_2 > 0$ small. Suppose that $z \in Q \cap \varphi_{n,a}(Q)$ and F is an $(m-1-w)$–subspace of $T_z\mathbb{R}^m$ such that

$\angle(F, \hat{E}_\mu) \leq \varepsilon_2$, meaning that $F = \text{graph}(u, W)$ with $\|(u, W - \hat{W}_\mu)\| \leq \varepsilon_2$. We claim that $\angle(D\varphi_{n,a}^{-1} \cdot F, \hat{E}_\mu)$ is also bounded by ε_2. In order to prove this we note first that

$$\begin{aligned} F_2 &= Df_\mu^{-n} D\theta_{n,\mu} \cdot F \\ &= \text{graph}(-\frac{a}{\alpha}\left(\frac{\rho_\mu}{\sigma_\mu^2}\right)^n u \cdot (\Lambda_\mu^s)^n, (\Lambda_\mu^w)^{-n} \cdot W \cdot (\Lambda_\mu^s)^n) \end{aligned} \tag{3}$$

and hence the angle between F_2 and E^{ss} can be made arbitrarily small by taking n sufficiently large. Therefore,

$$F_3 = Df_\mu^{-l} \cdot F_2 = \text{graph}(\tilde{u}_\mu, \tilde{W}_\mu) \text{ with } \|(\tilde{u}_\mu - \hat{u}_\mu, \tilde{W}_\mu - \hat{W}_\mu)\| \ll \varepsilon_2.$$

and so the claim follows from

$$D\varphi_{n,a}^{-1} \cdot F = D\theta_{n,\mu}^{-1} \cdot F_3 = \text{graph}(-\frac{\alpha}{a}\left(\frac{\sigma_\mu}{\rho_\mu}\right)^n \tilde{u}_\mu, \tilde{W}_\mu). \tag{4}$$

We also note that vectors in F are strongly expanded by $D\varphi_{n,a}^{-1}$

$$\|D\varphi_{n,a}^{-1} \cdot v\| \geq \text{const } \|\Lambda_\mu^s\|^{-n}\|v\| \text{ for every } v \in F. \tag{5}$$

Observe now that

$$Q \cap \varphi_{n,a}^{-1}(Q) = \{(x, Y) : g_{n,-}(Y) \leq x \leq g_{n,+}(Y)\}$$

with $g_{n,\pm}$ uniformly close to $\pm\sqrt{(1+A)/a}$ if n is large. Let an auxiliary $(m-1-w)$– subbundle $\tilde{E}_\mu$ on $Q \backslash \varphi_{n,a}^{-1}(Q)$ be constructed as follows. For $z \in \partial^u = \{(x, Y) : |x| = A\}$ we set simply $\tilde{E}_\mu(z) = \hat{E}_\mu(z)$. If $z \in \partial_\pm = \{(x, Y) : x = g_{n,\pm}(Y)\}$ then $\varphi_{n,a}(z) \in \partial^u$ and we take $\tilde{E}_\mu(z) = D\varphi_{n,a}^{-1} \cdot \tilde{E}_\mu(\varphi_{n,a}(z))$. By the claim above

$$\angle(\tilde{E}_\mu(z), \hat{E}_\mu(z)) \leq \varepsilon_2 \tag{6}$$

for every $z \in \partial^u \cup \partial_\pm$. Then, clearly, $\tilde{E}_\mu$ may be extended to a smooth subbundle on $Q \backslash \varphi_{n,a}^{-1}(Q)$ in such a way that property (6) is preserved. From now on $\tilde{E}_\mu$ denotes this extension. Now we consider the space $\mathcal{X}$ of all continuous $(m-1-w)$–dimensional subbundles E of $T_Q\mathbb{R}^m$ such that

- $\angle(\tilde{E}_\mu(z), \hat{E}_\mu(z)) \leq \varepsilon_2$ for every $z \in Q$;
- $E(z) = \tilde{E}_\mu(z)$ for $z \in Q \backslash \varphi_{n,a}^{-1}(Q)$;

and define the graph-transform operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ by

- $(\mathcal{G}E)(z) = D\varphi_{n,a}^{-1} \cdot E(\varphi_{n,a}(z))$ if $z \in Q \cap \varphi_{n,a}^{-1}(Q)$;
- $(\mathcal{G}E)(z) = \tilde{E}_\mu(z)$, otherwise.

The argument in the remark above shows that $\mathcal{G}$ is indeed well-defined, i.e. $\mathcal{G}(\mathcal{X}) \subset \mathcal{X}$. Moreover, the same calculations (recall (3), (4)) also imply that $\mathcal{G}$ is a contraction with respect to the sup-norm on $\mathcal{X}$. We take $E_{n,a}^{ss}$ to be its fixed point. Properties (a) and (b) in the statement follow immediately from our construction and property (c) is also an easy consequence:

$$\|D\varphi_{n,a}(z)|E_{n,a}^{ss}(z)\| \le \text{const } \|\Lambda_\mu^s\|^n,$$

recall (5). On the other hand, $E_{n,a}^{uw}$ can be constructed by a dual procedure (just iterating forward, instead of backwards). We get,

$$\|(D\varphi_{n,a}|E_{n,a}^{uw}(z))^{-1}\| \le \text{const } \|(\Lambda_\mu^w)^{-1}\|^n,$$

and property (d) follows immediately. In order to show that $E_{n,a}^{ss}$ is integrable, we note first that $\tilde{E}_\mu$ may be chosen to be integrable, let $\tilde{\mathcal{F}}_\mu$ be its integral foliation. Then we consider the space $\hat{\mathcal{X}}$ of all foliations $\mathcal{F}$ of Q whose tangent bundle $T\mathcal{F}$ belongs to $\mathcal{X}$ and define the graph transform $\hat{\mathcal{G}}: \hat{\mathcal{X}} \to \hat{\mathcal{X}}$ by

- $(\hat{\mathcal{G}}\mathcal{F})(z) =$ connected component of $D\varphi_{n,a}^{-1}(\mathcal{F}(\varphi_{n,a}(z)))$ containing z, if $z \in Q \cap \varphi_{n,a}^{-1}(Q)$;
- $(\hat{\mathcal{G}}\mathcal{F})(z) = \tilde{\mathcal{F}}_\mu(z)$, otherwise.

This is a contraction with respect to the sup-distance between tangent bundles and, clearly, the fixed point $\mathcal{F}_{n,a}^{ss}$ of $\hat{\mathcal{G}}$ must satisfy $T\mathcal{F}_{n,a}^{ss} = E_{n,a}^{ss}$. □

Remark 3.1. For future use we note that, by construction (recall (3), (4)),

$$\angle(E_{n,a}^{ss}(z), \tilde{E}_\mu) \le \text{const} \left(\left(\frac{\rho_\mu \|\Lambda_\mu^s\|}{\sigma_\mu^2} \right)^n + (\|(\Lambda_\mu^w)^{-1}\| \|\Lambda_\mu^s\|)^n + \left(\frac{\sigma_\mu}{\rho_\mu} \right)^n \right)$$

converges uniformly to zero as $n \to \infty$. Analogously, if n is large then $E_{n,a}^{uw}(z)$ is uniformly close to $E^{uw} = \mathbb{R}^2 \times \{0^{m-1}\}$.

Remark 3.2. A standard argument from hyperbolicity theory (see [**HP**])

shows that $E_{n,a}^{uw}$ and $E_{n,a}^{ss}$ are even Hölder-continuous but we will not need this fact here.

Clearly, $W^s(\Lambda_{n,a})$ is a union of leaves of $\mathcal{F}_{n,a}^{ss}$. Therefore, the theorem will be proved if we can find a transversal section of $\mathcal{F}_{n,a}^{ss}$ which is contained in $W^s(\Lambda_{n,a})$. For the sake of simplicity, we introduce here a last generic assumption on our family of diffeomorphisms: either (I) $w = 1$ and λ_1 is real or (II) $w = 2$ and λ_1, λ_2 are complex conjugate.

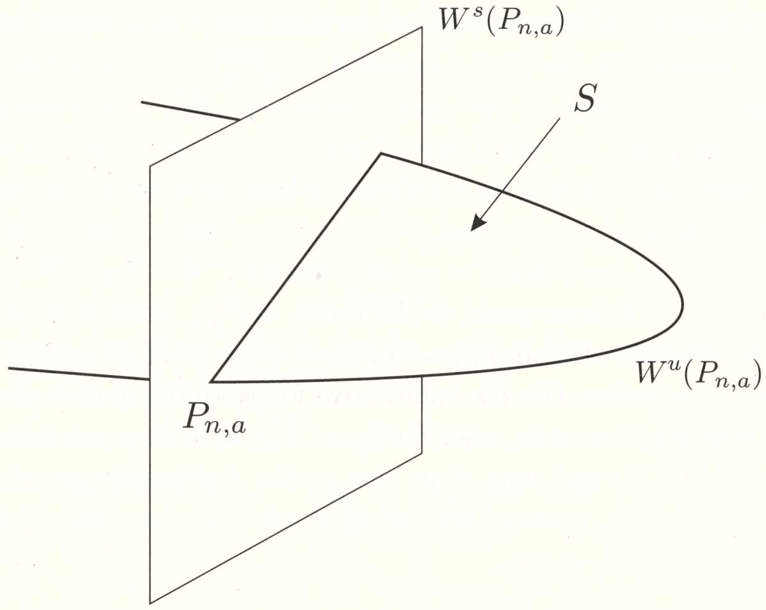

Figure 1

We treat case (I) in the following way. First, we construct a compact surface with boundary $S = S_{n,a} \subset R$, transversal to the foliation $\mathcal{F}_{n,a}^{ss}$ at every point and such that $\partial S_{n,a} \subset W^u(P_{n,a}) \cup W^s(P_{n,a})$. Then we show that for some $c > 0$ and every $j \ge 0$, $r > 0$ and $w \in \varphi_{n,a}^j(S)$ one has

$$\text{area}(B_r^{(j)}(w)) \ge cr^2 \quad if \quad B_r^{(j)}(w) \cap \partial(\varphi_{n,a}^j(S)) = \varnothing, \tag{7}$$

where $B_r^{(j)}(w)$ denotes the closed ball in $\varphi_{n,a}^j(S)$, with respect to the riemannian metric induced by the euclidean metric of R. Finally, we observe that these properties imply $S \subset W^s(\Lambda_{n,a})$.

In order to describe the construction of S and check its transversality to $\mathcal{F}^{ss}_{n,a}$ it is convenient to introduce auxiliary coordinates $(\tilde{x}, \bar{Y})$, where $\tilde{x}$ is as before and $\bar{Y} = (\bar{y}_1, \dots, \bar{y}_{m-1}) = (\sigma_\mu/\lambda_\mu)^n (H - \Lambda^n_\mu H_0)$. Note that this corresponds to taking $\rho_\mu = (\sigma_\mu/\lambda_\mu)$ in 2.3 and then, recall Remark 2.1, the expression of Φ^{n+l} in coordinates $(\nu, \tilde{x}, \bar{Y})$ converges to

$$\bar{\phi}: (\nu, \tilde{x}, \bar{y}_1, \dots, \bar{y}_{m-1}) \mapsto (\nu, \alpha\tilde{x}^2 + \beta\tilde{x} + \gamma + \nu, f_1\tilde{x} + v_1, 0^{m-2})$$

as $n \to \infty$. It is easy to check that assumption (1) above implies $f_1 \neq 0$. Hence, the unstable manifolds of the periodic points of $\bar{\phi}$ lie in a nondegenerate parabola in the $(\tilde{x}, \bar{y}_1)$−plane $\mathbb{R}^2 \times \{0^{m-1}\}$; on the other hand, local stable manifolds are just hyperplanes $\{\tilde{x}\} \times \mathbb{R}^{m-1}$. Recalling also that local invariant manifolds depend continuously on the map, it is now easy to see that, cf. figure, for n sufficiently large there exist $\tilde{S}$ a compact domain in the $(\tilde{x}, \bar{y}_1)$−plane and $g: \tilde{S} \ni (\tilde{x}, \bar{y}_1) \mapsto (\bar{y}_2, \dots, \bar{y}_{m-1})$ close to the null function, such that $S = \operatorname{graph}(g)$ has $\partial S \subset W^u(P_{n,a}) \cup W^s(P_{n,a})$. Observe that S is transversal to the leaves of $\mathcal{F}^{ss}_{n,a}$, since it is close to the $(\tilde{x}, \bar{y}_1)$−plane whereas $E^{ss}_{n,a} = T\mathcal{F}^{ss}_{n,a}$ is close to $\hat{E}_\mu = \operatorname{graph}(0, \hat{W}_\mu)$ in $(\tilde{x}, \bar{Y})$ coordinates; this last affirmative follows directly from the fact that Remark 3.1 remains valid for $\rho_\mu = (\sigma_\mu/\lambda_\mu)$. Now we return to coordinates (x, Y) and note that, due to the domination property in Proposition 3.2(d), positive iterates of S are uniformly transversal to $\mathcal{F}^{ss}_{n,a}$: actually, as $j \to +\infty$

$$|\text{angle}|(T_z\varphi^j_{n,a}(S), E^{uw}_{n,a}(z)) \to 0, \text{ uniformly on } z \in \varphi^j_{n,a}(S).$$

The uniform lower bound (7) for the area of balls follows from this, together with Remark 3.1. Observe then that, by sectional dissipativeness (SD), $\text{area}(\varphi^j_{n,a}(S)) \to 0$ and so, in view of (7),

$$\sup\{\text{dist}(z, \partial(\varphi^j_{n,a}(S))): z \in \varphi^j_{n,a}(S)\} \to 0 \text{ as } j \to +\infty.$$

Since $\partial(\varphi^j_{n,a}(S)) \subset W^u(P_{n,a}) \cup W^s(P_{n,a})$ and

$$\text{diam}(\partial(\varphi^j_{n,a}(S)) \cap W^s(P_{n,a})) \to 0,$$

it follows that, for every $w \in S$, $\text{dist}(\varphi^j_{n,a}(w), \Lambda_{n,a}) \to 0$ as $j \to +\infty$. This means that $S \subset W^s(\Lambda_{n,a})$, as we wanted to prove.

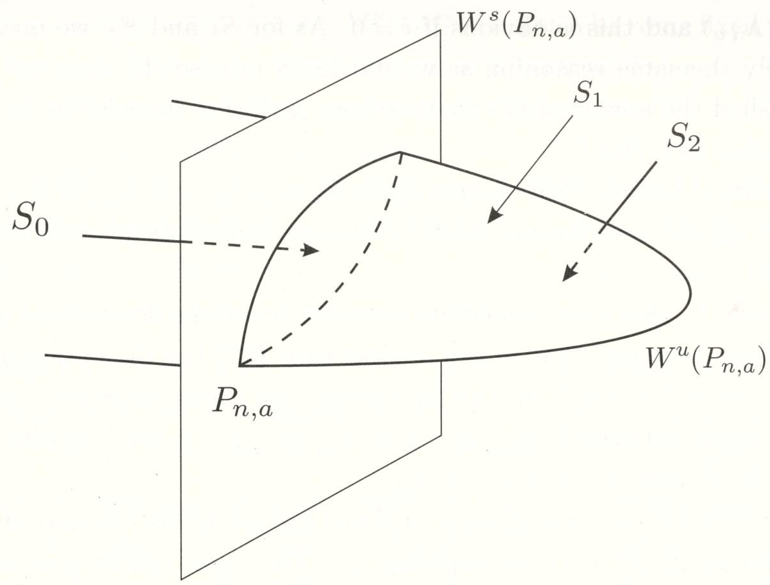

Figure 2

A variation of the same idea applies in case (II). We take $(\tilde{x}, \bar{Y})$ as above and then, according to Remark 2.1, the expression $\tilde{\varphi}_n$ of Φ^{n+l} in the coordinate system $(\nu, \tilde{x}, \bar{Y})$ satisfies $\lim_{n\to\infty} \|\tilde{\varphi}_n - R^n_\tau \circ \bar{\phi}\| = 0$, where

$$R^n_\tau \circ \bar{\phi}: (\nu, \tilde{x}, \bar{y}_1, \dots, \bar{y}_{m-1}) \mapsto (\nu, \alpha\tilde{x}^2 + \beta\tilde{x} + \gamma + \nu,$$
$$(f_1\tilde{x} + v_1)\cos n\tau - (f_2\tilde{x} + v_2)\sin n\tau,$$
$$(f_1\tilde{x} + v_1)\sin n\tau + (f_2\tilde{x} + v_2)\cos n\tau, 0^{m-3}).$$

Note also that in the present situation (1) implies $(f_1, f_2) \neq (0, 0)$. Therefore, for large n the local unstable manifold of $P_{n,a}$ is close to a nondegenerate (n−dependent) parabola contained in the $(\tilde{x}, \bar{y}_1, \bar{y}_2)$−space, whereas its local stable manifold is close to a hyperplane $\{\tilde{x}\} \times \mathbb{R}^{m-1}$. It follows that there is a compact domain $\tilde{S}$ in the $(\tilde{x}, \bar{y}_1, \bar{y}_2)$−space and a smooth function $g: \tilde{S} \ni (\tilde{x}, \bar{y}_1, \bar{y}_2) \mapsto (\bar{y}_3, \dots, \bar{y}_{m-1})$ close to zero, such that (cf. figure) the boundary of $S = \operatorname{graph}(g)$ is the union of three closed surfaces S_0, S_1, S_2 satisfying $S_0 \subset W^s(P_{n,a})$ and $\partial S_i \subset W^u(P_{n,a}) \cup W^s(P_{n,a})$ for $i = 1, 2$. The same argument as in the previous case shows that S is transversal to $\mathcal{F}^{ss}_{n,a}$ and so we are left to prove that $S \subset W^s(\Lambda_{n,a})$. For this it is sufficient to show that each S_i is contained in $W^s(\Lambda_{n,a})$ and this is obvious if $i=0$. As for S_1 and S_2, we may use precisely the same reasoning as we did for S in case (I), once we have established the analog of (7) for these two surfaces. In order to do this, we observe first that

(a) the angle between the tangent space $T_z\varphi^j_{n,a}(S_i)$, $z\in\varphi^j_{n,a}(S_i)$, $j\geq 0$, and $E^{ss}_{uw}(z)$ is uniformly small (even goes uniformly to zero, as $j\to+\infty$).

As in case (I), this is an immediate consequence of the domination property. Note, however, that in the present case $E^{uw}_{n,a}$ has dimension 3 and so (7) does not follow from this fact alone. We claim, moreover, that

(b) the angle between $T_z\varphi^j_{n,a}(S_i)$, $z\in\varphi^j_{n,a}(S_i)$, $j\geq 0$, and the direction of the $x-$axis is uniformly small.

Combined with (a), this gives a uniform (cylindric-like) bound on the geometry of $\varphi^j_{n,a}(S_i)$, $j\geq 0$, implying (7). The claim can be justified as follows. First, going back to the construction above, one notes that S_i may be taken so that in $(\tilde{x},\bar{Y})$ coordinates

(b_0) the angle between T_zS_i, $z\in S_i$, and the direction of the $x-$axis is uniformly small.

Then the same holds in (x,Y) coordinates, since changing from $(\tilde{x},\bar{Y})$ to (x,Y) coordinates does not increase (even diminishes) angles with respect to the $x-$direction. Finally, one checks from the form of

$$D\varphi_{n,a}=\begin{pmatrix} 2\alpha\tilde{x}+\beta\mu\sigma^n_\mu+\sigma^n_\mu\partial_\xi r & (-\alpha/a)(\sqrt{\lambda_\mu\sigma_\mu})^n(g+\partial_H r) \\ (-a/\alpha)(\sqrt{\sigma_\mu/\lambda_\mu})^n\Lambda^n_\mu(F+\partial_\xi R) & \Lambda^n_\mu(G+\partial_H R)\end{pmatrix}$$

that for two-dimensional subspaces whose angle to E^{uw} is small, the property in (b_0) is preserved by positive iteration. In view of (a) this completes the proof of the claim and so also of Theorem 3.1.

4. Contractive hyperplanes

Let, from now on, $\varphi=(\varphi_a)_a$ be a quadratic-like family, i.e. a family of diffeomorphisms satisfying properties (QL) and (SD) of Section 2, for a sufficiently small $b>0$. Here we construct for such a family the higher-dimensional analog of the contractive directions in **[BC2]** and [MV]. For $a\in(1,2)$, $z_1\in R$ and $\nu\geq 0$ we denote $M^\nu=M^\nu(a,z_1)=$

$D\varphi^\nu_a(z_1)$ and $w_\nu=w_\nu(a,z_1)=M^\nu(1,0^{m-1})$. Let λ be some fixed positive number, $\sqrt{b}\ll\lambda\leq 2$. The point z_1 is *$\lambda-$expanding up to time n* if $\|w_\nu\|\geq\lambda^\nu$ for all $1\leq\nu\leq n$. For such a z_1 we let $f^{(\nu)}=f^{(\nu)}(a,z_1)$ be a maximally expanding norm-1 vector: $\|M^\nu f^{(\nu)}\|\geq\|M^\nu u\|$ for all $u\in\mathbb{R}^m$ with $\|u\|=1$, in particular $\|M^\nu f^{(\nu)}\|\geq\lambda^\nu$. The $\nu-$th *contractive hyperplane* at z_1 is $E^{(\nu)}=E^{(\nu)}(a,z_1)=\{f^\nu(a,z_1)\}^\perp$. Note that $M^\nu E^{(\nu)}=\{M^\nu f^{(\nu)}\}^\perp$ and, by sectional dissipativeness, $\|M^\nu e\|\leq(Kb/\lambda)^\nu$ for every norm-1 vector $e\in E^{(\nu)}$.

Lemma 4.1. *There is $K_1=K_1(K,\lambda)$ such that for every $1\leq\mu\leq\nu\leq n$*

(a) $|\text{angle}|(E^{(\mu)},E^{(\nu)})=|\text{angle}|(f^{(\mu)},f^{(\nu)})\leq(K_1b)^\mu$;

(b) $\|M^\mu e\|\leq(K_1b)^\mu$ *for all* $e\in E^{(\nu)}$ *with* $\|e\|=1$.

Proof. In order to prove (a) it is sufficient to show that for every $2\leq\nu\leq n$ we have $|\text{angle}|(E^{(\nu-1)},E^{(\nu)})\leq K_1'(K_1''b)^{\nu-1}$, with K_1',K_1'' depending only on K and λ. Suppose $E^{(\nu-1)}\neq E^{(\nu)}$ and let $e^{(\nu-1)}\in E^{(\nu-1)}$, $e^{(\nu)}\in E^{(\nu)}$ be norm-1 vectors orthogonal to the intersection $E^{(\nu-1)}\cap E^{(\nu)}$. We write $e^{(\nu-1)}=\alpha e^{(\nu)}+\beta f^{(\nu)}$ and then $|\beta|\lambda^\nu\leq\|M^\nu e^{(\nu-1)}\|\leq K(Kb/\lambda)^{\nu-1}$. Thus,

$$\begin{aligned}|\text{angle}|(E^{(\nu-1)},E^{(\nu)})&=|\text{angle}|(e^{(\nu-1)},e^{(\nu)})\\&=|\arctan\frac{\beta}{\alpha}|\leq\frac{2K}{\lambda}\left(\frac{Kb}{\lambda^2}\right)^{\nu-1}.\end{aligned}$$

This proves (a) and (b) is now an easy consequence. □

We prove that at expanding points the contractive hyperplanes $E^{(\nu)}$ are nearly vertical (nearly parallel to the $Y-$plane $\{0\}\times\mathbb{R}^{m-1}$) and, moreover, vectors close to the horizontal direction are nearly maximally expanding. For $(v,V)\in\mathbb{R}\times\mathbb{R}^{m-1}$ we define $|\text{slope}|(v,V)=\|V/v\|$.

Lemma 4.2. *Let z_1 be such that $\|M^\nu u_0\|\geq\lambda^\nu$ for $1\leq\nu\leq n$ and some norm-1 vector u_0. Then, defining $f^{(\nu)}$ as before,*

(a) $|\text{slope}|(f^{(\nu)})\leq K_2\sqrt{b}$ *for some $K_2=K_2(K,\lambda)>0$ and every $1\leq\nu\leq n$;*

(b) $\|M^\nu u\|\geq\frac{1}{2}\|M^\nu\|$ *for $1\leq\nu\leq n$ and every norm-1 vector u with* $|\text{slope}|(u)\leq 1/10$.

Proof. Let $z_1=(x_1,Y_1)$. The assumption $\|Mu_0\|\geq\lambda\gg\sqrt{b}$ implies

$|x_1| \geq \lambda/5$ and so, by (QL), $|\,\text{slope}\,|(f^{(1)}) \leq \text{const}\,\sqrt{b}$. Then the same holds for every $f^{(\nu)}, 1 \leq \nu \leq n$, as a consequence of Lemma 4.1(a). This proves (a). Let now u be as in (b). Then we may write $u = \alpha e + \beta f^{(\nu)}$ with $e \in E^{(\nu)}$, $|\alpha| < 1$, $|\beta| > 2/3$ and so $\|M^\nu u\| \geq (2/3)\|M^\nu f^{(\nu)}\| - (K_1 b)^\nu \geq (1/2)\|M^\nu f^{(\nu)}\|$. □

In the sequel we denote $\xi_j = \varphi_a^j(\xi_0)$ whenever $\xi_0 \in R$ and $j \geq 1$.

Lemma 4.3. *Let $z_0, \zeta_0 \in R$ and let u, v be norm-1 vectors satisfying $\|z_0 - \zeta_0\| \leq \sigma^n$ and $\|u - v\| \leq \sigma^n$ for some $\sigma \leq (\lambda/10K^2)^2$. If $1 \leq \nu \leq n$ is such that $\|M^\nu(a, z_1)u\| \geq \lambda^\nu$ then*

(a) $\frac{1}{2} \leq \|M^\nu(a, z_1)u\|/\|M^\nu(a, \zeta_1)v\| \leq 2$

(b) $|\text{angle}|(M^\nu(a, z_1)u, M^\nu(a, \zeta_1)v) \leq (\sqrt{\sigma})^{2n-\nu} \leq (\sqrt{\sigma})^n$.

In particular, (a) and (b) hold for every $1 \leq \nu \leq n$ if z_1 is $\lambda-$expanding up to time n.

Proof. Analogous to [**MV, Lemma 6.3**]. □

Lemma 4.4. *Let $z_1, \zeta_1 \in R$ be such that z_1 is $\lambda-$expanding up to time n and $\|z_\nu - \zeta_\nu\| \leq \sigma^\nu$ for every $1 \leq \nu \leq n$, with $\sqrt{b} \leq \sigma \leq (\lambda/10K^2)^4$. Then, for $1 \leq \nu \leq n$ and any norm-1 vectors u, v satisfying $|\,\text{slope}\,|(u) \leq 1/10$, $|\,\text{slope}\,|(v) \leq 1/10$,*

(a) $\frac{1}{2} \leq \|M^\nu(a, z_1)u\|/\|M^\nu(a, \zeta_1)v\| \leq 2$

(b) $|\text{angle}|(M^\nu(a, z_1)u, M^\nu(a, \zeta_1)v) \leq (K_4\sqrt{\sigma})^{\nu+1}$,

for some $K_4 = K_4(K, \lambda)$

Proof. Analogous to [**MV, Lemma 6.4**], recall also Lemma 4.2 above. □

In what follows, we denote $Z = (a, z) = (a, x, Y) \in (1, 2) \times R$. For the sake of simplicity, we also use $D_Z = D_{(a,x,Y)}$ to denote derivation with respect to all variables $(a, x, Y) = (a, x, y_1, \ldots, y_{m-1})$. The main result in this section is

Lemma 4.5. *There is $K_5 = K_5(K, \lambda)$ such that, if z is $\lambda-$expanding up to time n then $\|D_Z f^{(\nu)}(Z)\| \leq K_5\sqrt{b}$ for every $1 \leq \nu \leq n$.*

In order to prove this lemma we deduce first a few auxiliary estimates. Let M_* denote the adjoint operator of M: $M_* u \cdot v = u \cdot Mv$ for all $u, v \in \mathbb{R}^m$. We denote $g^{(\nu)} = M^\nu f^{(\nu)}/\|M^\nu f^{(\nu)}\|$. Note that

$M_*^{-\nu} f^{(\nu)}$ is colinear to $M^\nu f^{(\nu)}$, since $\{M^\nu f^{(\nu)}\}^\perp = M^\nu\{f^{(\nu)}\}^\perp$, and so we also have $g^{(\nu)} = M_*^{-\nu} f^{(\nu)}/\|M_*^{-\nu} f^{(\nu)}\|$. Observe, moreover, that $\|M_*^{-\nu} f^{(\nu)}\| = \|M^\nu f^{(\nu)}\|^{-1} \leq \lambda^{-\nu}$.

Lemma 4.6. *There is $K_6 = K_6(K, \lambda) > 0$ such that if z is $\lambda-$expanding up to time n then*

(a) $\|D_Z f^{(\nu)}(Z)\| \leq K_6^\nu$ *and* $\|D_Z g^{(\nu)}(Z)\| \leq K_6^\nu$ *for every* $1 \leq \nu \leq n$;

(b) $\|D_Z(M^j f^{(\nu)}(Z))\| \leq K_6^\nu$ *and* $\|D_Z(M_*^{-j} f^{(\nu)}(Z))\| \leq K_6^\nu$ *for every* $1 \leq j \leq \nu \leq n$

Proof. Let $M_\#^\nu$ denote the map induced by M^ν on the bundle of unit spheres over $(1, 2) \times R$: $M_\#^\nu(Z)f = M^\nu(Z)f/\|M^\nu(Z)f\|$ for every norm-1 vector f. The property $M^\nu\{f^{(\nu)}\}^\perp = \{M^\nu f^{(\nu)}\}^\perp$ translates into

$$\mathcal{M}_\# f^{(\nu)} = f^{(\nu)}, \quad \text{where } \mathcal{M} = M_*^\nu M^\nu. \tag{1}$$

We let $F(Z, f) = \mathcal{M}_\#(Z)f$ and use the implicit function theorem in (1) to calculate $D_Z f^{(\nu)}$. Note first that

$$(\partial_f F)(Z, f)\dot{f} = \frac{\mathcal{M}\dot{f}}{\|\mathcal{M}f\|} - \frac{\mathcal{M}f \cdot \mathcal{M}\dot{f}}{\|\mathcal{M}f\|^2}\frac{\mathcal{M}f}{\|\mathcal{M}f\|} \quad \text{for } \dot{f} \in \{f\}^\perp.$$

This implies

$$(\partial_f F)(Z, f^{(\nu)}(Z)) = \left(\mathcal{M}\,|E^{(\nu)}(Z)\right)/\|\mathcal{M}f^{(\nu)}(Z)\|$$

and so

$$\|(\partial_f F)(Z, f^{(\nu)}(Z))\| \leq \|M\,|E^{(\nu)}(Z)\|^2/\|Mf^{(\nu)}(Z)\|^2 \leq (Kb/\lambda^2)^{2\nu} \ll 1.$$

Moreover,

$$(\partial_Z F)(Z, f)\dot{Z} = \frac{D_Z\mathcal{M}(\dot{Z}, f)}{\|\mathcal{M}f\|} - \frac{\mathcal{M}f \cdot D_Z\mathcal{M}(\dot{Z}, f)}{\|\mathcal{M}f\|^2}\frac{\mathcal{M}f}{\|\mathcal{M}f\|} \tag{2}$$

implying

$$\|(\partial_Z F)(Z, f^{(\nu)}(Z))\| \leq \|D_Z\mathcal{M}(\,\cdot\,, f^{(\nu)}(Z))\|/\|\mathcal{M}f^{(\nu)}(Z)\| \leq (K/\lambda)^{2\nu},$$

where we also use the fact that $\|D_Z\mathcal{M}\| \leq (\|\varphi\|_{C^2})^{2\nu} \leq K^{2\nu}$. As a consequence,

$$\|D_Z f^{(\nu)}(Z)\| = \|\partial_Z F \circ (Id - \partial_f F)^{-1}\|(Z, f^{(\nu)}(Z)) \leq 2\left(\frac{K}{\lambda}\right)^{2\nu},$$

proving the first part of (a). The second part follows from the same argument applied to $(M^\nu M_*^\nu)_\# g^{(\nu)} = g^{(\nu)}$. On the other hand,

$$\begin{aligned}\|D_Z(M^j f^{(\nu)}(Z))\| &\le \|D_Z M(D_Z\varphi^{j-1}(Z), M^{j-1} f^{(\nu)}(Z))\| + \\ &\quad + \|M(\varphi^{j-1}(Z))\| \|D_Z(M^{j-1} f^{(\nu)}(Z))\| \\ &\le K^{2j} + K\|D_Z(M^{j-1} f^{(\nu)}(Z))\|.\end{aligned}$$

By recurrence we find

$$\|D_Z(M^j f^{(\nu)}(Z))\| \le K^{2j} + \cdots + K^{j+1} + 2K^j (K/\lambda)^{2\nu},$$

which proves the first part of (b). Finally, the second part is be proved by a similar argument, starting with $M_*^{-j} f^{(\nu)} = M_*^{\nu-j} g^{(\nu)} \|M_*^{-\nu} f^{(\nu)}\| = M_*^{\nu-j} g^{(\nu)} / \|M^\nu f^{(\nu)}\|$. $\square$

Proof of Lemma 4.5. Throughout the proof const denotes a positive constant depending only on K and λ. First, we verify the conclusion of the lemma for $\nu = 1$. Observe that (2) gives for $f = f^{(1)} = f^{(1)}(Z)$

$$(\partial_Z F)(Z, f^{(1)})\dot{Z} = \frac{D_Z \mathcal{M}(\dot{Z}, f^{(1)})}{\|M f^{(1)}\|^2} - \left(\frac{D_Z \mathcal{M}(\dot{Z}, f^{(1)})}{\|M f^{(1)}\|^2} \cdot f^{(1)} \right) f^{(1)}, \quad (3)$$

where $\mathcal{M} = M_* M$. It follows from (QL) that we may write

$$\mathcal{M} = \hat{\mathcal{M}} + \mathcal{E} \text{ with } \hat{\mathcal{M}} = \begin{pmatrix} p(a,x) & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R} \times \mathbb{R}^{m-1})$$

and

$$\|\mathcal{E}\|_{C^1} \le \text{const}\ \sqrt{b}.$$

On the other hand, by Lemma 4.2, $f^{(1)}$ is (const $\sqrt{b}$)$-$close to the horizontal direction. Using these facts in (3) we obtain $\|(\partial_Z F)(Z, f^{(1)})\| \le$ const $\sqrt{b}$ and then the proof of Lemma 4.6 gives $\|D_Z f^{(1)}\| \le$ const $\sqrt{b}$. Now the lemma will follow if we prove that

$$\|D_Z f^{(\nu+1)}\| \le \|D_Z f^{(\nu)}\| + (\text{const } b)^\nu, \quad \text{for every } 1 \le \nu \le n-1 \quad 5$$

and we proceed to do this. Let $f^{(\nu+1)} = (f^{(\nu+1)} \cdot f^{(\nu)})\, f^{(\nu)} + h^{(\nu)}$ with $h^{(\nu)} \in E^{(\nu)}$. Then

$$D_Z f^{(\nu+1)} = (f^{(\nu+1)} \cdot f^{(\nu)})\, D_Z f^{(\nu)} + D_Z(f^{(\nu+1)} \cdot f^{(\nu)})\, f^{(\nu)} + D_Z h^{(\nu)}.$$

Clearly, $|f^{(\nu+1)} \cdot f^{(\nu)}| \le 1$ and $\|f^{(\nu)}\| \le 1$. On the other hand, $f^{(\nu)} \cdot D_Z f^{(\nu)} = 0$ because $\|f^{(\nu)}\| \equiv 1$. This, together with Lemmas 4.1(a) and 4.6(a), gives

$$|f^{(\nu+1)} \cdot D_Z f^{(\nu)}| \le (\text{const } b)^\nu \cdot \|D_Z f^{(\nu)}\| \le (\text{const } b)^\nu.$$

Moreover, the same argument applies to $|D_Z f^{(\nu+1)} \cdot f^{(\nu)}|$, proving that

$$\|D_Z(f^{(\nu+1)} \cdot f^{(\nu)})\| \le (\text{const } b)^\nu. \quad (6)$$

Now we are left to estimate $\|D_Z h^{(\nu)}\|$. Let

$$\begin{pmatrix} \alpha_* & 0 \\ 0 & \Delta_* \end{pmatrix}, \quad \alpha_* = \|M^\nu f^{(\nu)}\|^{-1}$$

and

$$\Delta_* = (\Delta_{ij})_{1 \le i,j \le m-1} \colon \{g^{(\nu)}\}^\perp \longrightarrow \{f^{(\nu)}\}^\perp,$$

be the matrix of M_*^ν with respect to fixed orthogonal frames

$$\{g^{(\nu)}(Z), u_1(Z), \ldots, u_{m-1}(Z)\} \text{ and } \{f^{(\nu)}(Z), e_1(Z), \ldots, e_{m-1}(Z)\}$$

satisfying $\|D_Z u_i\|$, $\|D_Z e_i\| \le$ const $^\nu$. Then we write

$$h^{(\nu)} = h_1 e_1 + \cdots + h_{m-1} e_{m-1}$$

and take $\bar{h}^{(\nu)} = \bar{h}_1 u_1 + \cdots + \bar{h}_{m-1} u_{m-1} \in M_*^\nu E^{(\nu)}$ to be such that $M_*^{-\nu} f^{(\nu+1)} = (f^{(\nu+1)} \cdot f^{(\nu)})\, M_*^{-\nu} f^{(\nu)} + \bar{h}^{(\nu)}$. Clearly, $h^{(\nu)} = M_*^\nu \bar{h}^{(\nu)}$ or, equivalently,

$$h_i = \|M^\nu f^{(\nu)}\| \sum_{j=1}^{m} (\alpha_* \Delta_{ij}) \bar{h}_j \quad \text{for } 1 \le i \le m-1. \quad (7)$$

Note that $\|\bar{h}^{(\nu)}\| \le$ const $^\nu$, since $\|M_*^{-i} f^{(i)}\| = \|M^i f^{(i)}\|^{-1} \le \lambda^{-i}$. It is also clear that $\|M^\nu f^{(\nu)}\| \le$ const $^\nu$. Moreover, Lemma 4.6 implies $\|D\bar{h}^{(\nu)}\| \le$ const $^\nu$ and $\|D(\|M^\nu f^{(\nu)}\|)\| \le$ const $^\nu$. We claim, in addition, that

$$|\alpha_* \Delta_{ij}| \le (\text{const } b)^\nu \text{ and } \|D_Z(\alpha_* \Delta_{ij})\| \le (\text{const } b)^\nu. \quad (8)$$

Taking derivatives in (7) and using all these estimates we get

$$\|D_Z h^{(\nu)}\| \le (\text{const } b)^\nu.$$

Together with (6) this concludes the proof of (5). Therefore, we have reduced the proof of the lemma to that of (8). Let $(M_{rs}^\nu)_{1 \le r,s \le m}$ be the matrix of M^ν with respect to the canonical basis of $\mathbb{R}^m$. Each $(\alpha_*\Delta_{ij})$, $1 \le i,j \le m-1$, may be written as a linear combination $\alpha_*\Delta_{ij} = \sum C_{rsuv} H^\nu(r,s,u,v)$, where $H^\nu(r,s,u,v) = (M^\nu_{rs}M^\nu_{uv} - M^\nu_{rv}M^\nu_{us})$ and $|C_{rsuv}| \le \text{const}^{\,\nu}$, $\|D_Z C_{rsuv}\| \le \text{const}^{\,\nu}$. Now,

$$H^\nu(r,s,u,v) = \sum_{i<j} H^1(r,i,u,j) H^{\nu-1}(i,s,j,v)$$

and, by (QL),(SD),

$$|H^1(r,i,u,j)| \le \text{const } b$$

and

$$\|D_Z H^1(r,i,u,j)\| \le \text{const } b$$

for every (r,i,u,j). Hence, by induction on ν,

$$\|H^\nu(r,s,u,v)\| \le (\text{const } b)^\nu$$

and

$$\|DH^\nu(r,s,u,v)\| \le (\text{const } b)^\nu.$$

This implies (8) and completes our argument. □

The same type of argument applies to the second order derivative, giving

Lemma 4.7. *There is $K_7 = K_7(K,\lambda) > 0$ such that if z is $\lambda-$expanding up to time n then $\|D_Z^2 f^{(\nu)}(Z)\| \le K_7\sqrt{b}$ for every $1 \le \nu \le n$.*

Finally, we also prove the following result, to be used in Section 8.

Lemma 4.8. *There is $K_8 = K_8(K,\lambda) > 0$ such that $\|D_Z(M^i e)(Z)\| \le (K_8 b)^i$ for every $1 \le i \le \nu \le n$ and any norm-1 vector field $e(Z)$ with $e(Z) \in E^{(\nu)}(Z)$ and $\|D_Z e(Z)\| \le 1$.*

Proof. We write $e = (e\cdot f^{(i)})\, f^{(i)} + \tilde{h}^{(i)}$ with $\tilde{h}^{(i)} \in E^{(i)}$ and then

$$D_Z(M^i e) = (e\cdot f^{(i)})\; D_Z(M^i f^{(i)}) + D_Z(e\cdot f^{(i)})\; M^i f^{(i)} + D_Z(M^i \tilde{h}^{(i)}).$$

We have $\|M^i f^{(i)}\| \le \text{const}^{\,i}$ and $\|D_Z(M^i f^{(i)})\| \le \text{const}^{\,i}$. Moreover, $|e\cdot f^{(i)}| \le (\text{const } b)^i$, as a consequence of Lemma 4.1. As in the proof of Lemma 4.5, we check that, for every $1 \le j \le n-1$, $f^{(j+1)} = c_j f^{(j)} + h^{(j)}$ with $|c_j| \in [1-(\text{const } b)^j, 1]$, $\|D_z c_j\| \le (\text{const } b)^j$, $\|h^{(j)}\| \le (\text{const } b)^j$ and $\|D_Z h^{(j)}\| \le (\text{const } b)^j$. It follows that $f^{(i)} = \hat{c} f^{(\nu)} + \hat{h}$ with $\|\hat{h}\| \le$

$(\text{const } b)^i$ and $\|D_Z\hat{h}\| \le (\text{const } b)^i$. Then $\|D_Z(e\cdot f^{(i)})\| = \|D_Z(e\cdot\hat{h})\| \le (\text{const } b)^i$. Finally, in order to estimate $\|D_Z(M^i\tilde{h}^{(i)})\|$, we let

$$\begin{pmatrix} \alpha & 0 \\ 0 & \Delta \end{pmatrix}, \quad \alpha = \|M^i f^{(i)}\| \text{ and } \Delta\colon \{f^{(i)}\}^\perp \longrightarrow \{g^{(i)}\}^\perp,$$

be the matrix of M^i with respect to orthogonal frames $\{f^{(i)}(Z), e_1(Z), \dots, e_{m-1}(Z)\}$ and $\{g^{(i)}(Z), u_1(Z), \dots, u_{m-1}(Z)\}$ as before. Then

$$M^i\tilde{h}^{(i)} = (\alpha\Delta)\tilde{h}^{(i)}/\|M^i f(i)\|$$

and now the same argument as in the proof of Lemma 4.5 gives

$$\|D_Z(M^i\tilde{h}^{(i)})\| \le (\text{const } b)^i.$$

Altogether, this shows that $\|D_Z(M^i e)\| \le (\text{const } b)^i$ as we wanted to prove. □

5. Critical points

Here we describe the algorithms to be used in the construction of critical points in $W^u(P_a)$. As explained in the Introduction, a main difference with respect to the two-dimensional situation results from the fact that, in the present setting, nearby disjoint segments of $W^u(P_a)$ need not have nearby tangent directions. Because of this, closeness of the tangent directions must be taken as an independent assumption here, see (2) below, and then be deduced directly from the binding construction, see Section 7.

5.1 Generation zero

We make use of the following notions. Let $\bar{z}_0$ be the point of $W^u(P_a) \cap \{(x,Y)\colon x=0\}$ closest to P_a in $W^u(P_a)$ and, for $j \ge 1$, let $\bar{z}_j = \varphi_a^j(z_0)$. We define $G_0 = [\bar{z}_1, \bar{z}_2] \subset W^u(P_a)$ and $G_g = \varphi_a^g(G_0) \setminus G_{g-1}$for $g \ge 1$. Points $z \in G_g$ are said to be of *generation* g. We assume that $\varphi = (\varphi_a)_a$ is close enough to the quadratic family ϕ so that the intersections of G_0 and G_1 with $\{(x,Y)\colon |x| < 1-\delta_0\}$, $\delta_0 = 5(2-\sup\Omega)$, are *$b-$flat curves*, i.e graphs of functions $Y\colon x \mapsto Y(x)$ with $\|\dot{Y}\|$, $\|\ddot{Y}\| \le b^{1/4}$. A point $z_0 \in W^u(P_a)$ is a *$\nu-$th critical approximation* if $W^u(P_a)$ is tangent to the $\nu-$th contractive hyperplane $E^{(\nu)}(z_1)$ at $z_1 = \varphi_a(z_0)$ and, moreover, the ρ^ν−neighbourhood of z_0 in $W^u(P_a)$, denoted by $\gamma(z_0,\rho^\nu)$, is a b−flat curve. The notation $z_0^{(\nu)}$ always corresponds to such a point. In Section 6 we fix the value of ρ but here we only need $0<b\ll\rho<1/2$.

It is easy to see that G_0 contains a 1−st critical approximation near $x=0$. In fact, let $x\mapsto z_0(x)=(x,Y(x))$ parametrize $G_0\cap\{(x,Y)\colon |x|\le 1-\delta_0\}$. The tangent direction to $\varphi_a(G_0)$ at $z_1(x)=\varphi_a(z_0(x))$ is given by

$$t(x)=(\alpha(z_0)+\beta(z_0)\cdot\dot Y(x),\Gamma(z_0)+\Delta(z_0)\cdot\dot Y(x)),$$

where

$$D\varphi_a=\begin{pmatrix}\alpha & \beta\\ \Gamma & \Delta\end{pmatrix}\in\mathcal{L}(\mathbb{R}\times\mathbb{R}^{m-1}).$$

For $x=0$ we have $|t(0)\cdot f^{(1)}(z_1(0))|\le\|t(0)\|\le 2K\sqrt{b}$, by (QL). Note that Lemma 4.5 implies $|t(x)\cdot D_x(f^{(1)}(z_1(x)))|\le\text{const}\,\sqrt{b}$ for every $|x|\le 1$. Now, $\dot t(x)=(-2a,0^{m-1})+\varepsilon(x)$ with $\|\varepsilon(x)\|\le\text{const}\,b$ and so $|\dot t(x)\cdot f^{(1)}(z_1(x))|\ge(3a/2)$, since $f^{(1)}$ is nearly horizontal at $z_1(x)$. Therefore, for $b>0$ sufficiently small, $|D_x(t(x)\cdot f^{(1)}(z_1(x)))|\ge a\ge 1$. It follows that there is $x^{(1)}$ with $|x^{(1)}|\le 2K\sqrt{b}$ such that $t(x^{(1)})\cdot f^{(1)}(z_1(x^{(1)}))=0$, i.e. $z_0^{(1)}=z_0(x^{(1)})$ is a 1−st critical approximation. Clearly the same argument applies also to G_1, starting with $\bar w_0=G_1\cap\{(x,Y)\colon x=0\}$, and we denote by $w_0^{(1)}$ the corresponding critical approximation.

5.2 Precision increasing

The same basic idea permits to show that, whenever $z_0^{(\nu)}$ is a ν−th critical approximation and $z_1^{(\nu)}=\varphi_a(z_0^{(\nu)})$ is expanding up to time $(\nu+1)$, there exists a $(\nu+1)$−st approximation $z_0^{(\nu+1)}$ near $z_0^{(\nu)}$. We think of $z_0^{(\nu)},z_0^{(\nu+1)}$ as approximations to the same *critical point* of φ_a. Let $\gamma(z_0,\rho^u)$ be parametrized by $x\mapsto z_0(x)=(x,Y(x))$ and let $z_0^{(\nu)}=z_0(x^{(\nu)})$. By definition $t(x^{(\nu)})\cdot f^{(\nu)}(z_1^{(\nu)})=0$ and so $|t(x^{(\nu)})\cdot f^{(\nu+1)}(z_1^{(\nu)})|\le 2K(K_1b)^\nu$, according to Lemma 4.1(a). As before, we get $|D_x(t(x)\cdot f^{(\nu+1)}(z_1(x)))|\ge 1$ and so there is $x^{(\nu+1)}$ with $|x^{(\nu)}-x^{(\nu+1)}|\le 2K(K_1b)^\nu$ such that $z_0^{(\nu+1)}=z_0(x^{(\nu+1)})$ is a $(\nu+1)$−st critical approximation.

Let $\delta>0$ be a small number and $\Omega\subset(1,2)$ be an interval close to

$a=2$ in the parameter space. We define $N\ge 1$ to be the maximum integer such that $\bar z_j,\bar w_j\in\{(x,Y)\colon |x|\ge 2\delta\}$ for every $1\le j<N$. Observe that N can be made arbitrarily large by taking Ω close to $a=2$ and b small. Then the construction above applies to the approximations $z_0^{(1)},w_0^{(1)}$ obtained in section 5, yielding sequences $z_0^{(1)},z_0^{(2)},\dots,z_0^{(N-1)}$ and $w_0^{(1)},w_0^{(2)},\dots,w_0^{(N-1)}$, in G_0 and G_1, respectively. Note that the $z_1^{(i)}$, $w_1^{(i)}$, are indeed expansive up to time $N-1$:

Lemma 5.1. *Given $0<c_0<\log 2$ and $\delta>0$ small then, for a close 2 and b sufficiently small, the following holds. Let v be a norm−1 vector with $|\text{slope}|(v)\le 1/10$ and let $z_i=(x_i,Y_i)$, $0\le i\le k+1$, be a segment of orbit of φ_a satisfying $|x_i|\ge\delta$ for $1\le i\le k$. Then*

(a) *$|\text{slope}|(D\varphi_a^i(z_1)\cdot v)\le\tilde K\sqrt{b}$ and $\|D\varphi_a^i(z_1)\cdot v\|\ge a|x_i|\|D\varphi_a^{i-1}(z_1)\cdot v\|$ for every $1\le i\le k$ and some $\tilde K=\tilde K(K,\delta)$;*

(b) *If $|x_0|<\delta$ or $|x_{k+1}|<\delta$ then $|D\varphi_a^k(z_1)\cdot v|\ge e^{kc_0}$.*

Proof. Analogous to [**MV, Lemma 7.2**], [**BC2, Lemma 4.6**]. □

For future use we let $\mathcal{C}_k=\{z_0^{(k-1)},w_0^{(k-1)}\}$ for $2\le k\le N-1$.

5.3 Higher generations

Critical approximations of generation $g>1$ are constructed using lower generation ones as starting points, in the following way. Let $\zeta_0^{(\nu)}$ be a ν−th critical approximation. We assume that $\zeta_1^{(\nu)}$ is λ−expanding up to time ν and $\tilde\gamma=\gamma(\zeta_0^{(\nu)},\ell)$ is b−flat, $\ell\ge 2\rho^\nu$. Let $\gamma=\gamma(z_0,\ell)$ be another b−flat segment of $W^u(P_{n,a})$, satisfying

$$\text{dist}(z_0,\zeta_0^{(\nu)})\le\frac{1}{2}\min\{\rho^\nu,\sigma^\nu\},\tag{1}$$

where $\sigma\le(\lambda/10K^2)^2$. By Lemma 4.3, all the points in $\varphi_a(\gamma(z_0,\sigma^\nu))$ are also expanding up to time ν. We assume, moreover, that the tangent directions to $\varphi_a(\tilde\gamma)$ and $\varphi_a(\gamma)$ satisfy

$$|\text{angle}|(\tilde t(\tilde x_0^{(\nu)}),t(x_0))\le\frac{1}{2}\min\{\rho^\nu,\sigma^\nu\},\tag{2}$$

where $\zeta_0^{(\nu)}=(\tilde x_0^{(\nu)},\tilde Y_0^{(\nu)})$ and $z_0=(x_0,Y_0)$. By definition,

$$\tilde t(\tilde x_0^{(\nu)})\cdot f^{(\nu)}(\zeta_1^{(\nu)})=0.$$

From (1), (2) and Lemma 4.5 we get

$$|t(x_0)\cdot f^{(\nu)}(z_1)| \le |t(x_0)-\tilde t(\tilde x_0^{(\nu)})| + |f^{(\nu)}(z_1)-f^{(\nu)}(\zeta_1)||\tilde t(\tilde x_0^{(\nu)})| \le \left(\frac{1}{2}+\text{const}\sqrt{b}\right)\min\{\rho^\nu,\sigma^\nu\} \le \min\{\rho^\nu,\sigma^\nu\}.$$

Hence, in the same way as before, there is $z_0^{(\nu)} = (x_0^{(\nu)}, Y_0^{(\nu)}) \in \gamma$ a critical approximation with $|x_0^{(\nu)} - x_0| \le \min\{\rho^\nu,\sigma^\nu\}$.

5.4 Contractive distributions

The natural substitute for the contractive vector fields of **[BC2]** and **[MV]** in our present setting are the distributions of contractive hyperplanes $E^{(\nu)}$ defined in Section 4. However, here we have to circumvent the problem that these distributions may not integrable. This is done in the following way. Let $z=(x,Y)$ be $\lambda-$expanding up to time n. Then, by Lemma 4.2, the contractive hyperplanes $E^{(\nu)}(z)$ are nearly vertical. For $1\le\nu\le n$ and any norm-1 vector $V\in\mathbb{R}^{m-1}$, we let $e_V^{(\nu)}(z)$ be the unique vector of the form (v,V) in $E^{(\nu)}(z)$.

Lemma 5.2. *For every $1\le\nu\le n$ and $V\in\mathbb{R}^{m-1}$ with $\|V\|=1$,*

(a) $|\text{angle}|(e_V^{(\nu)}(z), e_V^{(\nu-1)}(z)) \le (K_9 b)^\nu,\ K_9 = K_9(K,\lambda)$

(b) *The vector field $e_V^{(\nu)}$ admits an integral curve $[-b^{1/4}, b^{1/4}] \ni t \mapsto \Gamma_V^{(\nu)}(t)$ with $\Gamma_V^{(\nu)}(0)=z$.*

Proof. Write $e_V^{(\nu-1)}(z)=(v_1,V)$ and $e_V^{(\nu)}(z)=(v_2,V)$ and let $(1,F_1)$ and $(1,F_2)$ be colinear to $f^{(\nu-1)}(z)$ and $f^{(\nu)}(z)$, respectively. From $(v_i,V)\cdot(1,F_i)=0$ and Lemma 4.1 we get

$$|\text{angle}|(e_V^{(\nu-1)}(z), e_V^{(\nu)}(z)) \le \|v_1-v_2\| \le \|F_1-F_2\| \le 2(K_1 b)^\nu,$$

This proves (a) and (b) can now be proved by the same argument as in **[BC2, Lemma 5.8]** or **[MV, Section 7C]**. □

Notice that $\Gamma_V^{(\nu)}$ has the form $\Gamma_V^{(\nu)}(t) = (x_{\nu,V}(t), Y+tV)$. Suppose now that $z=(x,Y)$ is such that $|x|<1-2\delta_0$. By construction, $G_1\cap\{(x,Y)\colon |x|\le 1-\delta_0\}$ is a $b-$flat curve. Moreover, it is contained in $\{(x,Y)\colon \|Y\|\le\text{const}\sqrt{b}\}$, as a consequence of (QL) and Lipschitz dependence of unstable manifolds on the dynamics, see e.g. **[MV, Proposition 7.1]**. It follows that for some V the curve $\Gamma_V^{(n)}$ intersects G_1 in a point η. Since $\text{dist}(\varphi_a^\nu(\eta),\varphi_a^\nu(z)) \le \text{const}\sqrt{b}\,(\text{const}\, b)^\nu$ for $0\le\nu\le n$, we may apply Lemma 4.4 to $z_1=z$, $\zeta_1=\eta$ and $\sigma=\text{const}\sqrt{b}$, to conclude that

$$\frac{1}{2}\|D\varphi_a^\nu(\eta)\cdot v\| \le \|D\varphi_a^\nu(z)\cdot u\| \le 2\|D\varphi_a^\nu(\eta)\cdot v\| \tag{3}$$

$$|\text{angle}|(D\varphi_a^\nu(\eta)\cdot v, D\varphi_a^\nu(z)\cdot u) \le (\text{const}\, b^{1/4})^{\nu+1} \tag{4}$$

for every $1\le\nu\le n$ and any norm-1 vectors u and v with slope $\le 1/10$. In particular this holds for $u=(1,0^{m-1})$ and $v=$tangent to G_1 at η.

6. The induction

The proof of Theorem A is based on the construction of a sequence $(\mathcal{C}_k)_k$ of subsets of $W^u(P_a)$ with the following properties. Each $\mathcal{C}_k$ is formed by a finite (although unbounded) number of $(k-1)-$st critical approximations and the image $z_1^{(k-1)} = \varphi_a(z_0^{(k-1)})$ of every $z_0^{(k-1)}\in\mathcal{C}_k$ is e^c-expanding up to time k. The precision increasing procedure of Section 5.2 defines "canonical" one-to-one maps $\mathcal{C}_k \longrightarrow \mathcal{C}_{k+1}$, $z_0^{(k-1)}\mapsto z_0^{(k)}$, via which we may think of every $\mathcal{C}_k$ as a subset of $\mathcal{C}_{k+1}$. Then, each limit point $z_1=\lim_{k\to\infty}\varphi_a(z_1^{(k)})$ is e^c-expanding for all positive times and it is a *critical value* of φ_a in the sense that the tangent direction to $W^u(P_a)$ at z_1 is (exponentially) contracted by all positive iterates of $D\varphi_a$. The construction of each $\mathcal{C}_k$ requires a certain number of assumptions on the parameter, which are satisfied only by a subset S_k of values of $a\in\Omega$. Then we show that $S_\infty=\bigcap_{k\ge1}S_k$ has positive Lebesgue measure. Finally, for almost every $a\in S_\infty$ the orbit of some critical value z_1 as above is dense in $\Lambda_a=\text{closure}(W^u(P_a))$.

The objective of this section is to describe the induction procedure through which the *critical sets* $\mathcal{C}_k$ are constructed. This procedure was already initiated in Section 5.2, where the $\mathcal{C}_k$, $k\le N-1$, were defined. The inductive step, to be presented in Sections 6-8, is rather elaborate. It requires a detailed description of the operations performed at the previous stages of the construction, including the knowledge of several additional properties of the critical sets obtained during those stages. All this information must be part of the induction hypothesis, which, as a consequence, has a rather long statement. Here we give the precise content of this statement and describe a few other notions and auxiliary constructions involved in the induction. This follows closely [**MV, Sec. 8**] where more detailed motivation can be found. Parameter dependence is of secondary importance at this point and so, for the sake of notational simplicity, we omit reference to a in most instances below.

Let $n \geq N$. We assume that a set $\mathcal{C}_k$ consisting of $(k-1)$−st critical approximations of generation $g \leq \theta k$ has been defined for every $1 \leq k \leq n-1$. Here

$$\theta = \theta(b) = \frac{20 \log(10K^2)}{\log(1/b)}. \tag{1}$$

We fix a small number $\beta > 0$ and say that a point ξ_0 is *bound to* $\mathcal{C}_k$ *(up to time k)* if there is $z_0^{(k-1)} \in \mathcal{C}_k$ such that for every $1 \leq j \leq k$

$$\|\xi_j - z_j^{(k-1)}\| \leq h_k e^{-\beta j}, \text{ with } h_k = 2 - 2\sum_1^{k-1}(e^\beta/4)^i \in (1,2). \tag{B1}$$

By assumption, the image $\xi_1 = \varphi_a(\xi_0)$ of any such ξ_0 is e^c−expanding up to time k. We fix

$$\rho_0 = \left(\frac{1}{10K^2}\right)^2. \tag{2}$$

Given $z_0^{(k-1)} \in \mathcal{C}_k$, its $\rho_0^{\theta k}$−neighbourhood in $W^u(P_a)$, is assumed to be a b−flat curve. Recall that we denote this neighbourhood by $\gamma(z_0^{(k-1)}, \rho_0^{\theta k})$. If $z_0^{(k-1)}$ is of generation $g > 1$ we assume, in addition, that

- $\varphi_a^{1-g}(\gamma(z_0^{(k-1)}, \rho_0^{\theta k}))$ is contained in $\{(x,Y)\colon |x| \leq 1-\delta_0\} \cap G_1$ and
- any vector t tangent to it is expanded by $D\varphi_a^{g-1}$: $\|D\varphi_a^{g-1} t\| \geq \|t\|$.

Here, as before, $\delta_0 = 5(2 - \sup \Omega)$. Every $\mathcal{C}_{k+1}$ is derived from $\mathcal{C}_k$ in the following way.

(α) Let $\mathcal{C}'_{k+1}$ be the set of all k−th critical approximations $z_0^{(k)}$ obtained from the elements $z_0^{(k-1)}$ of $\mathcal{C}_k$ via the algorithm of Section 5.2. Note that $z_0^{(k-1)}$ and $z_0^{(k)}$ have the same generation $g \leq \theta k$ and, moreover,

$$\|z_0^{(k)} - z_0^{(k-1)}\| \leq 4K(K_1 b)^{k-1} \leq \left(\frac{1}{4K}\right)^k. \tag{3}$$

Observe that the first inequality also implies

$$\gamma(z_0^{(k)}, \rho_0^{\theta(k+1)}) \subset \gamma(z_0^{(k-1)}, \rho_0^{\theta k}).$$

(β) Let $\mathcal{C}''_{k+1}$ consist of all the k−th critical approximations $z_0^{(k)}$ of generation $\theta k < g \leq \theta(k+1)$ for which $\gamma(z_0^{(k)}, \rho_0^{\theta(k+1)})$ has the properties stated above and which can be obtained by applying the algorithm of Section 5.3 to points $\zeta_0^{(k)} \in \mathcal{C}'_{k+1}$, with the additional requirement that

$$\|z_0^{(k)} - \zeta_0^{(k)}\| \leq b^{g/10} \leq \left(\frac{1}{4K}\right)^k. \tag{4}$$

(γ) Then we take $\mathcal{C}_{k+1} = \mathcal{C}'_{k+1} \cup \mathcal{C}''_{k+1}$.

Remark 6.1. Note that $\mathcal{C}''_{k+1}$ is empty if $(\theta k, \theta(k+1)]$ contains no integer numbers. In particular, for $k < 1/\theta$ the critical set $\mathcal{C}_k$ consists only of the points $z_0^{(k-1)}$ and $w_0^{(k-1)}$ corresponding (by increase of precision) to the critical approximations in G_0, G_1 found in Sections 5.1 and 5.2. In this case we also replace $\rho_0^{\theta k}$ by 1/2 in the definition above.

The estimates of the parameter exclusions, to be performed in Section 9, are based on the fact that the exponential rate of growth of the number of points in $\mathcal{C}_k$ can be made arbitrarily small by taking b small. This is proved in the same way as (1) in [**MV, Sec. 8**].

Lemma 6.1. $\#\mathcal{C}_k \leq 8\left(\frac{K}{\rho_0}\right)^{\theta k}$ *for all k.*

Remark 6.2. In order to estimate these parameter exclusions we also need a parametrized version of the conditions in (β) above. This is also part of our definition of the critical sets but we postpone its statement until Section 9, where it fits more naturally.

We start the inductive step of our construction by defining $\mathcal{C}_n$ from $\mathcal{C}_{n-1}$, according to the procedure above. Then (3) and (4) (recall also our definition of h_k) assure that any point ζ_0 which is bound to $\mathcal{C}_n$ is also bound to $\mathcal{C}_{n-1}$. Hence, by induction, its image ζ_1 is e^c−expanding up to time $(n-1)$. Proving that (for many values of the parameter a) such a ζ_1 is also e^c−expanding at time n is the main part of the induction. Before that, we must introduce a few other notions which play an important role in the argument. It is part of our induction hypothesis that such notions have been defined for all times $\leq n-1$ in the way described below.

First we consider *returns, binding points* and *binding periods*. Recall that $\delta > 0$ is a small constant. Roughly, a return of a point ξ_0 is an iterate $\nu \geq 1$ for which $\xi_\nu \in \{(x,Y)\colon |x_\nu| < \delta\}$. To every such ν we assign a convenient element ζ_0 of $\mathcal{C}_k$, some $k \leq \nu$, close to ξ_ν, which we call the binding point of ξ_ν. Then the binding period associated to ν is the maximal interval of time $[\nu+1, \nu+p]$ during which the orbits of ξ_ν and ζ_0 remain close to each other, in the sense that $\|\xi_{\nu+j} - \zeta_j\| \leq e^{-\beta j}$ for all $1 \leq j \leq p$, $\beta > 0$ a small fixed number.

The precise definition is by recurrence. Let $n \geq N$ and assume that for every $1 \leq k \leq n-1$ and every point ξ_0 bound to $\mathcal{C}_k$, the returns $\nu \in [1,k]$ of ξ_0 have been defined and that a binding point $\eta_0 \in \bigcup_2^\nu \mathcal{C}_i$ and a binding period $[\nu+1, \nu+p]$ have been associated to each return ν. Let now ξ_0 be bound to $\mathcal{C}_n$ (and so also to $\mathcal{C}_{n-1}$). Suppose first that n belongs to some binding period associated to a return $\nu < n$ of ξ_0. Take such ν maximum and let ζ_0 be the binding point of ξ_ν. By definition, n is a *(bound) return* for ξ_0 if $(n-\nu)$ is a return for ζ_0 and the binding point of ξ_n is the same as that of $\zeta_{n-\nu}$. Moreover, the binding period of ξ_n is $[n+1, n+p]$ if the binding period of $\zeta_{n-\nu}$ is $[n-\nu+1, n-\nu+p]$. Suppose now that no binding period associated to a previous return of ξ_0 contains n. By definition, n is a *(free)return* for ξ_0 if $|x_n| < \delta$. Then, we take the binding point $\zeta_0 \in \mathcal{C}_n$ in such a way that a certain set of properties are satisfied. These properties are to be listed in (H1) below, after the necessary language has been introduced. Again, it is part of our induction hypothesis that these properties hold for all previous free returns. A construction of such a point ζ_0 is given in Section 7. The binding period $[n+1, n+p]$ corresponding to the free return n is defined as follows. First, we let q be the maximum integer such that

$$\|\xi_{n+j} - \zeta_j\| \leq h e^{-\beta j}, \quad h = \frac{1}{10} \exp\left(-50K \sum_1^\infty e^{-\alpha j}\right) \in (0,1), \qquad (B2)$$

for all $1 \leq j \leq q$. Then we take $p \leq q$ to be maximum such that $p+1$ is a

free iterate (i.e. it is neither a return nor part of a binding period) of ξ_n. Observe that this is well defined, since ξ_n is bounded to ζ_0 (and so also to $\mathcal{C}_q$) up to time q and, as we will show, $q < n$. In this way, the time $n+p+1$ immediately after a binding period is always a free iterate for ξ_0. All the binding periods associated to returns $\nu \leq n-1$ are assumed to be given in this way and to satisfy a few other properties whose statement we postpone to (H2). Section 8 is dedicated to showing that these properties also hold for all the binding periods starting at time n.

Actually, Sections 7 and 8 already require some restriction on the values of the parameter a. Given a return ν of a point ξ_0 we define $d_\nu(\xi_0) = \|\xi_\nu - \zeta_0\|$ where ζ_0 is the binding point of ξ_ν. (For completeness, we also set $d_\nu(\xi_0) = |x_\nu|$, $\xi_\nu = (x_\nu, Y_\nu)$ when ν not a return). We assume that whenever $1 \leq k \leq n$ is a free return of a point $z_0 \in \mathcal{C}_k$ then

$$d_k(z_k) \geq e^{-\alpha k}, \quad \text{where } \alpha \text{ is another small constant.} \qquad (BA)$$

Note that for $k = n$ this will be defined only after we have found the corresponding binding point ζ_0, see Definition 7.2.

Now we define *folding periods*. Heuristically, these are intervals $[\nu+1, \nu+l]$ corresponding to the time it takes for a fold in $W^u(P_a)$ created at a return ν to get flattened near the orbit of the point. The formal definition is, again, somewhat involved. As for the binding periods, we want folding periods to form a partially ordered family. We also want to assure that at time $(\nu+l+1)$ the point is in $\tilde{V} = \{(x,Y)\colon |x| \leq 1 - 2\delta_0\}$. Note that given any z we have $\varphi_a^i(z) \in \tilde{V}$ for some $0 \leq i \leq 4$. We consider in detail the case when n is a free return for a point ξ_0 bound to $\mathcal{C}_n$. This is extended to bound returns occurring at time n in precisely the same way as we did before for the binding period. Moreover, the same constructions and conclusions are supposed to apply to all returns $\nu \leq k$ of points bound to $\mathcal{C}_k$, $k \leq n-1$. Let $s \geq 1$ be defined by

$$s = \max\left\{\frac{10\log(1/d_n(\xi_0))}{\log(1/b)}, 4\right\}. \qquad (5)$$

Then the *folding period* is $[n+1, n+l]$, where $l \leq s$ is the maximum integer such that $\xi_{n+l+1} \in \tilde{V}$ and $l+1$ is a *fold-free iterate* (it is neither a return nor inside a folding period) of ξ_n. Note that these notions are already defined for ξ_n at this stage: (BA) implies $s \ll n$ and it is easy to check that ξ_n is bound to its binding point ζ_0 up to time s. Let us also state explicitly the following property of folding periods:

$$\|D\varphi_a^l(\xi_{n+1}) \cdot e\| \leq (\sqrt{b})^l d_n(\xi_0)^2 \quad \text{for every } e \in E^{(l)}(\xi_{n+1}),\ \|e\| = 1. \quad (6)$$

This is a direct consequence of Lemma 4.1 together with the fact that

$$l \geq \frac{s}{2} \geq \frac{5\log(1/d_n(\xi_0))}{\log(1/b)}.$$

In order to justify this last affirmative we note first that, up to choosing δ, $(2-a)$ and b sufficiently small, we may suppose that any point takes at least (say) 20 iterates in between two consecutive returns. Then case $s < 20$ above is obvious and from now on we suppose $s \geq 20$. By the definition and the inductive information on folding periods we have that, either $l > s-4$ or else there exists a folding period $[\mu+1, \mu+l_0]$ of ξ_n with $l+1 = \mu-1$ and $\mu + l_0 \geq s$. In the first case the conclusion follows immediately; in the second one we note that, by (BA), $l_0 \leq \max\{10\alpha\mu/\log(1/b), 4\}$ and so $s - l \leq l_0 + 2 \leq 10\alpha s/\log(1/b) + 6 \leq s/2$. This completes our argument.

Now we present the higher-dimensional form of the splitting algorithm in [**BC2**], [**MV**]. For $0 \leq \mu \leq k \leq n-1$ and ξ_0 bound to $\mathcal{C}_k$ we decompose $w_\nu = w_\nu(\xi_1) = \omega_\nu + \sigma_\nu$ in the following way.

1. Let $\omega_0 = w_0 = (1, 0^{m-1})$ and $\sigma_0 = 0$.
2. For $\mu \geq 1$, let $\tilde{\omega}_\mu = D\varphi_a(\xi_\mu) \cdot \omega_{\mu-1}$ and $\tilde{\sigma}_\mu = D\varphi_a(\xi_\mu) \cdot \sigma_{\mu-1}$.
3. If μ is a return for ξ_0, split $\tilde{\omega}_\mu = \beta_\mu(1, 0^{m-1}) + \alpha_\mu e_\mu$ with e_μ a norm-1 vector in the $l-$th contractive hyperplane $E^{(l)}(\xi_1)$, $l =$length of the folding period. Then take $\omega_\mu = \tilde{\omega}_\mu - \alpha_\mu e_\mu = \beta_\mu(1, 0^{m-1})$ and $\sigma_\mu = \tilde{\sigma}_\mu + \alpha_\mu e_\mu$.
4. If μ coincides with the end of $s \geq 1$ folding periods, $\mu = \mu_1 + l_1 = \cdots = \mu_s + l_s$, let $\omega_\mu = \tilde{\omega}_\mu + \sum_1^s \alpha_{\mu_i} D\varphi_a^{l_i}(\xi_{\mu_i+1}) e_{\mu_i}$ and $\sigma_\mu = \tilde{\sigma}_\mu - \sum_1^s \alpha_{\mu_i} D\varphi_a^{l_i}(\xi_{\mu_i+1}) e_{\mu_i}$.
5. If neither (3) nor (4) apply (by construction they never apply simultaneously), set $\omega_\mu = \tilde{\omega}_\mu$ and $\sigma_\mu = \tilde{\sigma}_\mu$.

At this point, we are in a position to state our remaining inductive hypotheses on the construction of binding points and binding periods.

Recall that we restrict ourselves to parameter values for which (BA) is satisfied at every free return.

(H1) For $k \leq n-1$, ξ_0 a point bound to $\mathcal{C}_k$ and $\nu \leq k$ a return of ξ_0, a binding point $\zeta_0 \in (\bigcup_2^\nu \mathcal{C}_i)$ is defined, in such a way that

$$a\, d_\nu(\xi_0) \leq \frac{|\beta_\nu(\xi_1)|}{\|\omega_{\nu-1}(\xi_1)\|} = \frac{\|\omega_\nu(\xi_1)\|}{\|\omega_{\nu-1}(\xi_1)\|} \leq 3a\, d_\nu(\xi_0) \quad (7)$$

$$|\alpha_\nu(\xi_1)| \leq 5K\sqrt{b}\, \|\omega_{\nu-1}(\xi_1)\| \quad (8)$$

If $\nu = k$ and it is a free return then we can even take slightly better factors $(3a/2)$, $(5a/2)$ in (7) and $4K\sqrt{b}$ in (8).

(H2) For $k \leq n-1$ and ξ_0 a point bound to $\mathcal{C}_k$, the binding period $[\nu+1, \nu+p]$ associated to a return $\nu \leq k$ satisfies $p \leq 5\alpha\nu < \nu$. Moreover, there are $\tau_1, \tau_2 > 0$ depending only on K, α and β, such that

$$\frac{1}{\tau_1} \leq \frac{\|\omega_{\nu+j}(\xi_1)\|}{|\beta_\nu(\xi_1)|\|\omega_j(\zeta_1)\|} \leq \tau_1 \text{ for } 0 \leq j \leq p-1 \quad (9)$$

$$\frac{\|\omega_{\nu+p}(\xi_1)\|\, d_\nu(\xi_0)}{\|\omega_\nu(\xi_1)\|} \geq \tau_2\, e^{(p+1)(c_1/3)} \quad (10)$$

where $\zeta_1 = \varphi_a(\zeta_0)$ and ζ_0 is the binding point of ξ_ν.

As we said in Section 2, we fix $1/2 < c < c_1 < c_0 < \log 2$. The following consequences of our assumptions are derived in the same way as in the two-dimensional case.

Lemma 6.2.

(a) ([**MV, Lemma 8.1**]) *For $0 \leq \mu \leq k \leq n-1$ and ξ_0 bound to $\mathcal{C}_k$*

$$|\,\mathrm{slope}\,|(\omega_\nu(\xi_1)) \leq \tilde{K}\sqrt{b},\ \tilde{K} = \tilde{K}(K, \delta).$$

(b) ([**MV, Lemma 8.2**]) *Given any $0 \leq \mu \leq k \leq n-1$, there are fold-free iterates $\mu_1 < \mu < \mu_2$ such that*

$$\mu_2 - \mu_1 \leq \max\left\{\frac{10\alpha\mu}{\log(1/b)}, 4\right\}.$$

Moreover, for any free iterate $\nu > \mu$,

$$\mu_2 - \mu_1 \leq \max\left\{\frac{50(\nu-\mu)\log K}{\log(1/b)}, 4\right\}.$$

(c) (see (6),(7),(9) in [**MV, Sec. 8**]) *Let* μ *be a return for a point* ξ_0 *bound to* $\mathcal{C}_k$, *with* $1 \le \mu \le k \le n-1$, *and let* p *be the length of the corresponding binding period. Then, denoting by* ζ_0 *the binding point of* ξ_μ,

(i) $p \ge \frac{1}{3\log K} \log d_\mu(\xi_0)^{-1}$;

(ii) $\|\xi_{\mu+p+1} - \zeta_{p+1}\| \ge he^{-2\beta(p+1)}$.

(d) ([**BC2, Lemma 7.6**],[**MV, Lemma 8.4**]) *For* $0 \le \mu < \nu \le k \le n-1$ *and* ξ_0 *bound to* $\mathcal{C}_k$

$$\|\omega_\nu(\xi_1)\| \ge \min_{\mu<j\le\nu}\left(\frac{\|\omega_j(\xi_1)\|}{\|\omega_{j-1}(\xi_1)\|}\right)\cdot\|\omega_\mu(\xi_1)\| \ge \min_{\mu<j\le\nu}(a\, d_j(\xi_0))\cdot\|\omega_\mu(\xi_1)\|.$$

(e) ([**BC2, Lemma 7.7**], [**MV, Lemma 8.3**]) *For* $1 \le \mu \le k \le n-1$ *and* ξ_0 *bound to* $\mathcal{C}_k$

$$K^{-5}e^{-\alpha\mu}\|\omega_\mu(\xi_1)\| \le \|w_\mu(\xi_1)\| \le K^5 e^{2\alpha\mu}\|\omega_\mu(\xi_1)\|.$$

(f) ([**BC2, Lemma 7.13**],[**MV, Lemma 9.4**]) *For* $1 \le \mu < \nu \le k \le n-1$ *and* ξ_0 *bound to* $\mathcal{C}_k$, *if* $(\nu+1)$ *is a free iterate of* ξ_0 *then*

$$\|w_\nu(\xi_1)\| \ge K^{-5}e^{(\nu-\mu)/10}\|w_\mu(\xi_1)\|.$$

In particular, part (c)(i) implies that binding periods are much longer than the corresponding folding periods. Note also that the hypothesis of (f) makes sense for $\nu = k$ too: it just means that every binding period starting in $[1,k]$ also ends in $[1,k]$.

We close this section by observing that, once the above properties have been extended to $k=n$, and up to an additional restriction on the parameter values, it follows that $\xi_1 = \varphi_a(\xi_0)$, every ξ_0 bound to $\mathcal{C}_n$, is e^c–expansive at time n. In order to see this, let $\nu_1 < \nu_2 < \cdots < \nu_s \le n$ be the free returns of ξ_0. For each $\nu = \nu_i$, $p = p_i$

$$\prod_{\nu}^{\nu+p}\frac{\|\omega_i(\xi_1)\|}{\|\omega_{i-1}(\xi_1)\|} = \frac{\|\omega_{\nu+p}(\xi_1)\|\, d_\nu(\xi_0)}{\|\omega_\nu(\xi_1)\|}\cdot\frac{\|\omega_\nu(\xi_1)\|}{d_\nu(\xi_0)\|\omega_{\nu-1}(\xi_1)\|} \ge 1$$

by (H1) and (H2). Here we use the fact that, according to Lemma 6.2(c), the binding period can be made large by taking δ small, while

keeping α and β fixed. On the other hand, denoting $\mu = \nu_{i+1}$ and $f = \mu - (\nu+p+1)$,

$$\prod_{\nu+p+1}^{\nu}\frac{\|\omega_i(\xi_1)\|}{\|\omega_{i-1}(\xi_1)\|} = \frac{\|\omega_{\mu-1}(\xi_1)\|}{\|\omega_{\nu+p}(\xi_1)\|} = \frac{\|w_{\mu-1}(\xi_1)\|}{\|w_{\nu+p}(\xi_1)\|} \ge e^{c_0 f}$$

as a consequence of Lemma 5.1. Hence, $\|\omega_n(\xi_1)\| \ge \exp(c_0 F_n(a,\xi_0) - \alpha n)$ where

$$F_n(a,\xi_0) = n - \sum_1^s p_i, \quad \text{resp. } F_n(a,\xi_0) = \nu_s - \sum_1^{s-1} p_i \ \text{ if } \ \nu_s \le n < \nu_s + p_s$$

is the *total free time* and the term $-\alpha n$ accounts for the period $(\nu_s + p_s, n]$, resp. $[\nu_s, n]$. We retain only the parameter values for which

$$F_n(a, z_0) \ge (1-\alpha)n \tag{FA}$$

for every $z_0 \in \mathcal{C}_n$ (and so for every ξ_0 bound to $\mathcal{C}_n$). Up to assuming $4\alpha \le (c_0 - c)$, it follows that $\|\omega_n(\xi_1)\| \ge e^{(c_0-2\alpha)n} \ge e^{(c+2\alpha)n}$. Then, using Lemma 6.2(e) for $k=n$, we get $\|w_n(\xi_1)\| \ge e^{cn}$, as we wanted to prove.

7. Returns. Binding Points

Let n be a free return for a point ξ_0 bound to $\mathcal{C}_n$. We describe here the construction of the binding point ζ_0 of ξ_n. The main concern in this construction is to get

$$|\text{angle}|(\xi_n - \zeta_0, \dot\gamma(\zeta_0)),\ |\text{angle}|(\omega_{n-1}(\xi_1), \dot\gamma(\zeta_0)) \ll \|\xi_n - \zeta_0\|$$

where $\gamma \subset W^u(P_a)$ is a b–flat segment centered at ζ_0. We say that $(\xi_n, \omega_{n-1}(\xi_1))$ and $(\zeta_0, \dot\gamma(\zeta_0))$ are in *tangential position*. Once this has been obtained, an essentially 1–dimensional calculation permits to deduce the properties corresponding to the hypothesis (H1).

Definition 7.1. *Fix* $\lambda_0 = (\delta/2)^2$. *Then* $1 \le r \le n$ *is a favourable iterate for* $z_0 \in \mathcal{C}_n$ *if*

1. r *is a fold-free iterate for* z_0;
2. $z_r \in \{(x,Y)\colon |x| < 1 - 2\delta_0\}$;
3. $d_j(z_r) \ge \lambda_0^{j+1}$ *for all* $0 \le j \le n-r-1$.

Lemma 7.1. *If n is a free return for $z_0 \in \mathcal{C}_n$ then there are $1 = m_1 < m_2 < \cdots < m_s \le n$, with $m_{i+1} \le 3m_i$ for $1 \le i \le s-1$ and $n \le 3m_s$, such that each $n - m_i$ is a favourable iterate for z_0.*

Proof. Analogous to [**MV, Lemmas 9.1, 9.2**]. □

The following consequence is deduced in the same way as (8) in [**MV, Sec. 9**].

Corollary 7.2. *For $1 \le i \le s$, z_{n-m_i} is $(\lambda_0/K)^5$–expanding up to time m_i.*

This means that we may apply Lemma 5.2 in order to get, for each $1 \le i \le s$, an integral contracting curve $\Gamma^{m_i}_{V_i}$ passing through z_{n-m_i} and cutting G_1 in a point $\eta_0^{[i]}$. We let $\gamma_0^{[i]} = \gamma(\eta_0^{[i]}, \rho_0^{m_i})$ and denote also $\eta^{[i]} = \varphi_a^{m_i}(\eta_0^{[i]})$ and $\gamma^{[i]} = \varphi_a^{m_i}(\gamma_0^{[i]})$.

Lemma 7.3. *For each $1 \le i \le s$, $\gamma_0^{[i]}$ is contained in $\{(x,Y): |x| \le 1-\delta_0\}$ and so it is b–flat. On the other hand, $\gamma^{[i]}$ is also b–flat and $\|D\varphi_a^{m_i} t\| \ge \|t\|$ for every tangent vector t of $\gamma_0^{[i]}$, implying that $\gamma^{[i]} \supset \gamma(\eta^{[i]}, \rho_0^{m_i})$.*

Proof. Fix $1 \le i \le s$ and let, for simplicity, $m = m_i$, $\gamma_0 = \gamma_0^{[i]}$, $\gamma = \gamma^{[i]}$, $\eta_0 = \eta_0^{[i]}$ and $\eta = \eta^{[i]}$. We write $z_j = (x_j, Y_j)$, $j \ge 0$, and $\eta_0 = (\bar{x}_0, \bar{Y}_0)$. Since $\Gamma^{m_i}_{V_i}$ is nearly vertical we may suppose $|\bar{x}_0 - x_{n-m}| < (\delta_0/2)$. On the other hand, it is easy to see that $(1 - |x_{j+1}|) \le 4(1-|x_j|) + 2(2-a)$ for every j. Since $1 - |x_{n-m}| \ge 10(2-a)$ and $1 - |x_n| \ge 1/2$, it follows that $(1 - |x_{n-m}|) \ge 1/4^{m+1}$. Then

$$1 - |\bar{x}_0| - \rho_0^m \ge 1 - |x_{n-m}| - \frac{\delta_0}{2} - \frac{1}{4^{m+2}} \ge \frac{3}{4}(1 - |x_{n-m}|) - \frac{\delta_0}{2} \ge \delta_0$$

and this proves the first part of the lemma. By Lemma 6.2(a)

$$w_{n-m-1}(z_1) = \omega_{n-m-1}(z_1)$$

has slope $\le 1/10$. Hence, we may apply 5.3, 5.4, to $u = \dot{\gamma}_0(\eta_0)$ = a norm-1 vector tangent to γ_0 at η_0 and

$$v = w_{n-m-1}(z_1)/\|w_{n-m-1}(z_1)\|$$

and get

$$\|D\varphi_a^m(\eta_0)\cdot\dot{\gamma}_0(\eta_0)\| \ge \|w_{n-1}(z_1)\|/2\|w_{n-m-1}(z_1)\| \quad (1)$$

$$|\text{angle}|(D\varphi_a^m(\eta_0)\cdot\dot{\gamma}_0(\eta_0), w_{n-1}(z_1)) \le (\text{const } b^{1/4})^{m+1} \le \text{const } \sqrt{b} \quad (2.)$$

Now (1), Lemma 4.3 (with $\lambda = 1$ and $\nu = m$) and Lemma 6.2(f) (with $\nu = k = n-1$ and $\mu = n-m-1$) give

$$\|D\varphi_a^m(\xi)\cdot\dot{\gamma}_0(\xi)\| \ge \frac{1}{4K^5}e^{m/10} \ge 1 \text{ for every } \xi \in \gamma_0, \quad (3)$$

at least if m is large; if m is small we also get $\|D\varphi_a^m(\xi)\cdot\dot{\gamma}_0(\xi)\| \ge 1$, directly from Lemma 5.1. On the other hand, (2) and Lemma 6.2(a) imply

$$|\text{ slope }|(D\varphi_a^m(\eta_0)\cdot\dot{\gamma}_0(\eta_0)) \le \text{const } \sqrt{b}. \quad (4)$$

In order to conclude the proof it is now sufficient to show that the curvature of γ satisfies $k(\gamma) \le$ const $\sqrt{b}$. Let $\gamma_j = \varphi_a^j(\gamma_0)$ and then $\dot{\gamma}_{j+1} = D\varphi_a(\dot{\gamma}_j)$ and $\ddot{\gamma}_{j+1} = D\varphi_a(\ddot{\gamma}_j) + D^2\varphi_a\cdot(\dot{\gamma}_j,\dot{\gamma}_j)$. From $k(\gamma) = \|(\dot{\gamma}\cdot\dot{\gamma})\ddot{\gamma} - \dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})\|/\|\dot{\gamma}\|^4 = |\det|(\dot{\gamma},\ddot{\gamma})/\|\dot{\gamma}\|^3$ we get

$$k(\gamma_{j+1}) = \left(\frac{\|\dot{\gamma}_j\|}{\|\dot{\gamma}_{j+1}\|}\right)^3 \left(\frac{|\det|(D\varphi_a(\dot{\gamma}_j), D\varphi_a(\ddot{\gamma}_j))}{\|\dot{\gamma}_j\|^3} + \frac{|\det|(D\varphi_a(\dot{\gamma}_j), D^2\varphi_a(\dot{\gamma}_j,\dot{\gamma}_j))}{\|\dot{\gamma}_j\|^3}\right)$$

and so

$$k(\gamma_{j+1}) \le K_j\left(\frac{|\det|(D\varphi_a(\dot{\gamma}_j), D\varphi_a(\ddot{\gamma}_j))}{|\det|(\dot{\gamma}_j,\ddot{\gamma}_j)}k(\gamma_j) + L_j\right)$$

with

$$K_j = \left(\frac{\|\dot{\gamma}_j\|}{\|\dot{\gamma}_{j+1}\|}\right)^3$$

and

$$L_j = |\det|(D\varphi_a(t_j), D^2\varphi_a(t_j,t_j)), \quad t_j = \frac{\dot{\gamma}_j}{\|\dot{\gamma}_j\|}.$$

Now, (QL) implies $L_j \le$ const $\sqrt{b}$ and thus, using also (SD),

$$k(\gamma_m) = (Kb)^m K_{m-1}\cdots K_0\, k(\gamma_0) + \sum_0^{m-1}(Kb)^{m-1-j}K_{m-1}\cdots K_j\,(\text{const } \sqrt{b}).$$

Finally, by Lemma 6.2(f),

$$K_{m-1}\cdots K_j = (\|\dot\gamma_j\|/\|\dot\gamma_m\|)^3 \le (\text{const } e^{(j-m)/10})^3$$

and replacing above we conclude that $k(\gamma_m) \le \text{const } \sqrt{b}$. □

Let also $m_0 = 0$, $\eta^{[0]} = G_1 \cap \{x = x_n\}$ and $\gamma^{[0]} = \gamma(\eta^{[0]}, 1/2)$. Note that $\gamma^{[i]} \subset G_{g_i}$, $g_i = 1 + m_i$, and $g_{i+1} \le 3g_i - 2$ for all $i \ge 0$. Note, moreover, that up to by taking δ and b sufficiently small we may always assume that $\gamma(\eta^{[0]}, \rho_0)$ contains the critical approximation $w_0^{(n-1)} \in \mathcal{C}_n \cap G_1$, recall Sections 5.1 and 5.2.

Definition 7.2. *Let $k \ge 0$ be maximum such that $g_k \le \theta n$ and $\gamma(\eta^{[k]}, \rho_0^{g_k})$ contains some element $\zeta_{0,k}$ of $\mathcal{C}_n$. The binding point of z_n is $\zeta_0 = \zeta_{0,k}$.*

As we said in Section 6, we restrict ourselves to the parameter values for which this construction yields (BA) $d_n(z_0) = \|z_n - \zeta_0\| \ge e^{-\alpha n}$. The measure of the set of parameters excluded by this condition is estimated in Section 9.

Lemma 7.4. *For $k \ge 0$ as in definition above*

$$\|z_n - \eta^{[k]}\| \le b^{3/8} d_n(z_0)$$

and

$$|\text{angle}|(\omega_{n-1}(z_1), \dot\gamma^{[k]}(\eta^{[k]})) \le b^{3/8} d_n(z_0).$$

In particular, there is a b−flat curve γ' passing through ζ_0 and z_n and tangent to $\gamma^{[k]}$ at ζ_0 and to $\omega_{n-1}(z_1)$ at z_n.

Proof. By construction,

$$\|z_n - \eta^{[i]}\| \le (\text{const } b)^{m_i} (\text{const } \sqrt{b}) \le b^{3/8 + g_i/10}$$

for all $i \ge 0$. Moreover, 5.4 yields

$$|\text{angle}|(\omega_{n-1}(z_1), \dot\gamma^{[i]}(\eta^{[i]})) \le (\text{const } b^{1/4})^{g_i} \le b^{3/8 + g_i/10}$$

at least if $g_i \ge 2$, i.e. $i \ge 1$. Actually, the same conclusion holds also for $i = 0$, as a consequence of Lemma 6.2(a) and the fact that G_1 varies in a Lipschitz fashion with the map. Therefore, the lemma will be proved if we show that $d_n(z_0) \ge b^{g_k/10}$. For $g_k \ge (\theta n/3)$ this follows immediately

from (BA) and our definition of θ, see Section 6, equation 6.1. For $g_k \le (\theta n/3)$ we even have the much stronger inequality

$$d_n(z_0) \ge \rho_0^{3g_k}. \tag{5}$$

We prove (5) as follows: assuming that it does not hold we show that $\gamma(\eta^{[k+1]}, \rho_0^{g_{k+1}})$ contains a point $\bar\zeta_0^{(n-1)} \in \mathcal{C}_n$; since $g_{k+1} \le \theta n$, this contradicts the choice of k. First, we take $\mu \ge 1$ such that $\theta\mu < g_{k+1} \le \theta(\mu+1)$ and let $\zeta_0^{(n-1)}, \zeta_0^{(n-2)}, \ldots, \zeta_0^{(\mu+1)}, \zeta_0^{(\mu)}$ be the sequence of critical approximations in $\gamma^{[k]}$ obtained by decreasing the precision of $\zeta_0^{(n-1)} = \zeta_0$. Note that $\zeta_0^{(i)} \in \mathcal{C}_{i+1}$ for all $\mu \le i \le n-1$ because $\zeta_0^{(n-1)} \in \mathcal{C}_n$ and $g_k \le \theta(\mu+1)$. We claim that there is $\bar z_0 \in \gamma^{[k+1]}$ such that

$$\begin{aligned} &\|\zeta_0^{(\mu)} - \bar z_0\| \le (\text{const } \sqrt{b})^{g_k} \quad \text{and} \\ &|\text{angle}|(\dot\gamma^{[k]}(\zeta_0^{(\mu)}), \dot\gamma^{[k+1]}(\bar z_0)) \le (\text{const } \sqrt{b})^{g_k}. \end{aligned} \tag{6}$$

We postpone the justification of (6) and proceed to complete the proof of the lemma. The claim means that we are in a position to use the algorithm of Section 5.3, with $\lambda = 1$ and $\rho = \sigma = \rho_0$, in order to construct a μ−th critical approximation $\bar\zeta_0^{(\mu)} \in \gamma^{[k+1]}$. Notice indeed that $(\text{const } \sqrt{b})^{g_k} \le \frac{1}{2}\rho_0^{\mu}$, as a consequence of our definition of θ. It is easy to check that $\bar\zeta_0^{(\mu)} \in \mathcal{C}''_{\mu+1}$: observe, in particular, that we get

$$\begin{aligned} \|\bar\zeta_0^{(\mu)} - \zeta_0^{(\mu)}\| &\le 4(\text{const } \sqrt{b})^{g_k} \\ &\le 2\rho_0^{\mu} \\ &< (1/4K)^{\mu} \end{aligned}$$

and so equation (4) of section 6 holds. It follows that $\gamma^{[k+1]}$ contains also the point $\bar\zeta_0^{(n-1)} \in \mathcal{C}_n$ obtained by increasing the precision of $\bar\zeta_0^{(\mu)}$. Actually, since we are supposing that (5) does not hold,

$$\begin{aligned} \|\bar\zeta_0^{(n-1)} - \eta^{[k+1]}\| &\le \|\bar\zeta_0^{(n-1)} - \zeta_0^{(n-1)}\| + \|\zeta_0^{(n-1)} - z_n\| + \|z_n - \eta^{[k+1]}\| \\ &\le 2\rho_0^{\mu} + 2(\text{const } b)^{\mu} + \rho_0^{3g_k} + (\text{const } b)^{m_{k+1}}(\text{const } \sqrt{b}) \\ &< \rho_0^{g_{k+1}}, \end{aligned}$$

where we also use $\mu \gg g_{k+1}$ and $g_{k+1} \le 3g_k - 2$. This means that $\bar\zeta_0^{(n-1)} \in \gamma(\eta^{[k+1]}, \rho_0^{g_{k+1}})$, contradicting the maximality of k.

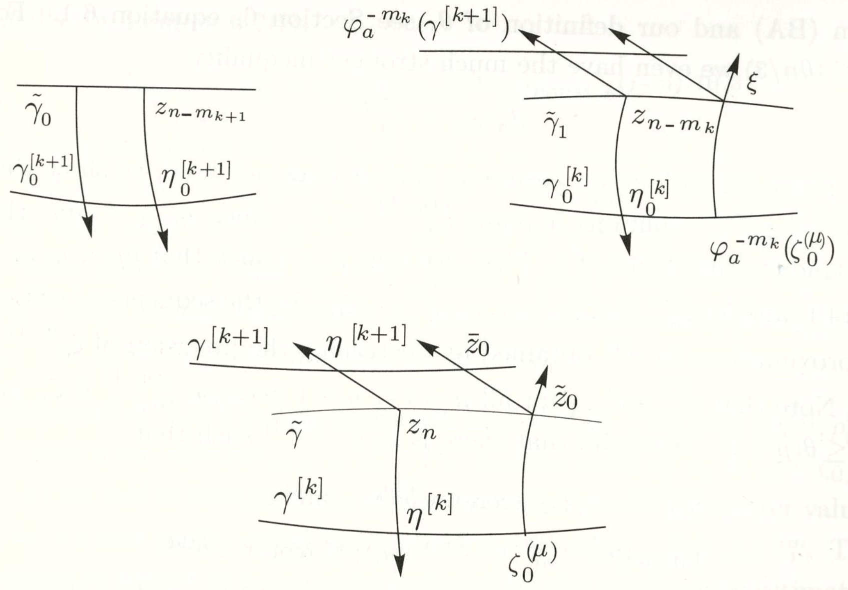

Figure 3

Finally, we prove the claim above. The fact that we have to bound both the distance and the angle requires a more delicate argument than that in the two-dimensional case. Let $\tilde{\gamma}_0$ be a horizontal straight segment of length $2\rho_0^{g_{k+1}}$ centered at $z_{n-m_{k+1}}$ and let $\tilde{\gamma} = \varphi_a^{m_{k+1}}(\tilde{\gamma}_0)$. It follows from the argument in Section 5.4 that for every $\tilde{z}_0 \in \tilde{\gamma}$ there exists $\bar{z}_0 \in \gamma^{[k+1]}$ such that $\|\tilde{z}_0 - \bar{z}_0\|$ and $|\text{angle}|(\dot{\tilde{\gamma}}(\tilde{z}_0), \dot{\gamma}^{[k+1]}(\bar{z}_0))$ are bounded by $(\text{const}\,\sqrt{b})^{g_{k+1}}$. Therefore, in order to prove (6) it is sufficient to show that it holds if $\bar{z}_0 \in \gamma^{[k+1]}$ is replaced by (some) $\tilde{z}_0 \in \tilde{\gamma}$. Note now that, according to (3),

$$\|\eta_0^{[k]} - \varphi_a^{-m_k}(\zeta_0^{(\mu)})\| \leq \|\eta^{[k]} - \zeta_0^{(\mu)}\| \leq \rho_0^{g_k} + (\text{const}\ b)^{\mu} < \rho_0^{m_k}. \qquad (7)$$

Hence, we can use the procedure in Section 5.4 to construct the m_k–th contractive lines $\Gamma_V^{(m_k)}$ passing through $\zeta_0^{(\mu)}$. In view of 5.3, 5.4, our argument will be complete if we show that $\tilde{\gamma}_1 = \varphi_a^{m_{k+1}-m_k}(\tilde{\gamma}_0)$ intersects some of the $\Gamma_V^{(m_k)}$. We do this in the following way. The same reasoning as in (4) above shows that $\tilde{\gamma}_1$ is nearly horizontal: $|\text{slope}|(\dot{\tilde{\gamma}}_1) \leq \text{const}\,\sqrt{b}$. Hence, we can take a nearly horizontal curve

$\tilde{\gamma}_2 \supset \tilde{\gamma}_1$ with radius $\geq 2\rho_0^{g_k}$ around z_{n-m_k}. Then, by (7), $\tilde{\gamma}_2$ intersects some $\Gamma_V^{(m_k)}$ in a point ξ and we are left to show that, actually, $\xi \in \tilde{\gamma}_1$. We take $\tilde{z}_0 = \varphi_a^{m_k}(\xi)$ and then

$$\begin{aligned}\|\tilde{z}_0 - z_n\| &\leq \|\tilde{z}_0 - \zeta_0^{(\mu)}\| + \|\zeta_0^{(\mu)} - \zeta_0^{(n-1)}\| + \|\zeta_0^{(n-1)} - z_n\| \\ &\leq (\text{const}\ b)^{m_k}(\text{const}\,\sqrt{b}) + (\text{const}\ b)^{\mu} + \rho_0^{3g_k} \\ &< \rho_0^{g_{k+1}}.\end{aligned}$$

Using once more the reasoning of (4), we get that $\tilde{\gamma}$ is also a nearly horizontal curve. Moreover, we find as in (3) that vectors tangent to $\tilde{\gamma}_0$ are expanded by $D\varphi^{m_{k+1}}$, so that the radius of $\tilde{\gamma}$ around z_n is larger than $\rho_0^{g_{k+1}}$. Altogether, this implies that $\tilde{z}_0 \in \tilde{\gamma}$ and so $\xi \in \tilde{\gamma}_1$. □

At this point the estimates corresponding to (H1) can be obtained in just the same way as in the two-dimensional setting.

Lemma 7.5. *Let ξ_0 be bound to $\mathcal{C}_n$ and n be a return for ξ_0.*

(a) ([**MV, Lemma 9.6, Corollary 10.4**]) *If n is a free return then*

$$|\alpha_n(z_1)| \leq 4K\sqrt{b}\|\omega_{n-1}(z_1)\|$$

and

$$\frac{3a}{2} d_n(z_0) \leq \frac{|\beta_n(z_1)|}{\|\omega_{n-1}(z_1)\|} \leq \frac{5a}{2} d_n(z_0)$$

(b) ([**MV, Lemma 9.7**]) *If n is a bound return then*

$$|\alpha_n(\xi_1)| \leq 5K\sqrt{b}\|\omega_{n-1}(\xi_1)\|$$

and

$$a\, d_n(\xi_1) \leq \frac{|\beta_n(\xi_1)|}{\|\omega_{n-1}(\xi_1)\|} \leq 3a\, d_n(\xi_1)$$

8. Binding periods

The part of our inductive step dealing with the information on binding periods (induction hypothesis (H2)) is performed in Lemma 8.1 below. This segment of our construction requires no new relevant ingredient: the arguments of [**MV, Sec. 10**] are independent of the dimension of the ambient manifold and so they apply in the present context to prove this lemma. As in [**MV**], we express our distortion estimates in terms of

$$\Theta_\nu = \Theta_\nu(\eta_0, \zeta_0) = \sum_1^\nu b^{(\nu-s)/4} \|\eta_s - \zeta_s\|.$$

Lemma 8.1.

(a) ([**MV, Lemma 10.2**]) *Let η_0 and ζ_0 be bound up to time $\mu \le k$ to a same element z_0 of $\mathcal{C}_k$, $k \le n$. Then, for all $1 \le \nu \le \mu$*

$$\frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|} \le \exp\left(8K \sum_1^\nu \frac{\Theta_s}{d_s(\zeta_0)}\right)$$

and

$$|\text{angle}|(\omega_\nu(\eta_1), \omega_\nu(\zeta_1)) \le 2b^{1/4}\Theta_\nu.$$

(b) ([**MV, Corollary 10.3**]) *Let n be a return for $z_0 \in \mathcal{C}_n$, ζ_0 be the binding point of z_n and p be the length of the corresponding binding period. For every $1 \le \nu \le \min\{p, 5\log(1/d_n(z_0))\}$*

$$\frac{1}{\tau_1} \le \frac{\|\omega_{n+\nu}(z_1)\|}{|\beta_n(z_1)|\|\omega_\nu(\zeta_1)\|} \le \tau_1$$

where $\tau_1 = \tau_1(K, \alpha, \beta) = 2\exp\left(100K \sum_0^\infty e^{j(\alpha-\beta)}\right)$. Moreover, the same holds for every point ξ_0 which remains bound to z_0 up to time $n + \nu$.

(c) ([**MV, Lemma 10.5**]) *Let η_0 and ζ_0 be bound up to time $\mu \le k$ to a same element z_0 of $\mathcal{C}_k$, $k \le n$. Then, for all $1 \le \nu \le \mu$*

$$\frac{\|w_\nu(\eta_1)\|}{\|w_\nu(\zeta_1)\|} \le \exp\left(8Ke^{2\alpha\nu} \sum_1^\nu \frac{\Theta_s}{d_s(\zeta_0)}\right) \quad \text{and}$$

$$|\text{angle}|(w_\nu(\eta_1), w_\nu(\zeta_1)) \le 4e^{2\alpha\nu} b^{1/4} \sum_1^\nu \frac{\Theta_s}{d_s(\zeta_0)}.$$

(d) ([**MV, Lemma 10.6**]) *Let $0 < \sigma < 2\delta$ and $[0, \sigma] \ni s \mapsto \eta_0(s) = (x_0 \pm s, Y(x_0 \pm s))$ be a $b-$flat curve with $\zeta_0 = \eta_0(0) \in \mathcal{C}_k$, $k \le n$. Let $\mu \le k - 1$ be such that*

$$\|\eta_\nu(s) - \zeta_\nu\| \le he^{-\beta\nu} \text{ for every } 0 \le s \le \sigma \text{ and } 1 \le \nu \le \mu.$$

(i) *Then $\|w_\nu(\zeta_1)\|\sigma^2 \le e^{-\beta\nu}$ for all $0 \le \nu \le \mu - 1$.*

(ii) *If, in addition, $\|w_\mu(\zeta_1)\|\sigma^2 \le h^2 e^{-2\beta\mu}$ then*

$$\|\eta_{\mu+1}(s) - \zeta_{\mu+1}\| \le he^{-2\beta(\mu+1)}$$

for all $0 \le s \le \sigma$.

(e) ([**MV, Corollary 10.7**]) *Let n be a return for $z_0 \in \mathcal{C}_n$ and let p be the length of the corresponding binding period. Then*

$$p \le 2/c\log(1/d_n(\xi_0)) \le 5\alpha n.$$

Moreover, for any point ξ_0 which remains bound to z_0 up to time $n + p$

$$\|\omega_{n+p}(\xi_1)\| d_n(\xi_0) \ge \tau_2 e^{c_1(p+1)/3} \|\omega_n(\xi_1)\|$$

where $\tau_2 = \tau_2(K, \alpha, \beta) = h/(2\tau_1)$.

Recall that we have defined $h = \frac{1}{10}\exp\left(-50K\sum_1^\infty e^{-\alpha j}\right)$ in Section 6. We also note that (e) gives $p \le 5\log(1/d_n(\xi_0))$ and so the conclusion of (b) really holds for every $1 \le \nu \le p$.

9. Parameter dependence. Exclusions

Now we develop the main tools (partitions and uniformity of $a-$derivatives) for proving that a set of parameters with positive Lebesgue measure remains after all the exclusions determined by conditions (BA) and (FA). This is also done in an inductive way and the initial step involves the points $z_0^{(i)} \in G_0$ and $w_0^{(i)} \in G_1$ of Sections 5.1 and 5.2.

Lemma 9.1. *Given $1 \le i \le N - 1$, $z_0^{(i)}$ and $w_0^{(i)}$ are defined for every $a \in \Omega$. Moreover, $\|\dot{z}_0^{(i)}(a)\| \le \text{const}\sqrt{b}$ and $\|\dot{w}_0^{(i)}(a)\| \le \text{const}\sqrt{b}$ for all $a \in \Omega$.*

Proof. Let the curve $G_0(a) \cap \{(x, Y) : |x| \le 1/2\}$ be parametrized by $x \mapsto z_0(a, x) = (x, Y(a, x))$. Note that $\|\partial_x Y\| \le \text{const}\sqrt{b}$, as a consequence of (QL) and the Lipschitz dependence of invariant manifolds on the map. Let also $t(a, x) = D\varphi_a(z_0(a, x)) \cdot (1, \partial_x Y(a, x))$. Then $z_0^{(i)}(a) = (x(a), Y(a, x(a)))$ where $x(a)$ is defined implicitly by

$$F(a, x) = t(a, x) \cdot f^{(i)}(a, z_1(a, x)) = 0, \quad z_1(a, x) = \varphi_a(z_0(a, x)).$$

Recall from Sections 5.1 and 5.2 that $|x(a)| \le \text{const}\,\sqrt{b}$. It follows, using (QL) once more,

$$\|\partial_a t(a, x(a))\| \le \text{const}\,\sqrt{b} \quad\text{and}\quad \|\partial_x t(a, x(a)) - (-2a, 0^{m-1})\| \le \text{const}\,\sqrt{b}.$$

Hence, by Lemma 4.5 and recalling also that $f^{(i)}$ is nearly horizontal,

$$|\partial_a F(a, x(a))| \le \text{const}\,\sqrt{b} \quad\text{and}\quad |\partial_x F(a, x(a))| \approx 2a > 1.$$

This proves the lemma for $z_0^{(i)}$ and the same argument applies to $w_0^{(i)}$. □

We consider the curves $z_0^{(i)}, w_0^{(i)} : \Omega \to \mathbb{R}^m$, $1 \le i \le N$, defined by this lemma. Up to this point we left the compact interval $\Omega \subset (1,2)$ essentially arbitrary (except for being close to $a = 2$) but now we fix it in such a way that the first return N is an *escape situation* for these critical curves: (see [**MV, Sec. 3**])

$$\text{length}(z_0^{(i)}(\Omega)) \ge \sqrt{\delta} \quad\text{and}\quad \text{length}(w_0^{(i)}(\Omega)) \ge \sqrt{\delta}.$$

For $n < N$ our set of good parameter values is, simply, $S_n = \Omega$. Let us describe the parameter exclusion procedure. At stage $n \ge N$ we assume that for each $k \le n-1$ a subset S_k of Ω has been defined. We assume, moreover, that for each $a_0 \in S_k$ and $\zeta_0^{(k-1)} \in \mathcal{C}_k(a_0)$ there exist an iterate $\nu \in [\frac{k+1}{2}, k+1]$ and an interval $\omega \subset \Omega$ with $K^{-3\nu/2} \le \text{length}(\omega) \le e^{-2c\nu/3}$, such that

- $\zeta_0^{(k-1)}$ admits a smooth continuation to ω (as a critical approximation) satisfying $\|\dot\zeta_0^{(k-1)}(a)\| \le \sum_1^g b^{i/15} + \sum_1^k b^{j/3} \le b^{1/20}$ for all $a \in \omega$ (here g = generation of $\zeta_0^{(k-1)}$ and $\dot\zeta$ denotes derivative with respect to the parameter a);
- $\zeta_0^{(k-1)}(a)$ satisfies all the conditions of Section 6 ((BA), (FA), expansiveness, binding and folding estimates, etc) for all $a \in \omega$ and all iterates $\le \nu - 1$;
- time ν is an escape situation for $\zeta_0^{(k-1)} : \omega \to \mathbb{R}^m$.

We also state the remaining conditions in our definition of critical sets, recall Remark 6.2. Let $a_0 \in S_k$ and $\zeta_0^{(k-1)} \in \mathcal{C}_k(a_0)$, $z_0^{(k)} \in \mathcal{C}''_{k+1}(a_0)$, be as in (β), Section 6. Let ω be the parameter interval associated to $\zeta_0^{(k-1)}$ as above. It is part of our definition of $\mathcal{C}_{k+1}$ that

the algorithm defining $z_0^{(k)}$ from $\zeta_0^{(k)}$ remains valid for all $a \in \omega$ and, moreover, equation (4) of section 6 holds on the whole ω:

$$\|z_0^{(k)}(a) - \zeta_0^{(k)}(a)\| \le b^{g/10} \le \left(\frac{1}{4k}\right)^k \quad \text{for every} \quad a \in \omega. \tag{1}$$

Remark 9.1. Naturally, we must take these additional assumptions in consideration in the proof of Lemma 7.4. More precisely, we must check that $\bar\zeta_0^{(\mu)}$ satisfies the conditions above, in order to be able to conclude that it belongs to $\mathcal{C}''_{\mu+1}$. It is crucial here that, while the parameter range to be dealt with is smaller than const$\,^\mu$, some const < 1, the construction of $\bar\zeta_0^{(\mu)}$ only involves iterates up to $g_{k+1} \approx \theta\mu$. We explain this in more detail. Let $\tilde\gamma_0$ be the horizontal straight line segment of radius $\rho_0^{g_{k+1}}$ around $y_0 = z_{n-m_{k+1}}(a_0)$. Note that if ω is the parameter interval associated to $\zeta_0^{(\mu)}$ then $\text{length}(\omega) \le e^{-(\mu+1)/6} \le e^{(-1/6\theta)g_{k+1}}$. It follows from an easy calculation that for any $a \in \omega$ the point y_0 is $\varphi_a - (\lambda_0^5/2k^5)-$expanding up to time m_{k+1}, the tangent vectors of $\tilde\gamma_0$ are $(1/2)-$expanded by the derivative of $\varphi_a^{m_{k+1}}$ and, moreover, we have $\|\varphi_a^{m_{k+1}-m_k}(y_0) - z_{n-m_k}(a_0)\| \ll \rho_0^{g_k}$. Now we prove that $\|\varphi_a^{-m_k}(\zeta_0^{(\mu)}(a)) - \varphi_{a_0}^{-m_k}(\zeta_0^{(\mu)}(a_0))\| \ll \rho_0^{g_k}$ for every $a \in \omega$. Let $S \ni s \mapsto \xi(a,s) = (s, H(a,s)) \in G_1(a)$ be a smooth parametrization, with $\xi(a_0, s_0) = \varphi_{a_0}^{-m_k}(\zeta_0^{(\mu)}(a_0))$ and $\xi(a_0, S) = \varphi_{a_0}^{-m_k}(\gamma(\zeta_0^{(\mu)}(a_0), \rho_0^{g_k}))$. Note that $\|\xi(a,s) - \xi(a_0,s)\|$, $\|\partial_s\xi(a,s) - \partial_s\xi(a_0,s)\| \le \text{const length}(\omega)$. It follows, in the same way as before, that for every $(a,s) \in \omega \times S$ the tangent vector $\partial_s\xi(a,s)$ is $(1/2)-$expanded by $D\varphi_a^{m_k}$ and, also, $\|D\varphi_a^{m_k}\partial_s\xi(a,s) - D\varphi_{a_0}^{m_k}\partial_s\xi(a_0,s)\| \ll 1$. Since $\gamma(\zeta_0^{(\mu)}(a_0), \rho_0^{g_k})$ is $b-$flat, we conclude that for every $a \in \omega$ the curve $\varphi_a^{m_k}(\xi(a,S))$ is close to being straight and horizontal: $|\,\text{slope}\,|(D\varphi_a^{m_k}\partial_s\xi(a,s)) \le 1/10$ for every $s \in S$. Thus, $\|\varphi_a^{m_k}(\xi(a,s)) - \varphi_a^{m_k}(\xi(a,s_0))\| \ge \frac{1}{4}|s - s_0|$. On the other hand,

$$\|\zeta_0^{(\mu)}(a) - \zeta_0^{(\mu)}(a_0)\| \le b^{1/20}\,\text{length}(\omega) \ll \rho_0^{g_k}$$

and so it must be $\varphi_a^{-m_k}(\zeta_0^{(\mu)}(a)) = \xi(a, s_0(a))$ for some $|s_0(a) - s_0| \ll \rho_0^{g_k}$. Altogether, this assures that the algorithm defining $\bar\zeta_0^{(\mu)}$ from $\zeta_0^{(\mu)}$ remains valid on the whole ω and gives $\|\bar\zeta_0^{(\mu)}(a) - \zeta_0^{(\mu)}(a)\| \le (\text{const}\,\sqrt{b})^{g_k} \le$

$b^{g_{k+1}/10}$ for every $a \in \omega$.

Let now $a_0 \in S_{n-1}$ and $z_0^{(n-1)} \in \mathcal{C}_n(a_0)$. Take $\zeta_0^{(n-2)} \in \mathcal{C}_{n-1}(a_0)$ to be related to $z_0^{(n-1)}$ in the sense of (α), (β), Section 6, and let $\nu \in [\frac{n}{2}, n]$ and $\omega \subset \Omega$ be associated to $\zeta_0^{(n-2)}$. The next lemma recovers for $z_0^{(n-1)}$ the bound on the norm of a-derivatives contained in the inductive information above.

Lemma 9.2. *Let $g =$ generation of $z_0^{(n-1)}$. Then*

$$\|\dot{z}_0(a)\| \le \sum_1^g b^{i/15} + \sum_1^n b^{j/3} \le b^{1/20} \text{ for all } a \in \omega.$$

Proof. By definition, $\zeta_0^{(n-1)}(a)$ and $z_0^{(n-1)}(a)$ are defined and satisfy

$$\begin{aligned} &\|\zeta_0^{(n-1)}(a) - \zeta_0^{(n-2)}(a)\| \le (\text{const } b)^{n-2} \le b^{n/2} \quad \text{and} \\ &\|z_0^{(n-1)}(a) - \zeta_0^{(n-1)}(a)\| \le b^{g/10} \end{aligned} \tag{2}$$

for all $a \in \omega$. Let $S \ni s \mapsto \xi(a,s) = (s, H(a,s)) \in G_1(a)$ be a smooth parametrization, with $\xi(a_0, S) = \varphi_{a_0}^{1-g}(\gamma(z_0(a_0), \rho_0^{\theta n}))$. We also let $x \mapsto (x, Y(a,x))$ parametrize $(z_0^{(n-1)}(a), \rho_0^{\theta n})$ and write $\zeta(a,s) = \varphi_a^{g-1}(\xi(a,s)) = (x(a,s), Y(a, x(a,s)))$. The same argument as in the remark above shows that $D\varphi_a^{g-1}\partial_s\xi(a,s) = \partial_s\zeta(a,s)$ satisfies

$$\|D\varphi_a^{g-1}\partial_s\xi(a,s)\| \ge \frac{1}{2}\|\partial_s\xi(a,s)\| \ge \frac{1}{2}$$

and

$$|\,\text{slope}\,|(D\varphi_a^{g-1}\partial_s\xi(a,s)) \le 1/10.$$

for every $(a,s) \in \omega \times S$. As a consequence, $|\partial_s x(a,s)| \ge 1/4$. On the other hand, clearly, $\|D_{(a,s)}\zeta\|$, $\|D^2_{(a,s)}\zeta\|$, $\|D^3_{(a,s)}\zeta\| \le \text{const}^{\,g}$ and, in view of the previous estimate, this implies $\|D^i_{(a,x)}Y\| \le \text{const}^{\,g}$ for $i = 1,2,3$. Recall that $z_0^{(n-1)}(a) = (x_0(a), Y(a, x_0(a)))$ is determined by the equation

$$D\varphi_a(x_0, Y(a,x_0))(1, \partial_x Y(a,x_0)) \cdot f^{(n-1)}(a, \varphi_a(x_0, Y(a,x_0))) = 0.$$

Then, using also Lemmas 4.5 and 4.7, an implicit function argument yields $\ddot{x}_0(a) \le \text{const}^{\,g}$ and so $\|\ddot{z}_0^{(n-1)}(a)\| \le \text{const}^{\,g}$. In the same way

we prove that $\|\ddot{\zeta}_0^{(n-2)}(a)\|$, $\|\ddot{\zeta}_0^{(n-1)}(a)\| \le \text{const}^{\,g}$. In particular, for every $a \in \omega$, $\|\ddot{\zeta}_0^{(n-1)}(a) - \ddot{\zeta}_0^{(n-2)}(a)\| \le \text{const}^{\,g}$. Since we also have $\text{length}(\omega) \ge K^{-3n/4}$, we are in a position to use Hadamard's lemma (see [**BC2, Lemma 8.7**]), to conclude that

$$\|\dot{\zeta}_0^{(n-1)}(a) - \dot{\zeta}_0^{(n-2)}(a)\| \le (\text{const } b^{1/2})^n \le b^{n/3}. \tag{3}$$

Now we distinguish two cases. If $z_0^{(n-1)} \in \mathcal{C}_n'$ then $z_0^{(n-1)}(a) = \zeta_0^{(n-1)}(a)$ and so the lemma is a consequence of (3) and the induction hypotheses. If $z_0^{(n-1)} \in \mathcal{C}_n''$ then we also apply Hadamard's lemma to $z_0^{(n-1)}(a)$ and $\zeta_0^{(n-1)}(a)$. From $\|\ddot{z}_0^{(n-1)}(a) - \ddot{\zeta}_0^{(n-1)}(a)\| \le \text{const}^{\,g}$ and the second part of (2) we get

$$\|\dot{z}_0^{(n-1)}(a) - \dot{\zeta}_0^{(n-1)}(a)\| \le (\text{const } b^{1/10})^g \le b^{g/15}. \tag{4}$$

Observe that in this case $g >$ generation of $\zeta_0^{(n-2)}$. Hence, the lemma follows from (3), (4) and the induction hypotheses. □

From now on we take $z_0 = z_0^{(n-1)}: \omega \to \mathbb{R}^m$ to be as above. In order to describe and estimate the exclusions of parameter values necessary for (BA) and (FA) to hold for z_0, we introduce partitions $\mathcal{P}_j(z_0)$ and subsets $S_j(z_0)$ of ω, $\nu - 1 \le j \le n$, as follows. For $j = \nu - 1$ we set $\mathcal{P}_j(z_0) = \{\omega\}$ and $S_j(z_0) = \omega$. Let $\nu \le j \le n$ and $\bar{\omega} \in \mathcal{P}_{j-1}(z_0)$, with $\bar{\omega} \subset S_{j-1}(z_0)$. We say that j is a *return situation* for $z_0\,|\bar{\omega}$ if it is a return for some $z_0(a)$, $a \in \bar{\omega}$. We call a return situation *free* if it does not belong to any binding period of $z_0(a)$, $a \in \bar{\omega}$. A free return situation is called *essential* if $\Delta(\bar{\omega}) = \{\|z_j(a) - \tilde{z}_0\| : a \in \bar{\omega}\}$ contains some interval

$$I_{r,i} = e^{-r} + \frac{e-1}{r^2}[i-1, i), \quad 1 \le i \le r^2, \quad r > |\log\delta|.$$

Here $\tilde{z}_0 = \tilde{z}_0(a_0)$ is the binding point of $z_0(a_0)$, any fixed $a_0 \in \bar{\omega}$. By definition,

(a) if j is an essential return situation then the elements of $\mathcal{P}_j(z_0)$ contained in $\bar{\omega}$ are the connected components of the sets $\omega_{r,i}$ and $\tilde{\omega}$ defined by (see also remark below)

$$a \in \omega_{r,i} \iff \Delta(a) = \|z_j(a) - \tilde{z}_0\| \in I_{r,i} \text{ and } \tilde{\omega} = \bar{\omega} \setminus \bigcup_{r,i} \omega_{r,i}$$

(b) otherwise, $\bar{\omega} \in \mathcal{P}_j(z_0)$.

Remark 9.2. This definition requires a few comments. Observe first that, due to (1), $z_0(a)$ is bound to $\zeta_0(a)$ up to time n and so all the notions (returns, binding periods) involved in the definition are indeed defined (by induction). We also make the following convention: for an interval ω' in a partition $\mathcal{P}_l(z_0')$, all the returns, binding periods and folding periods during $[1, l]$ are independent of $a \in \omega'$. Note that this is only a slight adjustment in the definitions, not affecting the estimates in Section 6. Indeed, by construction, $\log d_i(z_0(a))$, $1 \le i \le l$, is nearly constant on ω' (formally speaking, this last affirmative is also part of the induction). Note also that the particular choice of $a_0 \in \bar{\omega}$ above is irrelevant, because $\|\tilde{z}_0(a) - \tilde{z}_0(a_0)\| \le b^{1/20}\,\text{length}(\bar{\omega}) \ll e^{-\alpha j}$. Finally, for every interval $\omega' \in \mathcal{P}_j(z_0)$ having a return at time j, we need $\Delta(\omega')$ to contain some $I_{r,i}$ (and to be contained in at most three of these intervals). This requires the following exception in the definition of the $\omega_{r,i}$ in (a) above: if a connected component ω'' of $\Delta^{-1}(I_{r,i})$ has $\Delta(\omega'') \neq I_{r,i}$, instead of taking it to be an element of $\mathcal{P}_j(z_0)$, we join it to a nearby $\Delta^{-1}(I_{s,l})$ to form the corresponding $\omega_{s,l}$.

We take $(S_j(z_0) \setminus S_{j-1}(z_0)) \cap \bar{\omega}$ to be the union of the following intervals in $\mathcal{P}_j(z_0)$:

(BA) the connected components of all $\omega_{r,i} \subset \bar{\omega}$ with $r > j\alpha$ and

(FA) all the $\omega' \in \mathcal{P}_j(z_0)$ for which the total number of free iterates in $[1, j]$ is $< (1-\alpha)j$.

If $\nu < (n+1)/2$ we also exclude from $S_n(z_0)$ the $\omega' \in \mathcal{P}_n(z_0)$ which have no escape situation during $(\nu, n]$. This assures that the induction hypotheses stated earlier in this section are completely recovered for z_0 at time n.

A main ingredient in the estimation of the total measure of the excluded intervals is the fact that at free return situations $\dot{z}_j(a)$ is nearly horizontal and nearly uniform on $\bar{\omega}$. This is the content of part (d) of the lemma below, whose proof is contained in [**MV, Sec. 11**]. As in there, we denote $w_j(a) = w_j(a, z_1(a))$ and $\omega_j(a) = \omega_{j-1}(a, z_1(a))$.

Lemma 9.3.
(a) ([**MV, Lemma 11.3**], [**BC2, Lemmas 8.1, 8.4**]) *For every $\nu \le j \le n$ and $\bar{\omega} \in \mathcal{P}_{j-1}(z_0)$, $\bar{\omega} \subset S_{j-1}(z_0)$, we have*

$$\frac{1}{100} \le \frac{\|\dot{z}_j(a)\|}{\|w_{j-1}(a)\|} \le 100 \text{ for all } a \in \bar{\omega}.$$

If j is a free iterate we also have $|\text{angle}|(\dot{z}_j(a), w_{j-1}(a)) \le b^{1/4}$.

(b) ([**MV, Corollary 11.4**]) *For every $\nu \le j \le n$ and $\tilde{\omega} \in \mathcal{P}_j(z_0)$, $\tilde{\omega} \subset S_{j-1}(z_0)$, we have $K^{-3j/2} \le \text{length}(\tilde{\omega}) \le e^{-2cj/3}$.*

(c) ([**MV, Lemma 11.5**]) *There is $\tau_3 = \tau_3(K, \alpha, \beta, \delta) > 0$ such that if $j \in [\nu, n]$ is a free return situation for $\bar{\omega} \in \mathcal{P}_{j-1}(z_0)$, $\bar{\omega} \subset S_{j-1}(z_0)$, then for all $a_1, a_2 \in \bar{\omega}$*

$$\frac{\|w_{j-1}(a_1)\|}{\|w_{j-1}(a_2)\|} \le \tau_3 \text{ and } |\text{angle}|(w_{j-1}(a_1), w_{j-1}(a_2)) \le 5b^{1/4}.$$

(d) ([**MV, Corollary 11.6**]) *There is $\tau_4 = \tau_4(K, \alpha, \beta, \delta) > 0$ such that if $j \in [\nu, n]$ is a free return situation for $\bar{\omega} \in \mathcal{P}_{j-1}(z_0)$, $\bar{\omega} \subset S_{j-1}(z_0)$, then for all $a_1, a_2 \in \bar{\omega}$*

$$\frac{\|\dot{z}_j(a_1)\|}{\|\dot{z}_j(a_2)\|} \le \tau_4 \text{ and } |\text{angle}|(\dot{z}_j(a_1), \dot{z}_j(a_2)) \le 10b^{1/4}.$$

At this point the measure of the excluded set can be estimated in precisely the same way as in [**MV, Sec. 12**]. A crucial fact here is that each of the excluding rules above eliminates an exponentially small set of parameter values. More precisely, it follows from the same arguments as in [**BC2, Sec. 2**] or [**MV, Sec. 3**] that $m(S_j(z_0) \setminus S_{j-1}(z_0)) \le A_1 e^{-\alpha_1 j} m(\omega)$, with A_1 and α_1 depending on K, α, β, δ but not on N or b. Summing over all $\nu \le j \le n$, we find $m(S_n(z_0) \setminus \omega) \le A_2 e^{-2\alpha_2 n} m(\omega)$, for some A_2 and α_2 independent of N and b. On the other hand, by Lemma 6.1, the number of critical points we have to consider at each stage does not increase too fast: we set $S_n = S_{n-1} \setminus (\bigcup_{z_0} (S_n(z_0) \setminus \omega))$ and then

$$m(S_n \setminus S_{n-1}) \le 8A_2 e^{-2\alpha_2 n} \left(\frac{K}{\rho_0}\right)^{\theta n} m(\Omega) \le A_2 e^{-\alpha_2 n} m(\Omega)$$

if b is sufficiently small. Hence, $S_\infty = \bigcap_{n \ge N} S_n$ has positive Lebesgue measure

$$m(S_\infty) \ge m(\Omega) \left(1 - \sum_{n \ge N} A_2 e^{-\alpha_2 n}\right) > 0,$$

as long as N is also large enough (i.e. Ω is close enough to $a = 2$).

Finally, the reasoning in [**BC2, Sec. 10**] extends, in a straightforward way, to the present setting to show that for almost every $a \in S_\infty$ the orbit of the critical point $z_0(a)$ of generation zero (say) is dense in $W^u(P_a)$. This completes the proof of Theorem A.

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