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Note on exterior lines to two disjoint reguli

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Abstract. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint reguli in the projective 3-space over the field $GF(q)$ where $q = p^e$, p an odd prime. If p is not a point in neither of the two doubly-ruled quadrics associated to the given reguli, then there is at least one line through P which does not meet neither of the two reguli.

1. Introduction

Let Σ be the projective 3-space over the field $GF(q)$ where $q = p^e, p$ an odd prime. Let a, b, c be any three skew lines in Σ. Through any point on a there passes exactly one line that meets b and c. Such a line is called a *transversal* of a, b and c; there are exactly $q + 1$ transversals to any three skew lines, and they are mutually skew. Now let a', b', c' be any transversal to a, b and c. The set $\mathcal{R}$ of transversals to a', b' and c', which includes the lines a, b, c, is independent of the choice of a', b', c', and is called the *regulus* determined by a, b and c. It can be shown that the same regulus is determined by any three of its $q + 1$ lines. Moreover, the set of transversals to any three lines of a regulus $\mathcal{R}$ is independent of the particular choice of the three lines of $\mathcal{R}$, and forms a regulus $\mathcal{R}'$, called the *opposite* regulus to $\mathcal{R}$. Thus every line of $\mathcal{R}'$ meets every line of $\mathcal{R}$, and $\mathcal{R}$ and $\mathcal{R}'$ both cover the same $(q + 1)^2$ points of Σ, forming a *doubly-ruled quadric* $\mathcal{D}$. The lines of $\mathcal{R}$ and the lines of $\mathcal{R}'$ both lie in $\mathcal{D}$ [5].

The main purpose of this note is to show that through any point exterior to two disjoint reguli there is at least one line disjoint to both reguli. Although the

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proof is quite easy, the result we present appears to have been overlooked.

2. Desarguesian spreads in Σ

Let Σ be the projective 3-space over the field $GF(q)$ where $q = p^e$, p an odd-prime. A spread W in Σ is a set of q^2+1 lines in Σ which are such that each point of Σ lies on exactly one line of W. Thus the lines of W are all mutually skew. We shall call a spread W desarguesian if the regulus determined by any three lines of W is contained in W. The following, which we state without proof, appear in [4]:

2.1 Lemma. *The number of desarguesian spreads of Σ is $\frac{1}{2}q^4(q^3-1)(q-1)$.*

2.2 Lemma. *The number of reguli in Σ is $q^4(q^3-1)(q^2+1)$.*

2.3 Lemma. *Each desarguesian spread of Σ contains exactly $q(q^2+1)$ reguli and each regulus is contained in $\frac{1}{2}q(q-1)$ desarguesian spreads.*

2.4 Lemma. *Let W and W' be two desarguesian spreads of Σ and assume that $\mathcal{R}$ is a regulus contained in $W \cap W'$. If $|W \cap W'| > q+1$, then $W = W'$.*

3. The number of lines which do not meet a regulus

3.1 Lemma. *Let $\mathcal{R}$ be a regulus in Σ. If P is a point not contained in any line of $\mathcal{R}$, then there are exactly $\frac{1}{2}q(q-1)$ lines of Σ through the point P which do not meet the ruled quadric covered by $\mathcal{R}$.*

Proof. There are exactly $\frac{1}{2}q(q-1)$ desarguesian spreads containing $\mathcal{R}$, and P is contained in exactly one line of each desarguesian spreads. By (2.4) these are all different lines. On the other hand if ℓ is a line such that $\mathcal{R} \cup \{\ell\}$ consists of lines all mutually skew that $\mathcal{R} \cup \{\ell\}$ is contained in a unique desarguesian spread. Hence the number of lines through P which do not meet any line of $\mathcal{R}$ is exactly $\frac{1}{2}q(q-1)$.

3.2 Lemma. *Let $\mathcal{R}$ be a regulus in Σ. If P is a point not contained in any line of $\mathcal{R}$, then there are exactly $q+1$ lines of Σ through the point P which intersect a unique line of $\mathcal{R}$ and there are $\frac{1}{2}q(q+1)$ lines of Σ through the point P which intersects two lines of $\mathcal{R}$.*

Proof. By (3.1) there are

$$q^2+q+1-\frac{1}{2}q(q-1) = \frac{(q+2)(q+1)}{2}$$

lines of Σ through P which intersect lines of $\mathcal{R}$. Moreover, any one of these lines meet either one or two lines of $\mathcal{R}$. Let x denote the number of lines which meet one line of $\mathcal{R}$ and let y denote the number of lines which meet two lines of $\mathcal{R}$. Hence

$$x+2y = (q+1)^2 \quad \text{and} \quad x+y = \frac{(q+2)(q+1)}{2}$$

Thus $x = q+1$ and $y = \frac{1}{2}q(q+1)$.

4. Exterior lines to two disjoint reguli

We say that two reguli in Σ are disjoint if the corresponding ruled quadrics are disjoint. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be any two disjoint reguli. Coordinates may be chosen so that the lines of $\mathcal{R}_1$ have the form

$$\{(x,y)/x=(0,0)\} \bigcup \left\{(x,y)/y = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\right\}_{u\in GF(q)}$$

and (x,y) represents a 4-vector over $GF(q)$. Hence the lines of $\mathcal{R}_2$ have the form

$$\{(x,y)/y = xM_i\} \tag{4.1}$$

where M_i is a 2 by 2 matrix with entries in $GF(q)$, $M_i - M_j$ is a nonsingular matrix for $i \neq j$, and for each $i = 1, \dots, q+1$, $M_i - xI$ is a nonsingular matrix for all $x \in GF(q)$. Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let P be a point not in any line of $\mathcal{R}_1 \cup \mathcal{R}_2$.

It is well known that all reguli in Σ are in one orbit of $PGL(4,q)$. Moreover the stabilizer of a regulus in $PGL(4,q)$ is isomorphic to $PGL(2,q) \times PGL(2,q)$. In fact, the stabilizer of $\mathcal{R}_1$ is induced by the following subgroup of $GF(4,q)$

$$\left\{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} /A \in GL(2,q)\right\} \left\{\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix} /\alpha\delta - \beta\gamma \neq 0\right\}.$$

Also, the stabilizer of $\mathcal{R}_1$ acts transitively on the points not on any line of $\mathcal{R}_1$.

Thus, without loss of generality we may assume that the point P is represented by the vector $(1,0,0,1)$, and a line through the point P not intersecting any line of $\mathcal{R}_1$ has the form

$$\left\{(x,y) \mid y = x \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}\right\}$$

where $4c = \gamma - d^2$ and γ a nonsquare in $\mathrm{GF}(q)$. Note that $\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} - \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ must be nonsingular for all u in $\mathrm{GF}(q)$.

Let $\mathcal{E}_P$ the collection of all these lines. Next we calculate the number of lines in $\mathcal{E}_P$ disjoint to a given line in $\mathcal{R}_2$. Denote by $\mathcal{E}_P(\ell)$ the number of lines in $\mathcal{E}_P$ which meet ℓ.

4.2 Lemma. *Let $\mathcal{R}_2$ be a regulus represented as in (4.1). If*

$$\ell = \left\{(x,y) \mid y = x \begin{pmatrix} r & s \\ t & u \end{pmatrix}\right\}$$

is an arbitrary line of $\mathcal{R}_2$ then the following hold:

(a) *If $r^2 - (1-s)^2t - ur(1-s) = 0$ then $\mathcal{E}_P(\ell) = 0$.*

(b) *If $r^2 - (1-s)^2t - ur(1-s)$ is a nonzero square in $\mathrm{GF}(q)$ then $\mathcal{E}_P(\ell) = \dfrac{q-1}{2}$.*

(c) *If $r^2 - (1-s)^2t - ur(1-s)$ is a non square in $\mathrm{GF}(q)$ then $\mathcal{E}_P(\ell) = \dfrac{q+1}{2}$*

Proof. The line

$$\left\{(x,y) \mid y = x \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}\right\}$$

in $\mathcal{E}_P$ meets ℓ in and only if $\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} - \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ is a singular matrix. Now, replacing $4c = \gamma - d^2$ in the determinant of this matrix, this condition is equivalent to

$$r^2 - (1-s)^2t - ur(1-s) + \left(\frac{1-s}{2}\right)^2 \gamma$$

is square in $\mathrm{GF}(q)$. Let m and n denote $r^2 - (1-s)^2t - ur(1-s)$ and $\left(\dfrac{1-s}{2}\right)^2$ respectively. Hence, if $m = 0$, and since $s \neq 1$, we have that $n\gamma$ is a non square for each γ non square in $\mathrm{GF}(q)$, so (a) follows.

If $0 \neq m$ is a square in $\mathrm{GF}(q)$ then there are $\dfrac{q+1}{4}$ non squares γ in $\mathrm{GF}(q)$, in case $q-1$ is a non square in $\mathrm{GF}(q)$, for which $m + n\gamma$ is a square in $\mathrm{GF}(q)$,

and $\dfrac{q-1}{4}$ non squares γ in $\mathrm{GF}(q)$, in case -1 is a square in $\mathrm{GF}(q)$, for which $m + n\gamma$ is a square in $\mathrm{GF}(q)$ [1]. In the first case there are $1 + \left(\dfrac{q+1}{4} - 1\right)2$ such pairs of elements (c,d), and in the second case there are $2\dfrac{q-1}{4}$ such pairs (c,d). Thus (b) follows. If m is a non square in $\mathrm{GF}(q)$, then there are $\dfrac{q+1}{4}$ non squares γ in $\mathrm{GF}(q)$, in case -1 is a non square in $\mathrm{GF}(q)$, for which $m + n\gamma$ is a square in $\mathrm{GF}(q)$, and $\dfrac{q+3}{4}$ non squares γ in $\mathrm{GF}(q)$, in case -1 is a square in $\mathrm{GF}(q)$, for which $m + n\gamma$ is a square in $\mathrm{GF}(q)$. In the first case there are $2 \cdot \dfrac{q+1}{4} = \dfrac{q+1}{2}$ such pairs of elements (c,d), and in the second case there are $1 + 2 \cdot \left(\dfrac{q+3}{4} - 1\right) = \dfrac{q+1}{2}$ such pairs of elements (c,d). Thus (c) follows, and the proof of the lemma is complete.

4.3 Remark. Now, if ℓ is the line

$$\left\{(x,y) \mid y = x \begin{pmatrix} r & s \\ t & u \end{pmatrix}\right\},$$

then we denote by $\ell^1 = \{(c,d) \mid r(d-u) - (s-1)(c-t) = 0\}$. Thus, any pair (c,d) in ℓ^1 determines a line through P which meet ℓ. Clearly $|\ell^1 \cap m^1| \geq 2$ implies $\ell^1 = m^1$ for any two lines ℓ and m.

If $C_\gamma = \{(c,d) \mid 4c = \gamma - d^2\}$ for γ non square in $\mathrm{GF}(q)$, then we have proved in lemma 4.2 that $|\ell^1 \cap C_\gamma| \leq \dfrac{q+1}{2}$, for each line ℓ in $\mathcal{R}_2$.

Let $\mathcal{R}_2 = \{\ell_1, \ell_2, \ldots, \ell_{q+1}\}$. Assume that $\ell_i^1 \neq \ell_j^1$ for $i \neq j$, $i,j = 1,2, \ldots, q+1$.

If a pair (c,d) in C_γ determines a line through P which meets $\mathcal{R}_2$ then it determines a set $\{i,j\} \subseteq \{1,2,\ldots,q+1\}$ as follows: if t meets only the line ℓ_i we say that (c,d) determines $\{i,i\}$, and if t meets the lines ℓ_i and ℓ_j we say that (c,d) determines $\{i,j\}$. By assumption, if (c,d) determines $\{i,j\}$ and (c',d') determines $\{i',j'\}$ then $(c,d) \neq (c',d')$ implies $\{i,j\} \neq \{i',j'\}$.

Now let $\mathcal{H}$ be the set of all subsets $\{i,j\}, i \neq j = 1,2,\ldots,q+1$.

If $\mathcal{H}_\alpha$ denote the collection of all subsets S of $\mathcal{H}$ for which $|\{j|\{i,j\} \in S\}| \leq \alpha$ for each $i = 1,2,\ldots,q+1$, then it is clear that for each $S \in \mathcal{H}_\alpha$ we have

$$|S| \leq \alpha \cdot \frac{q+1}{2} \tag{4.4}$$

We are now ready to prove the following theorem.

4.5 Theorem. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two disjoint reguli in the projective 3-space over the field* GF(q) *where $q = p^e, p$ an odd prime. If P is not a point in neither of the two doubly-ruled quadrics associated to the given reguli, then there is at least one line through P which does not meet neither of the two reguli.*

Proof. Without loss of generality we may assume that the two reguli and the point P are as above. If the regulus $\mathcal{R}_2$ is such that there are lines ℓ_i and $\ell_j, i \neq j$ for which $\ell_i^1 = \ell_j^1$, then the number of lines through P that meet $\mathcal{R}_2$ is smaller or equal than the number of lines through P that meet a regulus for which this situation does not occur. Hence we consider only this case.

By (4.2), (4.3) and (4.4) the number of lines through P which does not meet $\mathcal{R}_1$ but meet $\mathcal{R}_2$ is smaller or equal to

$$q+1+\left(\frac{q+1}{2}\right)^2 = \frac{q^2+6q+5}{4}.$$

Note there is a set $\mathcal{I}$ of $q+1$ lines through P that are tangent to $\mathcal{R}_2$, and a set $\mathcal{D}$ of $\frac{1}{2}q(q-1)$ lines through P that are disjoint from $\mathcal{R}_1$. Of the lines of $\mathcal{D}$ there are at most $\frac{q+1}{2}$ lines which intersect a given line ℓ of $\mathcal{R}_2$. If $J \in \mathcal{D} - \mathcal{I}$ and J intersects ℓ then J also must intersect another line $m \neq \ell$ of $\mathcal{R}_2$. Thus the number of lines of $\mathcal{D}$ which intersect $\mathcal{R}_2 \leq 1+q+\frac{(q+1)^2}{2}$. Therefore, the number of lines through P which do not meet neither $\mathcal{R}_1$ and $\mathcal{R}_2$ is greater or equal to

$$\frac{q(q-1)}{2} - \frac{q^2+6q+5}{4} = \frac{q^2-8q-5}{4}$$

Hence if $q > 7$, there is at least one line through P which does not meet neither of the two reguli.

Assume now $q = 7$. Let α denote the number of lines through P which intersect a unique line of $\mathcal{R}_2$. Therefore the number of lines through P which intersect $\mathcal{R}_2$ does not exceed

$$\alpha + \frac{q+1-\alpha}{2} + \frac{q-1}{2}\frac{q+1}{2} = \frac{q^2+2q+1+1+2\alpha}{4} = 16 + \frac{1+2\alpha}{4}$$

where $0 \leq \alpha \leq 8$. Hence, there is at least one line through P which does not meet neither of the two reguli.

For $q = 3$ and 5 direct calculations show the existence of such lines.

4.6 Note. If two reguli are not disjoint then there is a Desarguesian plane containing them. If the two reguli of PG$(3,q)$ share precisely one line, then they are contained in exactly one regular spread of PG$(3,q)$ [3]. The case when the two reguli share two lines is considered in [2].

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