BOL. SOC. BRAS. MAT., VOL. 19, Nº 1 (1988), 21-37

EXISTENCE OF SOLUTION FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS AT DOUBLE RESONANCE

D. G. COSTA [1] AND A. S. OLIVEIRA [2]

1. Introduction

In this paper we consider a class of "doubly resonant" problems of the form

$$(*)\quad \begin{cases} -\Delta u = f(x,u) + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and we are given a function $h \in L^2(\Omega)$ and a Carathéodory function $f: \Omega \times \mathbb{R} \to \mathbb{R}$ such that $m_r(x) := \max_{|s| \leq r} |f(x,s)| \in L^2(\Omega)$ for each $r > 0$. We shall assume the conditions below are satisfied, where (f_1) expresses the fact that resonance may occur at two consecutive eigenvalues $\lambda_i < \lambda_{i+1}$ of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$:

(f_1) $$\lambda_i \leq \ell(x) := \liminf_{|s|\to\infty} \frac{f(x,s)}{s} \leq \limsup_{|s|\to\infty} \frac{f(x,s)}{s} := k(x) \leq \lambda_{i+1},$$

uniformly for a.e. $x \in \Omega$;

(F_1) $$\lambda_i \leq L(x) := \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \leq \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} := K(x) \leq \lambda_{i+1},$$

---

(1) Work partially supported by CNPq/Brasil.

(2) Work partially supported by CAPES/Brasil.

Recebido em 28/05/88.

uniformly for a.e. $x \in \Omega$, with strict inequalities $\lambda_i < L(x)$, $K(x) < \lambda_{i+1}$ holding on subsets of Ω of positive measure, where $F(x,s)$ denotes the primitive

$$F(x,s) = \int_0^s f(x,t)\,dt.$$

Remark. From the conditions above it follows that there exist constants a, $A > 0$ and functions $b \in L^2(\Omega)$, $B \in L^1(\Omega)$ such that

$$(1) \qquad |f(x,s)| \le a|s| + b(x), \quad x \in \Omega, \quad s \in \mathbb{R},$$

$$(2) \qquad |F(x,s)| \le As^2 + B(x), \quad x \in \Omega, \quad s \in \mathbb{R}.$$

Our main result states that, under hypotheses (f_1), (F_1), problem (*) is solvable for any given $h \in L^2(\Omega)$. To our knowledge, this generalizes many of the existing results for doubly resonant problems (cf. [4,7,9], e.g.). Corresponding general results for resonance at the first eigenvalue λ_1 have been recently obtained by Mawhin-Ward-Willem [16] and de Figueiredo-Gossez [10].

We remark that there is a rich literature dealing with resonant problems, starting with a very nice result due to Landesman-Lazer [14] on resonance at the first eigenvalue. Besides the above cited papers, we also refer the interested reader to e.g. [1-3,5,6,8,11,12,15,17,18] and their references.

Our approach to problem (*) is variational and uses the well-known Saddle Point Theorem of P. Rabinowitz [18]. We recall that, under condition (1), the functional

$$I(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - F(x,u) - hu\right] dx$$

is well-defined and of class C^1 on the Sobolev space $E = H_0^1(\Omega)$, with derivative $I'(u) \in E^* = H^{-1}(\Omega)$ given by

$$I'(u)\cdot\omega = \int_\Omega [\nabla u\cdot\nabla\omega - f(x,u)\omega - h\omega]\,dx$$

for all $u,\omega \in E$. Thus the critical points of I are precisely the weak solutions $u \in E$ of (*). We shall denote the norm in $E = H_0^1(\Omega)$ by $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$.

2. Preliminary Lemmas

Let $\{u_n\} \subset E$ be an unbounded sequence. Then, defining

$$v_n = \frac{u_n}{\|u_n\|},$$

we have $\|v_n\| = 1$ and, passing if necessary to a subsequence, we may assume that

$$(3) \qquad \begin{aligned} &v_n \longrightarrow v \quad \text{weakly in } E, \\ &v_n \longrightarrow v \quad \text{strongly in } L^2(\Omega), \\ &v_n(x) \longrightarrow v(x) \quad \text{a.e. in } \Omega, \end{aligned}$$

and $|v_n(x)| \le z(x)$ a.e., where $z \in L^2(\Omega)$.

Now, assuming (f_1), we obtain that the sequence $\{f(\cdot,u_n)/\|u_n\|\}$ is bounded in $L^2(\Omega)$, so we may assume that

$$(4) \qquad \frac{f(\cdot,u_n)}{\|u_n\|} \longrightarrow \tilde{f} \quad \text{weakly in } L^2(\Omega).$$

Lemma 1 ([4]). The function $\tilde{f}$ above satisfies

$$(5) \qquad \ell(x) \le \frac{\tilde{f}(x)}{v(x)} \le k(x) \quad \text{if } v(x) \ne 0,$$

$$(6) \qquad \tilde{f}(x) = 0 \quad \text{if } v(x) = 0,$$

where v and ℓ, k are given in (3) and (f_1), respectively.

Proof. See [4, Lemma 4].

Lemma 2. If $\|u_n\| \to \infty$ then

$$(7) \quad L(x)v(x)^2 \leq \liminf \frac{2F(x,u_n(x))}{\|u_n\|^2} \leq \limsup \frac{2F(x,u_n(x))}{\|u_n\|^2} \leq K(x)v(x)^2$$

for a.e. $x \in \Omega$, where v and K, L are given in (3) and (F_1), respectively.

Proof. We shall study the pointwise limits of the sequence

$$(8) \qquad \Psi_n(x) = \frac{2F(x,u_n(x))}{\|u_n\|^2}$$

on the sets $\Omega_* = \{x \in \Omega \mid v(x) \neq 0\}$ and $\Omega_0 = \{x \in \Omega \mid v(x) = 0\}$.

(i) On Ω_*. Since $|u_n(x)| = |v_n(x)|\|u_n\| \to +\infty$ a.e. in Ω_* we can write

$$\Psi_n(x) = \frac{2F(x,u_n(x))}{u_n(x)^2} v_n(x)^2$$

for all n sufficiently large (given $x \in \Omega_*$) and, hence, we obtain

$$\liminf \Psi_n(x) \geq (\liminf \frac{2F(x,u_n(x))}{u_n(x)^2}) v(x)^2 \geq L(x)v(x)^2,$$

$$\limsup \Psi_n(x) \leq (\limsup \frac{2F(x,u_n(x))}{u_n(x)^2}) v(x)^2 \leq K(x)v(x)^2,$$

in view of the definition of $L(x)$ and $K(x)$. The above inequalities prove (7) for a.e. $x \in \Omega_*$.

(ii) On Ω_0. For $x \in \Omega_0$ we distinguish two cases: if $|u_n(x)| \leq 1$ then $|F(x,u_n(x))| \leq A|u_n(x)|^2 + B(x) \leq A + B(x)$ and, since $\|u_n\| \to \infty$, we obtain $\lim \Psi_n(x) = 0 = L(x)v(x)^2 = K(x)v(x)^2$; if $|u_n(x)| \geq 1$ then

$$\frac{|F(x,u_n(x))|}{u_n(x)^2} \leq A + \frac{B(x)}{u_n(x)^2} \leq A + B(x)$$

and, in view of (8) and the fact that $v_n(x) \to v(x) = 0$, we obtain $\lim \Psi_n(x) = 0 = L(x)v(x)^2 = K(x)v(x)^2$. Therefore,

$$\lim \Psi_n(x) = 0 = L(x)v(x)^2 = K(x)v(x)^2 \quad \text{for a.e. } x \in \Omega_0$$

and the proof of (7) is complete. □

The next result is standard (cf. [15] e.g.).

Lemma 3. Let $m: \Omega \to \mathbb{R}$ be a measurable function satisfying $\lambda_i \leq m(x) \leq \lambda_{i+1}$, with $\lambda_i < m(x)$ and $m(x) < \lambda_{i+1}$ on subsets U_1 and U_2 of positive measure. If $v \in E$ is a weak solution of

$$-\Delta v = m(x)v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega,$$

then necessarily $v \equiv 0$.

3. The Main Result

Before we state and prove our main theorem we need to study the functional $I: E \to \mathbb{R}$ defined in the introduction. Throughout this section it will be assumed that conditions (f_1), (F_1) hold.

Proposition 1. The functional I satisfies the Palais-Smale condition (PS), that is, whenever $\{u_n\} \subset E$ is a sequence such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ then $\{u_n\}$ possesses a convergent subsequence.

Proof. Let $\{u_n\} \subset E$ be such that $|I(u_n)| \leq C$, $I'(u_n) \to 0$. Since $\nabla I(u) = u - T(u)$ where $T:E \to E$ is a compact operator, in order to show that $\{u_n\}$ has a convergent subsequence it suffices to show that $\{u_n\}$ is bounded.

Suppose by contradiction that $\|u_n\| \to \infty$. Then, as we observed in the previous section, (a subsequence of) $v_n = u_n/\|u_n\|$ is such that $v_n \longrightarrow v$ weakly in E, $v_n \longrightarrow v$ strongly in $L^2(\Omega)$ and $v_n(x) \to v(x)$ a.e. in Ω, with $|v_n(x)| \leq z(x)$, $z \in L^2(\Omega)$. Moreover, we showed that

(4) $$\frac{f(\cdot, u_n)}{\|u_n\|} \longrightarrow \tilde{f} \text{ weakly in } L^2(\Omega),$$

where $\tilde{f}$ satisfies

(5) $$\ell(x) \leq \frac{\tilde{f}(x)}{v(x)} \leq k(x) \text{ if } v(x) \neq 0,$$

(6) $$\tilde{f}(x) = 0 \quad \text{if} \quad v(x) = 0.$$

Let us define

$$m(x) = \begin{cases} \dfrac{\tilde{f}(x)}{v(x)}, & \text{if } v(x) \neq 0 \\ \bar{\lambda} = \dfrac{1}{2}(\lambda_i + \lambda_{i+1}), & \text{if } v(x) = 0. \end{cases}$$

Then $\tilde{f}(x) = m(x)v(x)$ and, by (5), (6), we have

(9) $$\ell(x) \leq m(x) \leq k(x) \quad \text{if} \quad v(x) \neq 0,$$

(10) $$m(x) = \bar{\lambda} \quad \text{if} \quad v(x) = 0,$$

so that $\lambda_i \leq m(x) \leq \lambda_{i+1}$ in view of (f_1).

Now, by hypothesis, we have $|I'(u_n)\cdot\omega| \leq \varepsilon_n\|\omega\|$ for all $\omega \in E$, where $\varepsilon_n \to 0$. Therefore

$$\frac{|I'(u_n)\cdot u_n|}{\|u_n\|^2} = \left|1 - \int \frac{f(x,u_n)}{\|u_n\|} v_n - \frac{1}{\|u_n\|}\int h v_n\right| \leq \frac{\varepsilon_n}{\|u_n\|} \to 0,$$

so that

$$\int \frac{f(x,u_n)}{\|u_n\|} v_n \to 1.$$

From this, using (4) and the fact that $v_n \to v$ in L^2, we obtain

(11) $$\int \tilde{f} v = 1,$$

so that $v \not\equiv 0$, necessarily.

On the other hand, for any $\omega \in E$ we have that

$$\frac{|I'(u_n)\cdot\omega|}{\|u_n\|} = \left|\int \nabla v_n \cdot \nabla\omega - \int \frac{f(x,u_n)}{\|u_n\|}\omega - \frac{1}{\|u_n\|}\int h\omega\right| \leq \varepsilon_n \frac{\|\omega\|}{\|u_n\|} \to 0,$$

from which, using (4) and the fact that $v_n \longrightarrow v$ weakly in E, we conclude

(12) $$\int \nabla v \cdot \nabla\omega - \int \tilde{f}\omega = 0 \quad \forall\, \omega \in E;$$

in other words, $v \in E$ is a weak solution of the problem

$$(13)\qquad \begin{cases} -\Delta v = \tilde{f} = mv & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Now, we have three cases to consider: (i) $m(x) \equiv \lambda_i$; (ii) $m(x) \equiv \lambda_{i+1}$; (iii) $\lambda_i \le m(x) \le \lambda_{i+1}$, with $\lambda_i < m(x)$ and $m(x) < \lambda_{i+1}$ on subsets of positive measure.

Case (iii). Since $v \not\equiv 0$, this case can not occur in view of Lemma 3.

Case (i). If $m(x) \equiv \lambda_i$, we obtain $\|v\|^2 = \lambda_i |v|^2_{L^2}$ from (13) and, in fact,

$$(14)\qquad \lambda_i |v|^2_{L^2} = 1 = \|v\|^2,$$

in view of (11). On the other hand, from $|I(u_n)| \le C$ we obtain

$$\frac{2I(u_n)}{\|u_n\|^2} = 1 - \int \frac{2F(x,u_n)}{\|u_n\|^2} - \frac{2}{\|u_n\|} \int h v_n \to 0,$$

so that

$$(15)\qquad \int \frac{2F(x,u_n)}{\|u_n\|^2} \to 1.$$

Therefore, combining (14), (15) and Fatou's Lemma yields

$$\lambda_i \int v^2 = \lim \int \frac{2F(x,u_n)}{\|u_n\|^2} \ge \int \liminf \frac{2F(x,u_n)}{\|u_n\|^2},$$

hence

$$\lambda_i \int v^2 \ge \int L(x)\, v^2,$$

in view of Lemma 2. But then, since $L(x) \ge \lambda_i$, we obtain $L(x) = \lambda_i$ a.e. in Ω which contradicts (F_1) and shows that case (i) can not occur.

Case (ii). Similarly to above, if $m(x) \equiv \lambda_{i+1}$ we obtain $1 = \lambda_{i+1}|v|^2_{L^2}$ and

$$\lambda_{i+1}\int v^2 = \lim \int \frac{2F(x,u_n)}{\|u_n\|^2} \le \int \limsup \frac{2F(x,u_n)}{\|u_n\|^2},$$

so that

$$\lambda_{i+1}\int v^2 \le \int K(x) v^2$$

by Lemma 2 and, as $K(x) \le \lambda_{i+1}$, we conclude that $K(x) = \lambda_{i+1}$ a.e. in Ω, again reaching a contradiction to (F_1).

Since neither one of cases (i)-(iii) can occur, this shows that any sequence $\{u_n\} \subset E$ for which $|I(u_n)| \le C$, $I'(u_n) \to 0$ must necessarily be bounded, so that the functional I satisfies the Palais-Smale condition. □

Now, let us decompose the space $E = H^1_0(\Omega)$ as

$$E = V \oplus W,$$

where V is the subspace spanned by the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_i$, $W = V^\perp$ and let us define the quadratic forms $\mu\colon V \to \mathbb{R}$, $\nu\colon W \to \mathbb{R}$ by

$$\mu(v) = \|v\|^2 - \int L(x) v^2, \quad v \in V,$$

$$\nu(\omega) = \|\omega\|^2 - \int K(x)\omega^2, \quad \omega \in W.$$

Following the ideas of [16, Lemma 1], we shall prove the following

Proposition 2. There exists $\delta > 0$ such that

(a) $\nu(\omega) \geq \delta\|\omega\|^2$ for all $\omega \in W$,

(b) $\mu(v) \leq -\delta\|v\|^2$ for all $v \in V$; moreover

(c) $I(\omega) \to +\infty$ as $\|\omega\| \to \infty$, $\omega \in W$ and

(d) $I(v) \to -\infty$ as $\|v\| \to \infty$, $v \in V$.

Proof. (a) Since $\|\omega\|^2 \geq \lambda_{i+1}\|\omega\|^2_{L^2}$ for all $\omega \in W$, we have

(16) $$\nu(\omega) \geq \int [\lambda_{i+1} - K(x)]\omega^2 \geq 0 \quad \forall\, \omega \in W.$$

Next we show that $\nu(\omega) = 0$ implies $\omega \equiv 0$. Indeed, if $\nu(\omega) = 0$ we use (16) to get that $\omega = 0$ on the set $\Omega_2 = \{x \in \Omega \; K(x) < \lambda_{i+1}\}$. But we also get

$$0 = \nu(\omega) = \|\omega\|^2 - \int K(x)\omega^2 \geq \|\omega\|^2 - \lambda_{i+1}|\omega|^2_{L^2} \geq 0,$$

hence $\|\omega\|^2 = \lambda_{i+1}|\omega|^2_{L^2}$, which shows that ω is a λ_{i+1}-eigenfunction. Therefore, since $\omega = 0$ on the set Ω_2 of positive measure, the unique continuation principle implies that $\omega \equiv 0$.

Now, suppose that (a) is false. Then we can find a sequence $\omega_n \in W$ such that $\|\omega_n\| = 1$ and $\nu(\omega_n) \to 0$. Passing, if necessary, to a subsequence we may assume that $\omega_n \longrightarrow \omega \in W$ weakly and $\omega_n \longrightarrow \omega$ in $L^2(\Omega)$, so that

$$\nu(\omega) \leq \liminf \nu(\omega_n) = 0$$

by the weak lower semicontinuity of the convex functional ν on W. Therefore, we get $\nu(\omega) = 0$ and, hence, $\omega = 0$. But, then, we have that $\omega_n \to \omega = 0$ in $L^2(\Omega)$, hence

$$\nu(\omega_n) = 1 - \int K(x)\omega_n^2 \to 1,$$

which contradicts $\nu(\omega_n) \to 0$. Therefore (a) holds.

(c) Let $0 < \varepsilon < \delta\lambda_{i+1}$ where δ is given above. By (F_1) there exists $B_\varepsilon \in L^1(\Omega)$ such that

$$2F(x,s) \leq (K(x) + \varepsilon)s^2 + B_\varepsilon(x)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Therefore, we obtain

$$2I(\omega) = \|\omega\|^2 - 2\int F(x,\omega) - 2\int h\omega \geq \|\omega\|^2 - \int (K(x)+\varepsilon)\omega^2 - 2\int h\omega - \int B_\varepsilon$$

$$= \nu(\omega) - \varepsilon\int \omega^2 - 2\int h\omega - \int B_\varepsilon \geq \delta\|\omega\|^2 - \varepsilon|\omega|^2_{L^2} - 2|h|_{L^2}|\omega|_{L^2} - |B_\varepsilon|_{L^1}$$

$$\geq \delta\|\omega\|^2 - \frac{\varepsilon}{\lambda_{i+1}}\|\omega\|^2 - C_1\|\omega\| - C_2$$

for all $\omega \in W$, that is,

$$2I(\omega) \geq (\delta - \frac{\varepsilon}{\lambda_{i+1}})\|\omega\|^2 - C_1\|\omega\| - C_2 \quad \forall\, \omega \in W,$$

where $\delta - \varepsilon/\lambda_{i+1} > 0$. It follows that $I(\omega) \to +\infty$ as $\|\omega\| \to \infty$, $\omega \in W$.

(b) Since $\|v\|^2 \leq \lambda_i|v|^2_{L^2}$ for all $v \in V$, we have

$$\mu(v) \leq \int [\lambda_i - L(x)]v^2 \leq 0 \quad \forall\, v \in V.$$

Similarly to (a), we prove that $\mu(v) = 0$ implies $v \equiv 0$, by showing that $v = 0$ on the set $\Omega_1 = \{x \in \Omega \mid \lambda_i < L(x)\}$ of positive measure and that v is a λ_i-eigenfunction. Then, supposing that (b) is false, we obtain $v_n \in V$ such that $\|v_n\| = 1$, $\mu(v_n) \to 0$, where we may assume that $v_n \to v \in V$ in H_0^1 since V is of finite dimension. Therefore we obtain $\mu(v) = 0$, so that $v = 0$ and, consequently, $\int L(x)v_n^2 \to 0$. But, then, it follows that

$$\mu(v_n) = 1 - \int L(x)v_n^2 \to 1,$$

contradicting $\mu(v_n) \to 0$. This shows that (b) holds.

Finally, to prove (d), we fix ε such that $0 < \varepsilon < \delta\lambda_1$ and use (F_1) to get the estimate

$$2F(x,s) \geq (L(x)-\varepsilon)s^2 - B_\varepsilon(x)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, which implies

$$\begin{aligned} 2I(v) &\leq \mu(v) + \varepsilon|v|^2_{L^2} + 2|h|_{L^2}|v|_{L^2} + |B_\varepsilon|_{L^1} \\ &\leq -(\delta - \frac{\varepsilon}{\lambda_1})\|v\|^2 + C_1\|v\| + C_2 \end{aligned}$$

for all $v \in V$. Since $\delta - \varepsilon/\lambda_1 > 0$ it follows that $I(v) \to -\infty$ as $\|v\| \to \infty$, $v \in V$. The proof of Proposition 2 is complete. □

We are now ready to prove our main result.

Theorem 1. Assume hypotheses (f_1), (F_1) given in the introduction. Then, for any given $h \in L^2(\Omega)$, problem

$$(*) \quad \begin{cases} -\Delta u = f(x,u) + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution $u \in H_0^1(\Omega)$.

Proof. We recall that the space $E = H_0^1(\Omega)$ is being decomposed as $E = V \oplus W$, where V is the subspace spanned by the λ_j-eigenfunctions, $j=1,\ldots,i$, and $W = V^\perp$. Also, our pertinent functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega [F(x,u) + hu]dx = \frac{1}{2}\|u\|^2 + N(u), \quad u \in E,$$

is weakly lower semicontinuous, being the sum of the weakly lower semicontinuous functional $\|u\|^2/2$ and the weakly continuous functional N. Therefore, since $I|W$ is coercive by Proposition 2(c), the infimum $\beta := \inf_W I > -\infty$ is attained. Now take $\alpha < \beta$. By Proposition 2(d) there exists $R > 0$ such that $I(v) \leq \alpha$ for all $v \in V$ with $\|v\| \geq R$. Therefore, since I satisfies (PS) by Proposition 1, we can use the Saddle Point Theorem of P. Rabinowitz [18] to conclude the existence of a critical point $u_0 \in E$ of I with $I(u_0) \geq \beta$. □

4. Some Examples

Example 1. As a first example, we consider the two-point boundary value problem

$$\begin{cases} -u'' = f(u) + h(x), & 0 < x < \pi \\ u(0) = u(\pi) = 0 \end{cases}$$

where $h \in L^2(0,\pi)$ and $f(s) = 6.5s + 2.5s\sin s$. This is a doubly resonant problem since

$$\ell := \liminf_{|s|\to\infty} \frac{f(s)}{s} = 4 = \lambda_2, \quad k := \limsup_{|s|\to\infty} \frac{f(s)}{s} = 9 = \lambda_3.$$

It was handled by a different method in [9] by means of a "density condition at infinity" for $f(s)$. Here, we compute $F(s) = 6.5s^2/2 + 2.5(\sin s - s\cos s)$, so that there exists the limit

$$\lim_{|s|\to\infty} \frac{2F(s)}{s^2} = 6.5,$$

with $\lambda_2 < 6.5 < \lambda_3$ and, hence, (F_1) is clearly satisfied. We remark [13] that condition (F_1) and the density condition in [9] seem to be equivalent in the x-independent case $f(x,s) = f(s)$.

Example 2. Now, we consider a PDE example where the nonlinearity $f(x,s)$ depends effectively on x and is such that

$$(f_2) \quad \ell(x) := \liminf_{|s|\to\infty} \frac{f(x,s)}{s} \equiv \lambda_i, \quad k(x) := \limsup_{|s|\to\infty} \frac{f(x,s)}{s} \equiv \lambda_{i+1}.$$

Let Ω_1, Ω_2 be measurable subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$, $\text{meas}(\Omega_1)\cdot\text{meas}(\Omega_2) > 0$ and $\text{meas}(\Omega \setminus \Omega_1 \cup \Omega_2) = 0$. Define

$$f(x,s) = \rho(x)s + \alpha(x,s)s,$$

where

$$\rho(x) := \chi_{\Omega_1}(x)\lambda_i + \chi_{\Omega_2}(x)\lambda_{i+1} = \begin{cases} \lambda_i & \text{if } x \in \Omega_1 \\ \lambda_{i+1} & \text{if } x \in \Omega_2, \end{cases}$$

$$\alpha(x,s) := [\chi_{\Omega_1}(x) - \chi_{\Omega_2}(x)]\theta(s) = \begin{cases} \theta(s) & \text{if } x \in \Omega_1 \\ -\theta(s) & \text{if } x \in \Omega_2, \end{cases}$$

and $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous functions satisfying

$$\liminf_{|s|\to\infty} \theta(s) = 0, \quad \limsup_{|s|\to\infty} \theta(s) = \lambda_{i+1} - \lambda_i,$$

$$\lim_{|s|\to\infty} \frac{A(s)}{s^2} = 0, \quad \text{with } A(s) = \int_0^s \theta(t)t\,dt. \quad \text{Then we obtain}$$

$$\frac{f(x,s)}{s} = \rho(x) + \alpha(x,s) = \begin{cases} \lambda_i + \theta(s) & \text{if } x \in \Omega_1 \\ \lambda_{i+1} - \theta(s) & \text{if } x \in \Omega_2, \end{cases}$$

so that (f_2) is satisfied. On the other hand, we have

$$\frac{2F(x,s)}{s^2} = \rho(x) + \frac{2}{s^2}\int_0^s \alpha(x,t)t\,dt = \begin{cases} \rho(x) + 2A(s)/s^2 & \text{if } x \in \Omega_1 \\ \rho(x) - 2A(s)/s^2 & \text{if } x \in \Omega_2, \end{cases}$$

hence

$$\lim_{|s|\to\infty} \frac{2F(x,s)}{s^2} = \rho(x) = \begin{cases} \lambda_i & \text{if } x \in \Omega_1 \\ \lambda_{i+1} & \text{if } x \in \Omega_2, \end{cases}$$

so that condition (F_1) is satisfied as $\text{meas}(\Omega_1)\cdot\text{meas}(\Omega_2) > 0$.

References

[1] S. Ahmad, A.C. Lazer, J.L. Paul - *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J., 25 (1976), 933-944.

[2] H. Amann, A. Ambrosetti, G. Mancini - *Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities*, Math. Zeit., 158 (1978), 179-194.

[3] H. Amann, G. Mancini - *Some applications of monotone operator theory to resonance problems*, Nonlinear Analysis, T.M.A., 3 (1979), 815-830.

[4] H. Berestycki, D.G. de Figueiredo - *Double resonance in semilinear elliptic problems*, Comm. Partial Diff. Equations, 6(1), 91-120 (1981).

[5] H. Brézis, L. Nirenberg - *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Sc. Norm. Sup. Pisa, Serie IV, 5 (1978), 225-326.

[6] D.G. Costa, J.V.A. Gonçalves - *Existence and multiplicity results for a class of nonlinear elliptic boundary value problems at resonance*, J. Math. Anal. Appl. 84 (1981), 328-337.

[7] D.G. de Figueiredo - *Resonance of a semilinear elliptic equation at two eigenvalues*, Trab. Mat. nº 133, Univ. Brasília, 1978.

[8] D.G. de Figueiredo - *Semilinear elliptic equations at resonance: higher eigenvalues and unbounded nonlinearities*, in "Recent Advances in Diff. Equations", Academic Press, 1981.

[9] D.G. de Figueiredo, J.-P. Gossez - *Conditions de non-résonance pour certains problèmes elliptiques semi-linéaires*, C.R. Acad. Sc. Paris, 302 (1986), 543-545.

[10] D.G. de Figueiredo, J.-P. Gossez - *Nonresonance below the first eigenvalue for a semilinear elliptic problem*, College on Variational Problems in Analysis (11 Jan/5 Feb 1988), I.C.T.P., Trieste-Italy.

[11] D.G. de Figueiredo, J.-P. Gossez - *Nonlinear perturbations of a linear elliptic problem near its first eigenvalue*, J. Diff. Equat., 30 (1978), 1-19.

[12] D.G. de Figueiredo, W.-M. Ni - *Perturbations of second order linear elliptic problems by nonlinearities without Landesman-Lazer condition*, Nonlinear Analysis, T.M.A., 3 (1979), 629-634.

[13] J.-P. Gossez - *Personal Communication*.

[14] E.M. Landesman, A.C. Lazer - *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech., 19 (1970), 609-623.

[15] J. Mawhin, J.R. Ward Jr. - *Nonresonance and existence for nonlinear elliptic boundary value problems*, Nonlinear Analysis, T.M.A., 6 (1981), 677-684.

[16] J. Mawhin, J. Ward Jr., M. Willem - *Variational methods and semilinear elliptic equations*, Arch. Rat. Mech. Anal., 95 (1986), 269-277.

[17] J. Mawhin, M. Willem - *Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance*, Rapport nº 65, Univ. Catholique de Louvain, 1985.

[18] P.H. Rabinowitz - *Some minimax theorems and applications to nonlinear partial differential equations*, in "Nonlinear Analysis", Cesari, Kannan and Weinberger ed., Academic Press, 1978, 161-177.

Universidade de Brasília
Departamento de Matemática
70.910 - Brasília-DF
Brasil

Universidade Federal da Paraíba
Departamento de Matemática-CCEN
58.059 - João Pessoa-PB
Brasil