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The fully non-linear Cauchy problem with small data

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Abstract. This paper is devoted to the Cauchy problem for a fully non-linear perturbation

$$(\partial_t^2 - \Delta)u + G(u', u'') = 0$$

of the wave equation with three space dimensions and small data $u = \varepsilon u_0$, $\partial_t u = \varepsilon u_1$; $u_j \in C_0^\infty(\mathbb{R}^3)$. Here $G \in C^\infty$ vanishes of second order at the origin. We give an explicit positive lower bound for $\underline{\lim}_{\varepsilon \to 0} \varepsilon \log T_\varepsilon$ where T_ε is the lifespan of the solution; it is equal to the lifespan of the limit as $\varepsilon \to 0$ of a rescaled solution. The main point is an exact determination of the lifespan for the solution of a non-linear first order differential equation in $\mathbb{R}^2$ of the form

$$\partial u/\partial t = a(\partial u/\partial x)^2 + 2bu\partial u/\partial x + cu^2$$

with $u(0, x) = u_0(x) \in C_0^\infty(\mathbb{R})$.

1. Introduction

As is well known, the Cauchy problem cannot in general be solved globally even for an ordinary differential equation. The simplest example is perhaps

$$du/dt = au^2, \quad u(0) = u_0, \tag{1.1}$$

with the solution $u(t) = u_0/(1 - au_0 t)$ when $au_0 t < 1$. The *lifespan* T of the solution, that is, the largest $T \in (0, \infty]$ such that the solution exists for $0 \le t < T$, is given by

$$T^{-1} = \max(au_0, 0).$$

By a change of scales it follows that for the Cauchy problem with *small data*

$$du/dt = F(u), \quad u(0) = \varepsilon u_0, \tag{1.2}$$

where $F \in C^\infty$, $F(0) = F'(0) = 0$, we have for the lifespan T_ε

$$\lim_{\varepsilon \to +0} (\varepsilon T_\varepsilon)^{-1} = \max(F''(0)/2, 0).$$

Next we recall the analogous results for the simplest non-linear hyperbolic

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partial differential equation

$$\partial u/\partial t = au\partial u/\partial x, \quad u(0,\cdot) = u_0 \in C_0^1(\mathbb{R}), \tag{1.3}$$

often called Burgers' (inviscid) equation although its study goes back at least to the beginning of the 19th century. The equation means that on the characteristic curves defined by

$$dx/dt = -au(t,x) \tag{1.4}$$

we have

$$du/dt = 0, \tag{1.5}$$

thus u = constant. This gives $u(t,x) = u_0(x + atu(t,x))$, a formula which Stokes [16] attributes to Poisson. Stokes observed that the formula means that the graph of $x \mapsto u(t,x)$ is equal to that of $y \mapsto u_0(y)$ referred to the new coordinates $y = x + atu_0$, $u = u_0$, which means that the maximum slope approaches infinity as $t \uparrow T, T^{-1} = \max_y \max(au_0'(y), 0)$, if $T < \infty$. Analytically this is also seen from the fact that differentiation of (1.3) gives

$$d\xi/dt = a\xi^2, \quad \xi = \partial u/\partial x \tag{1.6}$$

along the curves (1.4); this is an equation of the form (1.1). Thus T is the lifespan of u. Note that (1.3) has a homogeneity property similar to that of (1.1): if u_0 is replaced by εu_0 then the solution is replaced by $\varepsilon u(\varepsilon t, x)$, so the lifespan is multiplied by $1/\varepsilon$. By a change of scales one can obtain the asymptotic behavior of the lifespan for the solution of the Cauchy problem

$$\partial u/\partial t = a(u)\partial u/\partial x, \quad u(0,\cdot) = \varepsilon u_0, \tag{1.7}$$

when $\varepsilon \to 0$, even for systems for which $a(0)$ has real simple eigenvalues. (See John [6] and Hörmander [2].)

Much work has been devoted during the past decade to the Cauchy problem with small data for non-linear perturbations of the wave equation

$$\Box u + G(u, u', u'') = 0, \tag{1.8}$$

where $G \in C^\infty$ vanishes of second order at 0; this means that the equation is translation invariant and has the standard wave operator

$$\Box = \partial_t^2 - \Delta$$

in $\mathbb{R}^{1+n}$ as linearization at the solution $u = 0$. (For references up to 1983 see also Klainerman [11].) Fritz John pointed out that the Cauchy problem for (1.8) with data

$$u(0,\cdot) = \varepsilon u_0, \quad \partial_t u(0,\cdot) = \varepsilon u_1, \tag{1.9}$$

where $u_j \in C_0^\infty(\mathbb{R}^n)$, should have a much larger lifespan when the dimension n is large, because disturbances will then be quickly attenuated by spreading over a sphere with large area. This is indeed the case. A heuristic argument suggesting

the magnitude of the lifespan can be given as follows. For the solution u_L of the linear Cauchy problem

$$\Box u_L = 0, \quad u_L(0,\cdot) = u_0, \quad \partial_t u_L(0,\cdot) = u_1,$$

we have as $t \to +\infty$ (see e.g. [2])

$$u_L(t,x) = t^{(1-n)/2} F(\omega, r - t) + O(t^{-(1+n)/2}).$$

Here F is the Friedlander radiation field given for $n = 3$ by

$$F(\omega, q) = (R(\omega, q, u_1) - dR(\omega, q, u_0)/dq)/(4\pi),$$

where

$$R(\omega, q, \varphi) = \langle \varphi, \delta(\langle \cdot, \omega\rangle - q)\rangle$$

is the Radon transform of φ. When $n = 3 + 2k$ one obtains F apart from a constant factor by k additional differentiations; when $n = 2 + 2k$ this should be followed by convolution with $q \mapsto \max(0, -q)^{1/2}$. If one recalls that the basic energy estimate for the wave equation $\Box u = g$ states that

$$dE(t)/dt \le \|g(t,\cdot)\|$$

where $E(t) = \|u'(t,\cdot)\|$ and the norms are L^2 norms, this suggests that for a solution of (1.8), (1.9) one should have

$$E'(t) \le CE(t)\varepsilon(t+1)^{-(n-1)/2},$$

as would be the case if G were just a quadratic form in u' and the maximum of u' for fixed t is roughly as in the linear case. Integration of this inequality would give

$$E(t) \le E(0) \exp\left(\int_0^t C\varepsilon(t+1)^{-(n-1)/2} dt\right)$$

and suggests global existence if $n > 3$, existence when $\epsilon \log t < c$ if $n = 3$ and when $\epsilon t^{(3-n)/2} < c$ if $n = 1, 2$. To justify this argument one must deal not only with the equation (1.8) as it stands but also with the equations obtained by differentiating. It is not sufficient just to differentiate with respect to the coordinates in $\mathbb{R}^{1+n}$, for energy estimates can then at best lead to the conclusion that solutions are bounded in time, not that they decay as required above. However, Klainerman [9] has observed that the desired decay can be obtained by using all equations obtained by applying to (1.8) any product of the vector fields $\partial/\partial x_j, j = 0, \dots, n$ (we write $t = x_0$ here), the infinitesimal generators of the Lorentz group

$$\begin{aligned} Z_{jk} &= x_k \partial/\partial x_j - x_j \partial/\partial x_k, \quad j,k = 1, \dots, n, \\ Z_{0k} &= x_0 \partial/\partial x_k + x_k \partial/\partial x_0 = -Z_{k0}, \quad k = 1, \dots, n, \end{aligned} \tag{1.10}$$

which commute with $\Box$, and the radial vector field

$$Z_0 = \sum_0^n x_j \partial/\partial x_j. \tag{1.11}$$

This allows one to justify the statements on the lifespan made above when G is *independent* of u; terms depending on u cause difficulties because energy estimates do not give direct control of them. These arguments are simpler or give better results than the earlier ones of Klainerman [12], Klainerman-Ponce [13], Shatah [15] and John-Klainerman [8]. When $n \ge 5$ one can safely allow G to depend on u also (cf. Christodoulou [1] and Li-Chen [14]), but for lower dimensions this is not always true. In particular, John [5] has proved that for the equation $\Box u + u^2 = 0$ there are positive upper and lower bounds for $\varepsilon^2 T_\varepsilon$ as $\varepsilon \to 0$ if T_ε is the lifespan of the solution with Cauchy data (1.9).

When $n = 3$ (and $G = G(u', u'')$) it is particularly important to know also the size of the constant c in an estimate of the form $T_\varepsilon \ge \exp(c/\varepsilon)$. This question was first discussed by John [7] who gave an upper bound for $\overline{\lim}\, \varepsilon \log T_\varepsilon$ in a rather special rotationally symmetric case which could be treated by arguments close to the vector valued version of (1.7). For equations of the form

$$\sum_{j,k=0}^{3} g_{jk}(u')\partial_j\partial_k u = 0, \quad \sum_{j,k=0}^{3} g_{jk}(0)\partial_j\partial_k = \Box,$$

John [4] and Hörmander [2] independently obtained a lower bound for $\underline{\lim}\, \varepsilon \log T_\varepsilon$ which coincides with the upper bound for $\overline{\lim}\, \varepsilon \log T_\varepsilon$ of John [7] when it is applicable. More precisely, it was proved in [2] that

$$\underline{\lim}_{\varepsilon\to 0} \varepsilon \log T_\varepsilon \ge A = \left(\max \frac{1}{2} \sum g_{jkl} \hat{\omega}_j \hat{\omega}_k \hat{\omega}_l \partial^2 F(\omega, q)/\partial q^2 \right)^{-1}, \tag{1.12}$$

where $\omega \in S^2$ and $\hat{\omega} = (-1, \omega) \in \mathbf{R}^{1+3}$,

$$g_{jk}(\xi) = g_{jk}(0) + \sum_{0}^{3} g_{jkl}\xi_l + O(|\xi|^2).$$

Moreover,

$$\varepsilon^{-1} e^{s/\varepsilon} u_\varepsilon(e^{s/\varepsilon}, (e^{s/\varepsilon} + q)\omega) \to U(\omega, s, q), \quad \varepsilon \to 0, \tag{1.13}$$

locally uniformly in $S^2 \times (0, A) \times \mathbf{R}$, where $\partial U(\omega, s, q)/\partial q$ satisfies Burgers' equation (with parameters)

$$\partial U_q'(\omega, s, q)/\partial s = \frac{1}{2} \sum g_{jkl} \hat{\omega}_j \hat{\omega}_k \hat{\omega}_l U_q'(\omega, s, q)\, \partial U_q'(\omega, s, q)/\partial q,$$
$$U(\omega, 0, q) = F(\omega, q).$$

The lifespan of the approximation U for the rescaled solution is exactly equal to A, so (1.12) is optimal at least in a weak sense.

In addition to the methods of Klainerman [9] for proving that

$$\underline{\lim}_{\varepsilon\to 0} \varepsilon \log T_\varepsilon > 0,$$

the proof in [2] relied on the construction of the approximate solution implicit in (1.13). The main part of this paper will be devoted to an extension of these

results to a fully non-linear equation in $\mathbf{R}^{1+3}$

$$\Box u + G(u', u'') = 0 \tag{1.14}$$

where $G \in C^\infty$ vanishes of second order at 0 and does *not* depend on u. This requires a solution of the Cauchy problem for a generalized version of Burgers' equation in $\mathbf{R}^2$,

$$\partial u/\partial t = a\left(\partial u/\partial x\right)^2 + 2bu\partial u/\partial x + cu^2, \tag{1.15}$$

$$u(0, x) = u_0(x), \tag{1.16}$$

where $u_0 \in C_0^\infty(\mathbf{R})$. Here a, b, c are real constants. When $a = c = 0$ this is Burgers' equation. Note that $\varepsilon u(\varepsilon t, x)$ satisfies (1.15) if u does, and the Cauchy data are εu_0. Thus the lifespan of the solution with Cauchy data εu_0 is proportional to $1/\varepsilon$.

We can use Hamilton-Jacobi theory to study (1.15), (1.16). Set $\tau = \partial u/\partial t$ and $\xi = \partial u/\partial x$. Differentiation of (1.15) with respect to x gives as leading term the derivative of ξ along the curve where

$$dx/dt = -2(a\xi + bu). \tag{1.17}$$

Thus we obtain along this curve

$$d\xi/dt = 2\xi(b\xi + cu), \tag{1.18}$$

$$du/dt = -a\xi^2 + cu^2; \tag{1.19}$$

the last equation follows since

$$\tau + \xi dx/dt = a\xi^2 + 2bu\xi + cu^2 - 2\xi(a\xi + bu) = -a\xi^2 + cu^2.$$

Another differentiation of (1.15) with respect to x gives an equation for $U = \partial_x^2 u$,

$$dU/dt = 2aU^2 + 6b\xi U + 2cuU + 2c\xi^2. \tag{1.20}$$

It is linear if $a = 0$. For arbitrary a the equations obtained for derivatives of order higher than two are all linear, so questions concerning the lifespan of the solution of (1.15) can be answered as soon as we have studied the integration of the equations (1.18)–(1.20).

Section 2 is devoted to the differential equations (1.18)–(1.20). Somewhat miraculously they can be integrated explicitly so we gain full control of the solution in terms of the initial data

$$(x(0), u(0), \xi(0), U(0)) = (u, u_0(y), u_0'(y), u_0''(y)) \tag{1.21}$$

which occur in the integration of (1.15), (1.16). In Section 3 we deduce an explicit formula for the lifespan of the solution of (1.15), (1.16). We also discuss which derivatives of u that remain bounded at the blowup. Finally we sketch in Sections 4 and 5 the application to the equation (1.14). This will be done rather briefly since the arguments of [2] can be used with minor modifications once we have the basic facts on the equation (1.15).

2. Integration of the Hamilton equations

In this section we shall study the Cauchy problem for the system (1.18)–(1.20). The equations (1.18) and (1.19) for (ξ, u) involve a quadratic form in the right-hand side. With constants λ, μ to be determined we can write them in the form

$$d(\lambda\xi + \mu u)/dt = (2b\lambda - a\mu)\xi^2 + 2c\lambda u\xi + \mu c u^2.$$

The right-hand side is proportional to

$$(\lambda\xi + \mu u)^2$$

if

$$2b\lambda - a\mu/\lambda^2 = c/\mu,$$

that is, if

$$\lambda^2 c - 2b\lambda\mu + a\mu^2 = 0. \tag{2.1}$$

When this equation is fulfilled we obtain

$$\lambda\xi(t) + \mu u(t) = \frac{\mu(\lambda\xi(0) + \mu u(0))}{\mu - ct(\lambda\xi(0) + \mu u(0))} \tag{2.2}$$

Assume for a moment that $0 \neq ac \neq b^2$ so that (2.1) with $\mu = 1$ has two different roots λ_1, λ_2 not equal to 0. By eliminating $u(t)$ or $\xi(t)$ from the equations (2.2) with $(\lambda, \mu) = (\lambda_j, 1)$, we obtain

$$\xi(t) = \xi(0)/N(t), \quad u(t) = (u(0) - ct\prod_{j=1}^{2}(\lambda_j\xi(0) + u(0)))/N(t) \tag{2.3}$$

$$N(t) = \prod_{j=1}^{2}(1 - ct(\lambda_j\xi(0) + u(0))). \tag{2.4}$$

An elementary calculation gives

$$\begin{aligned} \xi(t) &= \xi(0)/N(t), \\ u(t) &= (u(0) - t(a\xi(0)^2 + 2bu(0)\xi(0) + cu(0)^2))/N(t), \end{aligned} \tag{2.3'}$$

$$N(t) = 1 - 2t(cu(0) + b\xi(0)) + t^2c(a\xi(0)^2 + 2bu(0)\xi(0) + cu(0)^2), \tag{2.4'}$$

and for reasons of continuity these formulas remain valid for arbitrary a, b, c.

We shall first consider the quasilinear case where $a = 0$, which is quite close to the standard Burgers' equation. Then the denominator N takes the simple form

$$N(t) = (1 - tcu(0))(1 - t(cu(0) + 2b\xi(0))),$$

so we have

Proposition 2.1. *When $a = 0$ the lifespan T of the solution of the Cauchy problem for* (1.18)–(1.19) *is given by*

$$T^{-1} = \max(cu(0), cu(0) + 2b\xi(0), 0). \tag{2.5}$$

In particular this means that the solution exists for $t \geq 0$ if and only if $cu(0) \leq 0$ and $cu(0) + 2b\xi(0) \leq 0$. Note that the linear equation (1.20) also has a solution for $0 \leq t \leq T$.

From now on we assume that $a \neq 0$. It is no restriction to assume that $a > 0$, for otherwise we can just change the signs of u, ξ, U, a, b, c. The reciprocals of the zeros of $N(t)$ are

$$1/t = cu(0) + b\xi(0) \pm \sqrt{b^2 - ac}\ \xi(0).$$

If $b^2 - ac > 0$ the largest one is $cu(0) + b\xi(0) + \sqrt{b^2 - ac}\ |\xi(0)|$, so the lifespan T of the solution of (1.18)–(1.19) is given by

$$T^{-1} = \max(cu(0) + b\xi(0) + \sqrt{b^2 - ac}\ |\xi(0)|,\ 0),$$

with the convention which will be used throughout that *non-real quantities shall be dropped.* However, we must also examine the solution of the equation (1.20) carefully, for the non-linear term $2aU^2$ in the right-hand side may cause a blowup. Set

$$V = U + 3b\xi/2a + cu/2a. \tag{2.6}$$

Then we obtain after some simple calculations

$$dV/dt = 2aV^2 + 3(ac - b^2)\xi^2/2a. \tag{2.7}$$

Here ξ is given by (2.3)′. It may very well happen that V blows up long before ξ becomes singular. We shall determine explicitly when and where that happens, starting with the simplest case where $ac - b^2 = 0$. Then we have

$$N(t) = (1 - t(cu(0) + b\xi(0)))^2,$$

and the equation (2.7) simplifies to

$$dV/dt = 2aV^2.$$

The solution is

$$V(t) = V(0)/(1 - 2atV(0)).$$

Hence we obtain

Proposition 2.2. *If $a > 0$ and $ac - b^2 = 0$, then the lifespan T of the solution of the Cauchy problem for the system* (1.18)–(1.20) *is given by*

$$T^{-1} = \max((cu(0) + b\xi(0)),\ (2aU(0) + 3b\xi(0) + cu(0)),\ 0). \tag{2.8}$$

In the general case where $ac - b^2 \neq 0$ we first note that $\xi(t) = \xi(0)/N(t)$

where

$$N(t) = 1 - 2t\gamma + t^2(\gamma^2+\beta)$$

$$(2.9)\qquad = (\gamma^2+\beta)(t-\gamma/(\gamma^2+\beta))^2 + \beta/(\gamma^2+\beta),$$

$$\beta = \xi(0)^2(ac-b^2), \quad \gamma = cu(0) + b\xi(0).$$

Here we have assumed that $\gamma^2+\beta\neq 0$; the discussion of the limiting case where that is not true is postponed to the end. If $\beta = 0$ we have the case already discussed in Theorem 2.3, so we may also assume that $\beta\neq 0$. Now set

$$(2.10)\qquad s = (t(\gamma^2+\beta)-\gamma)/\sqrt{|\beta|}, \quad W(s) = 2aV(t)\sqrt{|\beta|}/(\gamma^2+\beta).$$

Then $t=0$ corresponds to $s_0 = -\gamma/\sqrt{|\beta|}$, and

$$(2.11)\qquad \begin{aligned} &dW/ds = W^2 \pm 3(s^2\pm 1)^{-2}, \quad \pm = \operatorname{sgn}\beta,\\ &W(s_0) = 2aV(0)\sqrt{|\beta|}/(\gamma^2+\beta).\end{aligned}$$

The Riccati equations (2.11) can be integrated explicitly. To do so we set

$$\Phi(s) = (W(s)(s^2\pm 1)+s)/2$$

which transforms (2.11) to

$$\Phi'(s) = 2(\Phi^2\pm 1)/(s^2\pm 1).$$

For the positive sign we obtain $\arctan\Phi = \text{const} + 2\arctan s$, that is,

$$(2.12_+)\qquad \Phi(s) = ((s^2-1)K+2s)/(2sK-(s^2-1)).$$

Here K is a constant which is allowed to take the value ∞ also. Note that $(2.12)_+$ is symmetric in Φ and s, so it can also be written

$$(2.13_+)\qquad K = ((s^2-1)\Phi(s)+2s)/(2s\Phi(s)-(s^2-1)).$$

Similarly, we obtain for the minus sign the solution

$$(2.12_-)\qquad \Phi(s) = (K(s^2+1)+2s)/(2Ks+(s^2+1))$$

which is symmetric in Φ and K apart from a sign change, so that it can also be written

$$(2.13_-)\qquad K = (\Phi(s)(s^2+1)-2s)/(s^2+1-2s\Phi(s)).$$

In principle this finishes the integration of (2.7), but we have to examine the solution carefully to determine the lifespan.

(i) We shall now assume that

$$ac-b^2>0$$

so that $\beta>0$ and (2.11) holds with the plus sign. If the solution $(2.12)_+$ blows up when $s=S$, it follows from $(2.13)_+$ that $K=(S^2-1)/2S$, that is,

$$(2.14_+)\qquad S^2-2SK-1=0.$$

Given K this equation has two solutions $S = K\pm\sqrt{1+K^2}$, one positive and one negative with product -1. An exception is the solution $\Phi(s) = (s^2-1)/2s$ corresponding to $K=\infty$ and $W(s) = -1/(s(s^2+1))$ displayed in Fig. 1; it blows up at 0 only. Note that (2.11) can be written

$$d(-1/W)/ds = 1\pm 3(s^2\pm 1)^{-2}(-1/W)^2.$$

If W blows up at S and $S^2\pm 1\neq 0$ it follows that $W(s)(s-S)\to -1$ as $s\to S$. In particular, it follows that $W\to+\infty$ when $s\uparrow S$. For the Cauchy problem (2.11) (with the plus sign) we conclude by comparing with the solution corresponding to $K=\infty$ that if $W(s_0) < -1/(s_0(s_0^2+1))$ then no blowup can occur to the right of s_0 if $s_0>0$; if $s_0<0$ then the only blowup to the right of s_0 occurs when $S = K+\sqrt{1+K^2}$. If $W(s_0) > -1/(s_0(s_0^2+1))$ and $s_0\ge 0$ the solution also blows up at $K+\sqrt{1+K^2}$, but if $s_0<0$ it blows up the first time at $K-\sqrt{1+K^2}$.

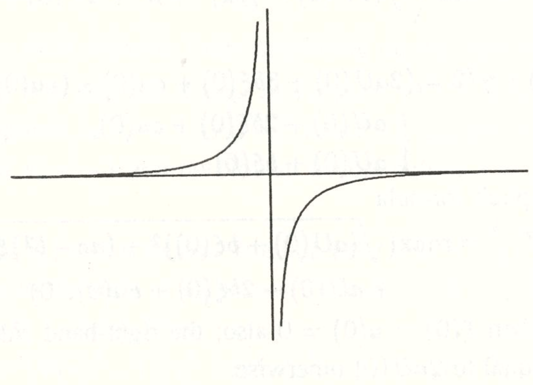

Fig. 1

Summing up,

$$S = \begin{cases} \infty, & \text{if } 2s_0\Phi(s_0)-s_0^2+1\le 0,\ s_0>0\\ K+\sqrt{1+K^2}, & \text{if } 2s_0\Phi(s_0)-s_0^2+1>0,\\ 0, & \text{if } 2s_0\Phi(s_0)-s_0^2+1=0,\ s_0<0\\ K-\sqrt{1+K^2}, & \text{if } 2s_0\Phi(s_0)-s_0^2+1\le 0, s_0<0.\end{cases}$$

Note that $W(s)s(s^2+1)+1 = s(2\Phi(s)-s)+1 = 2s\Phi(s)+1-s^2$ has the same sign as the denominator in $(2.13)_+$. Since

$$K^2+1 = (s^2+1)^2(\Phi(s)^2+1)/(2s\Phi(s)-s^2+1)^2,$$

$$(s_0^2-1)\Phi(s_0)+2s_0 - s_0(2s_0\Phi(s_0)-s_0^2+1) = (1+s_0^2)(s_0-\Phi(s_0)),$$

and

$$\frac{s_0-\Phi(s_0)+\sqrt{\Phi(s_0)^2+1}}{2s_0\Phi(s_0)-s_0^2+1} = (\sqrt{\Phi(s_0)^2+1}+\Phi(s_0)-s_0)^{-1}$$

we can sum up the list by

$$(S-s_0)^{-1} = \max\left(\sqrt{\Phi(s_0)^2+1}+\Phi(s_0)-s_0,\ 0\right)/(s_0^2+1).$$

For $s_0=-\gamma/\sqrt{\beta}$ and $W(s_0)$ as in (2.11) we have

$$\Phi(s_0)=\frac{1}{2}(2aV(0)\sqrt{\beta}(\gamma^2/\beta+1)/(\gamma^2+\beta)-\gamma/\sqrt{\beta})$$
$$=(aV(0)-\gamma/2)/\sqrt{\beta},$$

$$\Phi(s_0)-s_0=(aV(0)+\gamma/2)/\sqrt{\beta},\quad s_0^2+1=1+\gamma^2/\beta,$$

$$\sqrt{\Phi(s_0)^2+1}=\sqrt{(aV(0)-\gamma/2)^2+\beta}/\sqrt{\beta}.$$

Thus the lifespan $T=(S-s_0)\sqrt{\beta}/(\gamma^2+\beta)$ is given by

(2.15) $$T^{-1}=\max\left(\sqrt{(aV(0)-\gamma/2)^2+\beta}\ +aV(0)+\gamma/2,\ 0\right).$$

Here

(2.16)
$$aV(0)\pm\gamma/2=(2aU(0)+3b\xi(0)+cu(0)\pm(cu(0)+b\xi(0))/2$$
$$=\begin{cases}aU(0)+2b\xi(0)+cu(0),\\ aU(0)+b\xi(0).\end{cases}$$

This gives the explicit formula

(2.15′)
$$T^{-1}=\max(\sqrt{(aU(0)+b\xi(0))^2+(ac-b^2)\xi(0)^2}\ +$$
$$+aU(0)+2b\xi(0)+cu(0),\ 0).$$

It remains true when $\xi(0)=u(0)=0$ also; the right-hand side is equal to 0 if $aU(0)<0$ and equal to $2aU(0)$ otherwise.

(ii) Finally we must study the case where

$$ac-b^2<0.$$

At first we assume that $\xi(0)\neq 0$, hence $\beta<0$, and that $\gamma^2+\beta\neq 0$. We can then transform to the equation (2.11) with the minus sign. If a solution W given by $(2.12)_-$ blows up at $s=S\neq\pm 1$, then $K=-(S^2+1)/2S$ by $(2.13)_-$, that is,

(2.14_-) $$S^2+2SK+1=0.$$

This equation for S has no real root if $|K|<1$, the double root $S=-K$ if $K=\pm 1$, and two different roots $-K\pm\sqrt{K^2-1}$ with product 1 if $|K|>1$; neither is equal to ± 1 and both have the sign of $-K$. For $K=\pm 1$ we have $\Phi=\pm 1$ and $W(s)=(\pm 2-s)/(s^2-1)$. When $K=\infty$ we have the solution $\Phi(s)=(s^2+1)/2s$, that is, $W(s)=1/(s(s^2-1))$. The three curves with $K=\pm 1$ and $K=\infty$ are displayed in Fig. 2.

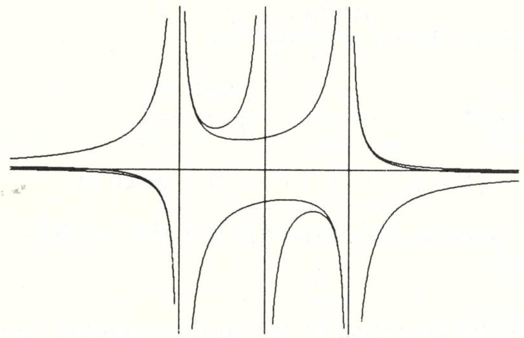

Fig. 2

To proceed we must separate cases depending on the sign of $\gamma^2+\beta$ and γ.

$(\text{ii})_+$ If $\gamma^2+\beta>0$ then the change of variables (2.10) preserves the orientation. $t=0$ corresponds to $s_0=-\gamma/\sqrt{-\beta}$, hence $s_0^2>1$ and s_0 has the sign of $-\gamma$. If $\gamma>0$ then $s_0<-1$ and no blowup occurs before $s=-1$ if $\Phi(s_0)\le 1$; set

$S=-1$ then. If $\Phi(s_0)>1$ it follows from $(2.13)_-$ that $K\in(1,-(s_0^2+1)/2s_0)$, so the first blowup to the right of s_0 occurs when $S=-K-\sqrt{K^2-1}$; the other root of $(2.14)_-$ is in $(-1,0)$. Now

$$K^2-1=(\Phi(s)^2-1)(s^2-1)^2/(s^2+1-2s\phi(s))^2,$$
$$-\Phi(s)(s^2+1)+2s-s(s^2+1-2s\Phi(s))=(\Phi(s)-s)(s^2-1),$$
$$(\Phi(s)-s)^2-(\Phi(s)^2-1)=s^2+1-2s\Phi(s).$$

Since $s_0^2+1-2s_0\Phi(s_0)>0$ we conclude that

$$(S-s_0)^{-1}=(\sqrt{\Phi(s_0)^2-1}+\Phi(s_0)-s_0)/(s_0^2-1).$$

This expression is $>(-1-s_0)^{-1}$ if and only if

$$\sqrt{\Phi(s_0)^2-1}\ +\Phi(s_0)-s_0>-(s_0^2-1)/(s_0+1)=1-s_0,$$

that is, $\Phi(s_0)>1$, so we see that in both cases

$$(S-s_0)^{-1}=\max\left(\frac{\sqrt{\Phi(s_0)^2-1}+\Phi(s_0)-s_0}{s_0^2-1},\frac{1}{-1-s_0}\right),$$

with the interpretation that a non-real quantity should be dropped. For $s_0=-\gamma/\sqrt{-\beta}$ and $W(s_0)=2aV(0)\sqrt{-\beta}/(\gamma^2+\beta)$ as in (2.11) we have $s_0^2-1=-(\gamma^2+\beta)/\beta$ and

$$\Phi(s_0) = \frac{1}{2}\left(\frac{2aV(0)}{\sqrt{-\beta}} - \frac{\gamma}{\sqrt{-\beta}}\right) = \frac{aV(0)-\gamma/2}{\sqrt{-\beta}},$$

$$\Phi(s_0) - s_0 = \frac{aV(0)+\gamma/2}{\sqrt{-\beta}},$$

$$\Phi(s_0)^2 - 1 = \frac{(aV(0)-\gamma/2)^2+\beta}{-\beta},$$

so the lifespan $T = (S-s_0)\sqrt{-\beta}/(\gamma^2+\beta)$ is given by

$$T^{-1} = \max(\sqrt{(aV(0)-\gamma/2)^2+\beta} + aV(0)+\gamma/2, \quad \gamma+\sqrt{-\beta}). \tag{2.17}$$

Here $\gamma+\sqrt{-\beta}$ is the reciprocal of the smallest positive root of $N(t)$. Using (2.16) we can easily express (2.17) in terms of the original quantities $u(0)$, $\xi(0)$, $U(0)$.

We shall now consider the case where $\gamma < 0$, hence $s_0 > 1$. Comparison with the solution corresponding to $K = \infty$ shows that there is no blowup if $\Phi(s_0) \le (s_0^2+1)/2s_0$. When $\Phi(s_0) > (s_0^2+1)/2s_0 > 1$ then $K \in (-\infty, -(s_0^2+1)/2s_0)$ by (2.13)$_-$, so (2.14)$_-$ has a unique solution $S = -K+\sqrt{K^2-1} > 1$; it is necessarily $> s_0$ since no blowup can occur between 1 and s_0. The lifespan $T = (S-s_0)\sqrt{-\beta}/(\gamma^2+\beta)$ can now be computed as before. We have a plus sign in front of $\sqrt{K^2-1}$ now, but since $s_0^2+1-2s_0\Phi(s_0) < 0$ now this difference disappears and we obtain as before

$$T^{-1} = \max\left(\sqrt{(aV(0)-\gamma/2)^2+\beta} + aV(0)+\gamma/2,\ 0\right), \tag{2.18}$$

provided that $\Phi(s_0) > (s_0^2+1)/2s_0$. To verify (2.18) otherwise we must show that

$$\sqrt{\Phi(s_0)^2-1} + \Phi(s_0) - s_0 < 0 \text{ if } \begin{cases} 1 < \Phi(s_0) \le (s_0^2+1)/2s_0 \\ \text{or} \\ \Phi(s_0) < -1. \end{cases}$$

This is clear since

$$s_0 - \Phi(s_0) \ge s_0 - (s_0^2+1)/2s_0 = (s_0^2-1)/2s_0 \ge 0,$$
$$\Phi(s_0)^2 - 1 - (\Phi(s_0)-s_0)^2 = 2s_0\Phi(s_0) - 1 - s_0^2 \le 0.$$

(ii)$_-$ If $\gamma^2+\beta < 0$ then the change of variables (2.10) reverses the orientation so we have to look for a blowup to the *left* of $s_0 = -\gamma/\sqrt{-\beta}$. Note that $s_0^2 < 1$. If $\Phi(s) \to \infty$ as $s \downarrow S \in (-1, s_0)$ then $\Phi(s) \to +\infty$, so there is no such blowup if $\Phi(s_0) \le 1$; define $S = -1$ then. If $\Phi(s_0) > 1$ then the root S of (2.14)$_-$ in $(-1, s_0)$ must be $-K+\sqrt{K^2-1}$ if $K > 0$, for the other one is < -1. If $K < 0$ then $s_0 > 0$ and $1 < \Phi(s_0) < (s_0^2+1)/2s_0$; we must then take the smallest root

$-K-\sqrt{K^2-1}$ for the other one is > 1. We can sum up by

$$S = -K + K\sqrt{1-1/K^2}.$$

Since

$$1 - 1/K^2 = (\Phi(s_0)^2-1)(s_0^2-1)^2/(\Phi(s_0)(s_0^2+1)-2s_0)^2$$

and $\Phi(s_0)(s_0^2+1) - 2s_0 > 0$, we have

$$K\sqrt{1-1/K^2} = \sqrt{\Phi(s_0)^2-1}(1-s_0^2)/(s_0^2+1-2s_0\Phi(s_0)).$$

Hence

$$S - s_0 = (1-s_0^2)(s_0-\Phi(s_0)+\sqrt{\Phi(s_0)^2-1})/(s_0^2+1-2s_0\Phi(s_0)),$$

$$(S-s_0)^{-1} = (s_0-\Phi(s_0)-\sqrt{\Phi(s_0)^2-1})/(1-s_0^2).$$

This expression is $> (-1-s_0)^{-1}$ if and only if

$$s_0 - \Phi(s_0) - \sqrt{\Phi(s_0)^2-1} > s_0 - 1,$$

that is,

$$\sqrt{\Phi(s_0)^2-1} < 1 - \Phi(s_0)$$

which is true if $\Phi(s_0) < -1$ and false if $\Phi(s_0) > 1$. Thus we have in all cases

$$(S-s_0)^{-1} = \min\left(\frac{s_0-\Phi(s_0)-\sqrt{\Phi(s_0)^2-1}}{1-s_0^2}, \frac{1}{-1-s_0}\right).$$

For the lifespan $T = (S-s_0)\sqrt{-\beta}/(\gamma^2+\beta)$ we obtain since $\gamma^2+\beta < 0$

$$T^{-1} = \max(\sqrt{(aV(0)-\gamma/2)^2+\beta} + aV(0)+\gamma/2, \gamma+\sqrt{-\beta}), \tag{2.19}$$

where $\gamma+\sqrt{-\beta}$ is the reciprocal of the positive zero of $N(t)$ and we have the usual convention of dropping non-real quantities.

We can now sum up (2.15), (2.17), (2.18), and (2.19) in one expression

$$T^{-1} = \max(\sqrt{(aV(0)-\gamma/2)^2+\beta} + aV(0)+\gamma/2, \quad \gamma+\sqrt{-\beta}, 0). \tag{2.20}$$

In fact, when $\beta > 0$ the expression $\gamma+\sqrt{-\beta}$ is non-real and drops out, which gives (2.15), and similarly we obtain the other cases. We can make (2.20) explicit by using (2.16) and

$$\gamma+\sqrt{-\beta} = cu(0) + b\xi(0) + |\xi(0)|\sqrt{b^2-ac}.$$

Using continuity arguments to cover the excluded cases where $\beta = 0$ or $\gamma^2+\beta = 0$

Proposition 2.3. *The lifespan T of the solution of the Cauchy problem for the system* (1.18)–(1.20) *is given by*

$$T^{-1} = \max(\sqrt{a(aU(0)^2 + 2bU(0)\xi(0) + c\xi(0)^2)} + aU(0) + 2b\xi(0) + cu(0),\ cu(0) + b\xi(0) + |\xi(0)|\sqrt{b^2 - ac},\ 0). \tag{2.21}$$

Here it is understood that non-real quantities should be dropped.

Note that Proposition 2.3 contains Propositions 2.1 and 2.2 as special cases; they could have been obtained as limiting cases of the generic case of Proposition 2.3.

3. The generalized Burgers' equation.

Since the solution of (1.18), (1.19) vanishes if the initial data do, we see from (1.17) that $x(t)$ is also constant then. Hence the support of $u(t,\cdot)$ is always equal to the support of u_0, as long as a classical solution exists. In what follows we shall always assume that $u_0 \not\equiv 0$.

If u is a C^∞ solution of (1.15)–(1.16) for $0 \le t < T$ then $u, \xi = \partial u/\partial x, U = \partial^2 u/\partial x^2$ satisfy (1.18)–(1.20) for $0 \le t < T$ along the integral curves of (1.17), with initial conditions (1.21). Denote these solutions by $x^y(t)$, $u^y(t)$, $\xi^y(t)$, $U^y(t)$. If they are uniformly bounded for $0 \le t < T$ it follows that the higher order derivatives with respect to x are bounded too, for they satisfy linear equations. Hence u can then be extended to a C^∞ function for $0 \le t \le T$. By the local existence theorems the solution of (1.15) can therefore be extended to a larger t interval. Thus T is the lifespan of u if and only if there is no uniform bound for $u^y(t), \xi^y(t), U^y(t)$ when $0 \le t < T$, that is, the lifespan T is the infimum with respect to y of the lifespans of these solutions of (1.18)–(1.20). It can therefore be obtained from Propositions 2.1-2.3. It will turn out that the expression involving the highest derivative always dominates. For the proof we need some elementary lemmas.

Lemma 3.1. *Let $0 \not\equiv v \in C_0^1(\mathbf{R})$. If $x_0 \in \mathbf{R}, \varepsilon > 0$ and $v(x) < v(x_0)$ when $x_0 < x < x_0 + \varepsilon$, it follows that*

$$\varliminf_{x \downarrow x_0} v'(x)/(v(x_0) - v(x)) = -\infty. \tag{3.1}$$

If instead $v(x) < v(x_0)$ when $x_0 - \varepsilon < x < x_0$, then

$$\varlimsup_{x \uparrow x_0} v'(x)/(v(x_0) - v(x)) = +\infty. \tag{3.2}$$

Proof. If (3.1) is not valid we obtain for some δ with $0 < \delta < \varepsilon$ and some constant K that

$$v'(x) > -K(v(x_0) - v(x)) \quad \text{when } x_0 < x < x_0 + \delta.$$

Hence $(v(x) - v(x_0))e^{-Kx}$ is negative and increasing in $(x_0, x_0 + \delta)$ and 0 at the left endpoint which is impossible. This proves (3.1), and (3.2) is (3.1) applied to $v(-x)$.

Lemma 3.2. *Let $0 \not\equiv v \in C_0^1(\mathbf{R})$. Then the convex hull K of*

$$J(v) = \{(v(x), v'(x)),\ x \in \mathbf{R}\} \subset \mathbf{R}^2$$

is a convex set with the interval $(\min v, \max v) \times \{0\}$ in its interior and the end points on the boundary. K is differentiable at these points. The whole boundary of K is in C^1 if $v \in C_0^2$.

Proof. Let $M = \max v$ and choose x_0 with $v(x_0) = M$; then $v'(x_0) = 0$, which proves that $(M,0) \in J(v)$. Since $v \le M$ we conclude that $(M,0) \in \partial K$. If $M > 0$ we can choose x_0 maximal or minimal with $v(x_0) = M$; if $M = 0$ we can choose x_0 locally maximal or minimal. In both cases it follows from Lemma 3.1 that K has a vertical tangent at $(M,0)$. Replacing v by $-v$ we make the same conclusion at $(\min v, 0)$.

Assume now that $v \in C_0^2$ and that K has a corner at a point (v_0, v_1) with $v_1 \ne 0$. Then $(v_0, v_1) = (v(x_1), v'(x_1))$ for some x_1. We can choose c_0, c_1 different from 0 and $\varepsilon > 0$ such that

$$c_0 v(x) + c_1 v'(x) \le c_0 v(x_1) + c_1 v'(x_1) - \varepsilon(|v(x) - v(x_1)| + |v'(x) - v'(x_1)|).$$

Leting $x \to x_1$ noting that

$$|v(x) - v(x_1)|/|x - x_1| \to |v'(x_1)| \ne 0$$

we obtain a contradiction since $c_0 v(x) + c_1 v'(x)$ is differentiable at x_1. The proof is complete.

Theorem 3.3. *If $a = 0$, then the lifespan T of the solution of the Cauchy problem* (1.15), (1.16) *is given by*

$$T^{-1} = \max(cu_0(y) + 2bu_0'(y)). \tag{3.3}$$

If $b \ne 0$ then $T < \infty$ and u is bounded for $0 \le t < T$. If $b = 0$ then $T < \infty$ and u is unbounded for $0 \le t < T$ unless $cu_0 \le 0$.

Proof. If $b \ne 0$ it follows from Lemma 3.2 that

$$\max cu_0(y) < \max(cu_0(y) + 2bu_0'(y)),$$

so this is an immediate consequence of Proposition 2.1.

Theorem 3.4 *If $a > 0$ and $ac - b^2 = 0$ then the lifespan T of the solution of* *the Cauchy problem* (1.15), (1.16) *is finite and is given by*

$$T^{-1} = \max(2au_0''(y) + 3bu_0'(y) + cu_0(y)). \tag{3.4}$$

Proof. By Proposition 2.2 we only have to prove that

$$\max(cu_0(y) + bu_0'(y)) < \max(2au_0''(y) + 3bu_0'(y) + cu_0(y)). \tag{3.5}$$

This is clear if $b = 0$, hence $c = 0$, so we may assume that $b \neq 0$. Write $v = cu_0 + bu_0'$, thus $av' = b(bu_0' + au_0'')$ and

$$2au_0'' + 3bu_0' + cu_0 = 2ab^{-1}v' + v.$$

This reduces the inequality (3.5) to

$$\max v < \max(2ab^{-1}v' + v),$$

which is a consequence of Lemma 3.2.

We shall now examine the case where $ac - b^2 > 0$. Then we have

$$\begin{aligned} T^{-1} = \max(&\sqrt{a(au_0''(y)^2 + 2bu_0''(y)u_0'(y) + cu_0'(y)^2)} + \\ &+ au_0''(y) + 2bu_0'(y) + cu_0(y)) \\ \geq \max&(bu_0'(y) + cu_0(y)). \end{aligned}$$

This is positive if $b \neq 0$. If $b = 0$ then $c > 0$, and $T = \infty$ if and only if

$$\sqrt{(au_0''(y))^2 + acu_0'(y)^2} + au_0''(y) + cu_0(y) \leq 0. \tag{3.6}$$

This implies that $cu_0 \leq 0$, hence $u_0 \leq 0$. (3.6) is then automatically true where $u_0 = 0$, for $u_0' = 0$ and $u_0'' \leq 0$ there, so we only have to examine the condition where $u_0 < 0$. Differentiation shows that

$$\sqrt{U^2 + acu_0'(y)^2} + U + cu_0(y)$$

is an increasing function of U, and it is equal to 0 precisely when

$$acu_0'(y)^2 = c^2u_0(y)^2 + 2cUu_0(y),$$

that is

$$U = (au_0'(y)^2 - cu_0(y)^2)/2u_0(y).$$

Thus (3.6) is equivalent to

$$au_0''(y) \leq (au_0'(y)^2 - cu_0(y)^2)/2u_0(y)$$

when $u_0(y) < 0$, which implies that everywhere

$$2au_0''(y)u_0(y) \geq au_0'(y)^2 - cu_0(y)^2.$$

Set $u_0(y) = -f(y)^2$ where $f \geq 0$. If $u_0 \in C_0^\infty$ then f is piecewise in C^∞ in the complement of the set E of points where u_0 vanishes of infinite order without vanishing in a neighborhood. In this set $f \in C^1$ except at double zeros y_0 of u_0, where $f'(y_0+0) - f'(y_0-0) = \sqrt{-2u''(y_0)}$. Moreover, f is Lipschitz continuous on $\mathbb{R}$ by the well known inequality $|u_0'| \leq C\sqrt{|u_0(y)|}$ which follows from the

positivity of $-u_0$. The condition (3.6) can then be written

$$4af'' + cf \geq 0, \quad f \geq 0. \tag{3.7}$$

The first inequality is clear in the complement of E. To prove it in general we choose $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ equal to 1 at distance $< \varepsilon$ from E, equal to 0 at distance $> 2\varepsilon$ from E, so that $0 \leq \chi_\varepsilon \leq 1$ and $\chi_\varepsilon^{(j)} = O(\varepsilon^{-j})$. Since

$$\chi_\varepsilon f'' = (\chi_\varepsilon f)'' - 2(\chi_\varepsilon' f)' + \chi_\varepsilon'' f$$

converges to 0 as $\varepsilon \to 0$, it follows that $4af'' + cf$ is a positive measure on $\mathbb{R}$, and also that the mass at distance $< \varepsilon$ from E is $O(\varepsilon^N)$ for every N. This implies that $f'(y) \to 0$ as the distance from y to E converges to 0, so $f \in C^1$ except at the double zeros of u_0.

Proposition 3.5. *Let I be a compact interval on $\mathbb{R}$ with length $|I|$. If $|I| < 2\pi\sqrt{a/c}$ then there is no $f \in C(\mathbb{R})$ with $\operatorname{supp} f \subset I$ satisfying* (3.7) *in the sense of distribution theory. If $|I| \geq 2\pi\sqrt{a/c}$ then the set K of all such f with $\int f\,dx = 1$ is compact and convex, and it is the closed convex hull of all functions in K of the form $kf_0(y - y_0)$, where k is a positive constant and*

$$f_0(y) = \begin{cases} \cos(\sqrt{c/a}\,y/2), & |y| < \pi\sqrt{a/c} \\ 0, & |y| \geq \pi\sqrt{a/c} \end{cases}$$

or the maximum of two such functions with supports overlapping in an interval of positive length. Hence every $f \in K$ is monotonic in any interval of length $\pi\sqrt{a/c}$ such that $f = 0$ at one end point. Every component of the set where $f > 0$ is an interval of length $\geq 2\pi\sqrt{a/c}$.

Proof. Set $g = 4af'' + cf$, which by hypothesis is a positive measure with support in I. Since f has compact support, g is orthogonal to the solutions $y \to \cos(\sqrt{c/a}(y - y_0)/2)$ of the homogeneous equation. This proves that $g = 0$, hence $f = 0$, if $\sqrt{c/a}|y - y_0| < \pi$ in I for some y_0, which gives the first statement.

Now assume that $|I| \geq 2\pi\sqrt{c/a}$, and let K be the convex set of all $f \in C(\mathbb{R})$ with $\operatorname{supp} f \subset I$ satisfying (3.7). We have

$$\langle g, 1\rangle = c\langle f, 1\rangle = c,$$

so the total mass of the positive measure g is equal to c. Since f is the convolution of g and the fundamental solution, we obtain a uniform bound for f and for f', which proves that K is compact. By the Krein-Milman theorem we just have to determine the extremal elements in K.

If f has a zero at an interior point x_0 of I, we define $f_-(y) = f(y)$ when $y < x_0$, $f_-(y) = 0$ when $y \geq x_0$ and claim that f_- and $f_+ = f - f_-$ are non-negative and satisfy (3.7). The only point which is not obvious is whether a Dirac measure or its derivative might appear at x_0. To examine that possibility we note that for $\varphi \in C_0^\infty(\mathbf{R})$

$$\langle f''_-, \varphi\rangle = \lim_{\varepsilon\to+0}\int_{-\infty}^{x_0-\varepsilon} f(y)\varphi''(y)dy$$
$$= \lim_{\varepsilon\to+0} -f'(x_0-\varepsilon)\varphi(x_0-\varepsilon) + \int_{-\infty}^{x_0-\varepsilon}\varphi(y)f''(y)dy$$

where f'' should be interpreted as a measure. Since we can let $\varepsilon \to 0$ through a sequence such that $f'(x_0-\varepsilon) \le 0$ we conclude that the first term is a non-negative multiple of $\varphi(x_0)$. Hence f_- satisfies (3.7), and the proof is the same for f_+. If f is extremal it follows that f_- or f_+ must be equal to 0, for if $c_\pm = \int f_\pm(y)dy$ we would otherwise obtain

$$f = c_+(f_+/c_+) + c_-(f_-/c_-),$$

contradicting the extremality.

For an extremal f the support is thus an interval $J \subset I$ such that $f > 0$ in the interior. We claim that $\operatorname{supp} g$ can at most contain three points. If not, then we can write $g = \sum_1^5 g_j$ where $g_j \ge 0$ and g_1, g_2, g_3, g_4 are different from 0 with disjoint supports. The set of $(\lambda_1,\dots,\lambda_4)\in\mathbf{R}^4$ with $\sum_1^4\lambda_j g_j$ orthogonal to $\cos(\sqrt{c/a}y/2)$ and $\sin(\sqrt{c/a}y/2)$ has dimension ≥ 2, so we can find some such $(\lambda_1,\dots,\lambda_4)$ not proportional to $(1,\dots,1)$. For sufficiently small $|\varepsilon|$ it follows that

$$G_\varepsilon = \sum_1^4(1+\varepsilon\lambda_j)g_j + g_5 \ge 0,$$

and that $G_\varepsilon = 4aF''_\varepsilon + cF_\varepsilon$ for some F_ε with support in J. It is obtained by convolution of G_ε with the fundamental solution with support on the positive (or negative) axis, which is positive near the origin. Hence it is clear that F_ε is positive near the end points of J; in the rest of J this is true for small ε since $F_0 = f > 0$ there. It follows that f is not extremal.

Suppose now that $\operatorname{supp} g$ consists of two or three points. Two of them must be end points of $J = [y_1, y_2]$. It these are the only points in $\operatorname{supp} g$ then f must be of the form $kf_0(y-y_0)$. If there is a third point y_3 between y_1 and y_2 in the support, then

$$f(y) = \begin{cases} k_1\sin(\sqrt{c/a}(y-y_1)/2), & \text{if } y_1\le y\le y_3,\\ k_2\sin(\sqrt{c/a}(y_2-y)/2), & \text{if } y_3\le y\le y_2.\end{cases}$$

Since $f \ge 0$ we must have

$$\sqrt{c/a}(y_3-y_1) < 2\pi, \quad \sqrt{c/a}(y_2-y_3) < 2\pi;$$

the constants k_1, k_2 are then determined so that f is continuous and has integral 1. The mass of g at y_3 is positive since a solution of the homogeneous equation

$4af'' + cf = 0$ cannot have two zeros in an interval of length $< 2\pi\sqrt{a/c}$. This completes the determination of the extremal elements in K. They are all increasing (decreasing) at distance $\le \pi\sqrt{a/c}$ from the left (right) end point of I. This proves the last two assertions, for we may replace I by the closure of any component of the set where $f > 0$. The proof is complete.

If $|I| = 2\pi\sqrt{a/c}$ there is just one element in K. It is not even in C^1, but by regularization of this element we can find solutions of (3.7) in $C_0^\infty(I)$ for any I of larger length. Returning to $u_0 = -f^2$ we then obtain functions $u_0 \in C_0^\infty(\mathbf{R})$ satisfying (3.6). We have proved

Theorem 3.6. *If $a > 0$ and $ac - b^2 > 0$, $u_0 \in C_0^\infty(\mathbf{R})$, $u_0 \not\equiv 0$, then the lifespan T of the solution u of the Cauchy problem (2.1), (2.2) is finite,*

$$T^{-1} = \max(\sqrt{a(au_0''(y)^2 + 2bu_0''(y)u_0'(y) + cu_0'(y)^2)} + au_0''(y) + 2bu_0'(y) + cu_0(y)) \tag{3.8}$$

unless $b = 0$ and u_0 satisfies (3.6). The length of $\operatorname{supp} u_0$ must then exceed $2\pi\sqrt{a/c}$, and (3.6) means that $u_0 = -f^2$ with f as in Proposition 3.5. When $T < \infty$ then u and $\partial u/\partial x$ remain bounded for $0 \le t < T$.

Remark. In general we can only assert that f is Lipschitz continuous even if $u_0 \in C_0^\infty$. To construct an example we let

$$f_t(y) = \max(\sin(\sqrt{c/a}\,y/2),\ \sin(\sqrt{c/a}(y-t)/2))$$

for $0 \le y \le 2\pi\sqrt{a/c} + t$ and $f_t(y) = 0$ elsewhere. Here $0 < t < 2\pi\sqrt{a/c}$ so that we have one of the extremal functions in Proposition 3.5. Then

$$4af_t'' + cf_t = 2\sqrt{ac}(\delta_0 + \delta_{2\pi\sqrt{a/c}+t} + 2\sin(\sqrt{c/a}\,t/4)\delta_{\pi\sqrt{a/c}+t/2}).$$

Choose $\chi \in C_0^\infty(0, 2\pi\sqrt{a/c})$ with $0 \le \chi \le 1$, $\int \chi dt = 1$, and set

$$f(y) = \int f_t(y)\chi(t)dt.$$

Then

$$4af'' + cf = 2\sqrt{ac}(\delta_0 + \chi(.-2\pi\sqrt{a/c}) + 4\sin(\sqrt{c/a}(.-\pi\sqrt{a/c})/2)\chi(2(.-\pi\sqrt{a/c})),$$

so $f \in C^\infty$ except at 0 where $f(y) = \sin(\sqrt{c/a}y/2)$ in a right neighborhood. Hence

$$u_0(y) = -f(y)^2 - f(-y)^2$$

is a C^∞ function satisfying (3.6) although $\sqrt{-u_0(y)} = f(y) + f(-y)$ is not differentiable at 0. However, it is always true that f is differentiable except at isolated points separated by at least a distance $2\pi\sqrt{a/c}$ where the derivative has a simple jump.

Finally assume that $b^2 - ac > 0$. By Lemma 3.2 we have

$$M = \max(cu_0(y) + bu_0'(y) + \sqrt{b^2 - ac}|u_0'(y)|) > 0,$$

so T is finite. We shall prove that $1/T > M$, which implies that u and $\partial u/\partial x$ remain bounded for $0 \le t < T$ and that T is given by (3.8) with the maximum taken over all y such that the square root is real. What we must show is just that this maximum is larger than M. Choose β with $\beta^2 = b^2 - ac$ so that

$$M = \max v(y), \quad v(y) = cu_0(y) + (b+\beta)u_0'(y).$$

Assume for the time being that $c \neq 0$. Thus

$$(b-\beta)v = c((b-\beta)u_0 + au_0'),$$
$$au_0'' + bu_0' = (b-\beta)v'/c + \beta u_0',$$

which implies that

$$\begin{aligned} a(au_0''(y)^2 + 2bu_0''(y)u_0'(y) + cu_0'(y)^2) &= \\ = (au_0''(y) + bu_0'(y))^2 - \beta^2 u_0'(y)^2 \\ = ((b-\beta)v'(y)/c)((b-\beta)v'(y)/c + 2\beta u_0'(y)). \end{aligned} \tag{3.9}$$

Let y_0 be a point where $v(y_0) = M$, hence $v'(y_0) = 0$ and $\beta u_0'(y_0) > 0$. Then we have $(b-\beta)v'(y)/c + 2\beta u_0'(y) > 0$ in a neighborhood ω of y_0, so the square root in (3.8) is real in ω when $(b-\beta)v'(y)/c \ge 0$. If the maximum in (3.8) is $\le M$ it follows that

$$M \ge au_0''(y) + 2bu_0'(y) + cu_0(y) = (b-\beta)v'(y)/c + v(y)$$

for all $y \in \omega$ with $(b-\beta)v'(y)/c \ge 0$. Since $v(y) \le M$ this inequality is obvious when $(b-\beta)v'(y)/c < 0$. Thus we have

$$(b-\beta)v'(y)/c + v(y) \le M, \quad v(y) \le M, \quad \text{if } y \in \omega.$$

Choosing y_0 (locally) minimal (resp. maximal) when $(b-\beta)/c > 0$ (resp. $(b-\beta)/c < 0$), we obtain a contradiction with Lemma 3.1 which proves that the supremum in (3.8) exceeds M.

The case where $c = 0$ is easier. Then we have

$$M = \max v + |v| > 0, \quad v = bu_0'.$$

If the maximum in (3.8) is not larger than M, then

$$\sqrt{a^2v'^2/b^2 + 2avv'/b} + av'/b + 2v \le M$$

when the square root is real, hence when

$$a^2v'/b + 2av \ge 0, \text{ and } bv' \ge 0.$$

Thus $av'/b + 2v \le M$ if $av'/b + 2v \ge 0$ and $bv' \ge 0$; the same inequality is obvious when $bv' \le 0$ since $2v \le M$. The inequalities $2v \le M$, $v' + 2v \le M$ contradict Lemma 3.2 since $(v(y), v'(y)) = (M/2, 0)$ for some y. Hence we have proved:

Theorem 3.7. *If $a > 0$ and $ac - b^2 < 0$, $u_0 \in C_0^\infty(\mathbf{R})$, $u_0 \not\equiv 0$, then the lifespan T of the solution of the Cauchy problem (1.15), (1.16) is finite and given by (3.8), with the maximum taken over the set where the square root is real. Both u and $\partial u/\partial x$ remain bounded for $0 \le t < T$.*

Note that (3.3) and (3.4) are special cases of (3.8); in the latter case one should also recall that $cu_0 + bu_0 < 1/T$.

4. An approximate solution of the Cauchy problem

We shall look for an approximate solution of the Cauchy problem (1.14), (1.9) in $\mathbf{R}^{1+3}$ of the form

$$u(t, r\omega) = \varepsilon r^{-1} U(\omega, \varepsilon \log t, r - t), \quad |\omega| = 1, \quad r > 0. \tag{4.1}$$

This is motivated by the asymptotic formula $\varepsilon r^{-1} F(\omega, r-t)$ for the solution of the unperturbed wave equation mentioned in the introduction, and the expectation that $\varepsilon \log T_\varepsilon$ should have a fixed lower bound if T_ε is the lifespan of the true solution. Since

$$\Box u = r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega) ru,$$

with Δ_ω denoting the Laplacian in S^2, the main term in $\Box u$ is obtained when $\partial_t + \partial_r$ acts on the argument $s = \varepsilon \log t$ and $\partial_t - \partial_r$ acts on $q = r - t$, which gives

$$-2\varepsilon^2 (tr)^{-1} U_{sq}''(\omega, s, q). \tag{4.2}$$

When derivatives act on ω or s the inner derivative gives a factor $O(1/t)$ so the main contributions from the non-linear terms are expected when all derivatives act on q. We can write the non-linear part of (1.14) in the form

$$G(u', u'') = \sum_{1 \le |\alpha|, |\beta| \le 2} g_{\alpha\beta} \partial^\alpha u \partial^\beta u + O(|u'|^3 + |u''|^3). \tag{4.3}$$

Since $q'^\alpha = \hat\omega^\alpha$, $\hat\omega = (-1, \omega)$, the main non-linear terms in (1.14) are

$$\varepsilon^2 r^{-2} \sum_{1 \le |\alpha|, |\beta| \le 2} g_{\alpha\beta} \hat\omega^{\alpha+\beta} \partial_q^{|\alpha|} U \partial_q^{|\beta|} U \tag{4.4}$$

when u is of the form (4.1). With the notation

$$\begin{aligned} a(\omega) &= \frac{1}{2} \sum_{|\alpha| = |\beta| = 2} g_{\alpha\beta} \hat\omega^{\alpha+\beta}, \\ 2b(\omega) &= \frac{1}{2} \sum_{|\alpha| + |\beta| = 3} g_{\alpha\beta} \hat\omega^{\alpha+\beta}, \\ c(\omega) &= \frac{1}{2} \sum_{|\alpha| = |\beta| = 1} g_{\alpha\beta} \hat\omega^{\alpha+\beta} \end{aligned} \tag{4.5}$$

we now obtain from (4.2), (4.4) and (4.5) the following approximation to the equation (1.14)

$$\partial U_q'/\partial s = a(\omega)(\partial U_q'/\partial q)^2 + 2b(\omega)\partial U_q'/\partial q U_q' + c(\omega)(U_q')^2. \tag{4.6}$$

When t is large but $\varepsilon \log t$ is still small we see that it is natural to require the initial condition

$$U(\omega, 0, q) = F(\omega, q), \tag{4.7}$$

where F is the Friedlander radiation field for the solution of the unperturbed equation, given by (see e.g. [2])

$$F(\omega, .) = \frac{1}{4\pi}(R(\omega, .; u_1) - R(\omega, .; u_0)'), \tag{4.8}$$

with the differentiation taken in the variable q indicated by a dot. Here R is the Radon transform,

$$R(\omega, q; g) = \int \delta(q - \langle \omega, y\rangle) g(y) dy = \int_{\langle \omega, y\rangle = q} g(y) dS(y). \tag{4.9}$$

Theorem 4.1. *Let $|x| \le M$ when $x \in \operatorname{supp} u_0 \cup \operatorname{supp} u_1$. Then the Cauchy problem (4.6), (4.7) has a unique solution $U \in C^\infty(S^2 \times [0, A) \times \mathbb{R})$ vanishing in $S^2 \times [0, A) \times (M, \infty)$, if*

$$1/A = \max_{\omega, q}\left(\sqrt{I_1(\omega, q)} + I_2(\omega, q)\right), \tag{4.10}$$

where

$I_1(\omega,q) = a(\omega)(a(\omega)F_{qqq}'''(\omega,q)^2 + 2b(\omega)F_{qqq}'''(\omega,q)F_{qq}''(\omega,q) + c(\omega)F_{qq}''(\omega,q)^2),$

$I_2(\omega,q) = a(\omega)F_{qqq}'''(\omega,q) + 2b(\omega)F_{qq}''(\omega,q) + c(\omega)F_q'(\omega,q).$

Here the maximum is taken over all (ω, q) such that the square root is real. All derivatives of order ≤ 3 are not bounded as $s \nearrow A$. For $q < -M$ we have $U(\omega, s, q) = U_-(\omega, s)$.

Proof. For fixed ω the equation (4.6) is an equation for $V = U_q'$ of the form discussed in Section 3, though depending on parameters. Hence a unique solution exists for $0 \le s < A$. The Hamilton-Jacobi theory shows at once that it is a C^∞ function of (ω, s, q). The support of V is contained in $\operatorname{supp} F \times [0, A) \subset S^2 \times [0, A) \times [-M, M]$, so the integral U with respect to q which vanishes for $q > M$ has the required properties and is uniquely determined. For $q < -M$ we have

$$U(\omega, s, q) = U_-(\omega, s)$$

$$= -\int_{-\infty}^{\infty} V(\omega, s, q) dq$$

$$= -\iint_{0<\sigma<s} (a(\omega)(\partial V(\omega, \sigma, q)/\partial q)^2 + c(\omega)V(\omega, \sigma, q)^2) dq d\sigma. \tag{4.11}$$

This completes the proof.

When $a(\omega) \equiv c(\omega) \equiv 0$, as in the case studied in [2], it follows from (4.11) that $U_- = 0$, but this cannot be expected otherwise. That will perhaps be the main change in the following constructions. To define an approximate solution to (1.14), (1.9) we shall cut U off near the t axis by taking a function $\psi(t, x)$ in C^∞ which is homogeneous of degree 0, equal to 1 in a conic neighborhood of the light cone and equal to 0 in a conic neighborhood of the t axis. Let $0 < B < A$ where A is defined by (4.10). When ε is sufficiently small we have

$$|\partial^\alpha(\psi(t, x)U_-(\omega, \varepsilon \log(\varepsilon t)))| \le C_{\alpha, B} t^{-|\alpha|}, \tag{4.12}$$

if $\varepsilon t \ge 1$, $\varepsilon \log t \le B$. This is clear since all terms obtained by differentiation have a factor homogeneous of degree $-|\alpha|$. Choose $\chi \in C^\infty(\mathbb{R})$ decreasing, equal to 1 in $(-\infty, 1)$ and equal to 0 in $(2, \infty)$ and denote by εw_0 the solution of the unperturbed wave equation with Cauchy data (1.9). We shall prove that $\varepsilon w_\varepsilon$ is a good approximate solution of (1.14), (1.9) for $\varepsilon \log t \le B$ if

$$w_\varepsilon(t, x) = \chi(\varepsilon t) w_0(t, x) + (1 - \chi(\varepsilon t)) r^{-1} \psi(t, x) U(\omega, \varepsilon \log t, r - t), \tag{4.13}$$

where $r = |x|$, $\omega = x/r$. Defining $R_\varepsilon(t, x)$ by

$$\varepsilon \Box w_\varepsilon + G(\varepsilon w_\varepsilon', \varepsilon w_\varepsilon'') = \varepsilon R_\varepsilon, \tag{4.14}$$

we shall estimate w_ε, R_ε and their derivatives.

As mentioned in the introduction, the methods introduced by Klainerman [9] require that one considers not only differentiation with respect to the constant vector fields $\partial/\partial x_j$ but also the infinitesimal generators (1.10) of the Lorentz rotations and the radial vector field (1.11). By Z^I we shall denote any product of $|I|$ such vector fields.

Theorem 4.2. *With w_ε and R_ε defined by (4.13) and (4.14) we have for any I and for $0 < B < A$, defined by (4.10), if $\varepsilon \log t \le B$ and ε is small*

$$\left|Z^I w_\varepsilon(t, x)\right| \le C_{I,B}(1 + t)^{-1}, \tag{4.15}$$

$$\left|Z^I R_\varepsilon(t, x)\right| \le C_{I,B}\varepsilon(1 + t)^{-2}(1 + \varepsilon t)^{-1}. \tag{4.16}$$

Proof. The estimate (4.15) for $\varepsilon = 0$ is contained in [2], formula (2.1.13), and it implies that (4.15) holds for the first term in (4.13) since $Z^I \chi(\varepsilon t)$ has a uniform bound when $|x| \le t + M$. When $\varepsilon t \le 1$ we also obtain (4.16) since

$$\varepsilon R_\varepsilon = G(\varepsilon w_0', \varepsilon w_0'')$$

and the factor $(1 + \varepsilon t)^{-1}$ is $\ge 1/2$ then. To prove (4.15) when $\varepsilon t \ge 1$ we note that $Z \log(\varepsilon t)$ is homogeneous of degree ≤ 0, and that Zq is either homogeneous of degree 0 or else equal to $-\omega_j q$, if $Z = x_j \partial_t + t\partial_j$. This implies that Z^I applied to the second term in (4.13) is a sum of derivatives of U multiplied by functions homogeneous of degree ≤ -1 and powers of $q = r - t$; the latter only occur if U is differentiated with respect to q and then we have $|q| \le M$ in the support. Hence (4.15) follows.

Next we prove (4.16) in the transition zone where $1 \le \varepsilon t \le 2$. In addition to the arguments given above when $\varepsilon t \le 1$ we must then also estimate

$$\Box w_\varepsilon = \Box(w_\varepsilon - w_0) =$$
$$= ((1-\chi(\varepsilon t))\Box - 2\varepsilon\chi'(\varepsilon t)\partial_t - \varepsilon^2\chi''(\varepsilon t))(\tfrac{1}{r}\psi(t,x)U(\omega,\varepsilon\log(\varepsilon t), r-t) - w_0(t,x)).$$

In the term where χ is differentiated twice the desired bound $O(\varepsilon^3)$ is immediately clear. In the term where χ is differentiated once we use that

$$r^{-1}(\psi(t,x)U(\omega,\varepsilon\log(\varepsilon t), r-t) - F(\omega, r-t)) = O(\varepsilon r^{-1}),$$
$$r^{-1}F(\omega, r-t) - w_0(t,x) = O(r^{-2}),$$

and that these bounds still hold after multiplication by any Z^I. The first estimate follows from the proof of (4.15) since $0 \le \log(\varepsilon t) \le \log 2$ and $F(\omega, r-t) = \psi(t,x)F(\omega, r-t)$. The second one follows from (2.1.16) in [2]. What remains is to study

$$(1-\chi(\varepsilon t))r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega)(\psi(t,x)U(\omega,\varepsilon\log(\varepsilon t), r-t)).$$

Here $\partial_t + \partial_r$ must either act on ψ or else act on $\varepsilon\log(\varepsilon t)$, in the latter case producing a factor ε/t, which gives the desired bound if again we recall the proof of (4.15). When $\partial_t + \partial_r$ acts on ψ we just obtain a factor $O(1/t)$, but in the support of a derivative of ψ we have $U(\omega, 0, r-t) = F(\omega, r-t) = 0$, and since $0 \le \log(\varepsilon t) \le \log 2$ this shows that $U = O(\varepsilon)$ there. These arguments also yield the estimate (4.16) when $|I| \ne 0$.

Finally we must study the case where $2/\varepsilon \le t \le e^{B/\varepsilon}$. Then

$$\Box w_\varepsilon = r^{-1}((\partial_t - \partial_r)(\partial_t + \partial_r) - r^{-2}\Delta_\omega)(\psi(t,x)U(\omega,\varepsilon\log(\varepsilon t), r-t)).$$

In the terms where ψ is not differentiated we have just observed that $\partial_t + \partial_r$ must act on $\varepsilon\log(\varepsilon t)$, which yields a factor ε/t. Writing $s = \varepsilon\log(\varepsilon t)$ and $q = r-t$, we obtain

$$\left|\Box w_\varepsilon + 2\varepsilon r^{-1}t^{-1}U''_{sq}(\omega,s,q)\right| \le Ct^{-3},$$

for when ψ is differentiated we can replace U by U_- and conclude that every differentiation contributes a factor homogeneous of degree -1. With the notation in (4.4) we have

$$\left|\partial^\alpha w_\varepsilon - r^{-1}\hat\omega^\alpha\partial_q^{|\alpha|}U(\omega,s,q)\right| \le Cr^{-2},$$
$$|G(\varepsilon w'_\varepsilon, \varepsilon w''_\varepsilon) - \varepsilon^2 r^{-2}\sum_{1\le|\alpha|,|\beta|\le 2} g_{\alpha\beta}\hat\omega^{\alpha+\beta}\partial_q^{|\alpha|}U(\omega,s,q)\partial_q^{|\beta|}U(\omega,s,q)| \le$$
$$\le C\varepsilon^2 r^{-3}.$$

Recalling that by (4.6)

$$2U''_{sq} = \sum_{1\le|\alpha|,|\beta|\le 2} g_{\alpha\beta}\hat\omega^{\alpha+\beta}\partial_q^{|\alpha|}U(\omega,s,q)\partial_q^{|\beta|}U(\omega,s,q)$$

and that

$$\left|r^{-1}t^{-1} - r^{-2}\right| = |q|/tr^2 \le M/tr^2$$

in supp U'_q, we conclude that $|R_\varepsilon| \le C\varepsilon t^{-3}$. This proves (4.16) when $|I| = 0$, and using (4.15) we obtain (4.16) for arbitrary I.

5. A lower bound for the lifespan

Apart from minor differences of notation the estimates (4.15) and (4.16) are identical to the estimates (2.4.10) and (2.4.11) given in [2] for a special quasilinear case. However, there is an essential difference in that the measure of the supports of w_ε and R_ε for fixed t is now only $O(1+t)^3$, whereas it was $O(1+t)^2$ in the case discussed in [2]. This means that with L^2 norms

$$\left\|Z^I R_\varepsilon(t,.)\right\| \le C_{I,B}\varepsilon(1+t)^{-1/2}(1+\varepsilon t)^{-1}, \quad \text{if } \varepsilon\log t \le B,$$

which implies that

$$\int_0^{e^{B/\varepsilon}} \left\|Z^I R_\varepsilon(t,.)\right\| dt \le C_{I,B}\varepsilon^{1/2}$$

where we had an estimate by $\varepsilon\log(1/\varepsilon)$ in [2]. However, this does not affect the ideas of the proof in [2] of estimates for the derivatives of the difference $v = u_\varepsilon - \varepsilon w_\varepsilon$ where u_ε is the solution of (1.14), (1.9). (When estimating v' one should factor the non-linear term $G(v' + \varepsilon w'_\varepsilon, v'' + \varepsilon w''_\varepsilon) - G(\varepsilon w'_\varepsilon, \varepsilon w''_\varepsilon)$ and regard it as a linear function of v', v''.) The conclusion is that for small ε a C^∞ solution exists when $\varepsilon\log t \le B$, and that

$$\left\|Z^I(u'_\varepsilon(t,.)/\varepsilon - w'_\varepsilon(t,.))\right\| \le C_{I,B}\varepsilon^{1/2}.$$

This implies that

$$\left|Z^I(u'_\varepsilon(t,.)/\varepsilon - w'_\varepsilon(t,.))\right| \le C_{I,B}\varepsilon^{1/2}(1+t)^{-1}(1+|q|)^{-1/2},$$

hence by integration from the set where $|x| > t + M$,

$$|u_\varepsilon(t,.)/\varepsilon - w_\varepsilon(t,.)| \le C_{I,B}\varepsilon^{1/2}(1+|q|)^{1/2}/(1+t),$$

so we obtain

Theorem 5.1. *If $0 < B < A$, where A is defined by (4.10), and if ε is small, then the Cauchy problem (1.14), (1.9) has a solution u_ε when $0 \le t \le e^{B/\varepsilon}$, and*

$$(5.1)\quad \varepsilon^{-1}(e^{s/\varepsilon} + q)u_\varepsilon(e^{s/\varepsilon}, (e^{s/\varepsilon} + q)\omega) - U(\omega,s,q) = O(\varepsilon^{1/2})(1+|q|)^{1/2}$$

as $\varepsilon \to 0$, *uniformly when* $\varepsilon \log(2/\varepsilon) \le s \le B$ *and* $q > -\frac{1}{2}e^{s/\varepsilon}$. *Hence*

$$\varliminf_{\varepsilon\to 0} \varepsilon \log T_\varepsilon \ge A, \tag{5.2}$$

if T_ε *is the lifespan of* u_ε.

When $a(\omega) \equiv 0$, $b(\omega) \equiv 0$, and $c(\omega) \equiv 0$ then $A = \infty$ in (4.10). If the equation is quasi-linear it has then been proved by Christodoulou [1] and Klainerman [10] (see also Hörmander [3]) that there is a global solution for small ε. Also u is then allowed to occur in terms of order higher than 2. However, it is an open question if there is global existence in the other cases where Theorem 3.6 allows A to be infinite.

6. Some open problems

Theorem 5.1 is unsatisfactory in several ways. First of all one would like to know if (5.2) can be strengthened to $\lim_{\varepsilon\to 0} \varepsilon \log T_\varepsilon = A$, which is known only in a simple rotationally symmetric case (John [7]; see also Hörmander [2]). A proof of this should at the same time give much more precise information on the way that blowup occurs. When the perturbation G depends on u also one does not even know the order of magnitude of T_ε in all cases when $n \le 4$. (The statements made in Klainerman [11], John-Klainerman [8] in this respect have been withdrawn by Klainerman.)

Finally there is of course the much harder question of defining and proving existence of solutions with appropriately restricted discontinuities beyond the time of blowup. There are few results on that except when there is only one space variable; even then there are numerous open questions, so the field cannot be expected to be exhausted in the near future.

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