CORRECTION OF COMPLETE HYPERSURFACES IN THE SPACE FORM WITH THREE PRINCIPAL CURVATURES

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In the proofs of following two theorems in [2], some mistakes were found out by Professor Pawel Walczak and Professor Fabiano Brito.

Theorem I. Let M be a complete hypersurface in $\bar{M}(c)$ with constant mean curvature, where $c \geq 0$. If M has three non-simple principal curvatures, then M is isoparametric and $c > 0$.

Theorem II. Let M be a complete minimal hypersurface in S^{n+1} with three principal curvatures. If $n \geq 4$, then M is either isoparametric, of type (B_1), or of type (B_2) in [1].

Here we correct both proofs following the original method. The essential miss is in the proof of case (iii) appeared in p. 352 of [2]. By the way, we give another proof for Theorem I in compact case using the results in [3]. This is equivalent to prove that

Theorem III. If a compact embedded Dupin hypersurface M in $\bar{M}(c)$ $(c \geq 0)$ with three principal curvatures has a constant mean curvature, then M is isoparametric and $c > 0$.

In this paper, we follow the notation in [2]. Recall that $\bar{M}(c) = E^{n+1}$ or $S^{n+1} = S^{n+1}(1)$ according to $c = 0$ or $c = 1$. Principal curvatures of M are λ, μ, ν satisfying

$$m_1\lambda + m_2\mu + m_3\nu = nh,$$

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where we may assume that h is non-negative constant by choosing a suitable unit normal vector field on M. Corresponding curvature distributions are given by

$$\mathcal{D}(\lambda) = \{e_a \mid 1 \leq a \leq m_1\},$$

$$\mathcal{D}(\mu) = \{e_i \mid m_1+1 \leq i \leq m_1+m_2\},$$

$$\mathcal{D}(\nu) = \{e_r \mid m_1+m_2+1 \leq r \leq n\},$$

where e_a, e_i, e_r are suitably chosen orthonormal frame of M. Range of indices is always: $1 \leq a,b,c \leq m_1 < i,j,k \leq m_1+m_2 < r,s,t \leq n$.

1. Preliminaries

Let M be as in Theorem I. Note that if all leaves of foliations $\mathcal{D}(\lambda)$, $\mathcal{D}(\mu)$, $\mathcal{D}(\nu)$ are compact, then M is compact. Since a leaf L^λ of $\mathcal{D}(\lambda)$ is an m_1-sphere of $\bar{M}(c)$, whose radius ρ is given by

$$\rho = \left\{\sum_i (\Lambda^{\ i}_{aa})^2 + \sum_r (\Lambda^{\ r}_{aa})^2 + \lambda^2 + c\right\}^{-\frac{1}{2}},$$

M is compact if $c > 0$. When $c = 0$, L^λ is an m_1-plane of E^{n+1} if and only if $\lambda = \Lambda^{\ i}_{aa} = \Lambda^{\ r}_{aa} \equiv 0$ on L^λ.

For later use, we show

Lemma 1.1. Let M be a complete hypersurface in $\bar{M}(c)$ with three principal curvatures λ, μ, ν satisfying $m_1\lambda+m_2\mu+m_3\nu = nh$ (a constant). Let $m_1 \geq 2$ or let λ be constant along any leaf of $\mathcal{D}(\lambda)$. If a leaf L^λ of $\mathcal{D}(\lambda)$ is compact and if $2\lambda \neq \mu+\nu$ everywhere on L^λ, then μ and ν are constant on L^λ.

Proof. It is sufficient to prove that μ is constant on L^λ. Since L^λ is compact, there exists a point p of L^λ such that $\mu|_{L^\lambda}$ is critical at p, or $\Lambda^{\ a}_{ii}(p) = 0$ for all $1 \leq a \leq m_1$. We first show that $\mu|_{L^\lambda}$ depends only on the distance (of L^λ) from p.

Let γ be a geodesic of L^λ through p. Let $e_1 = \dot\gamma$ and e_b be any parallel vector field (with respect to ∇^λ, the connection of L^λ) along γ such that $e_1 \perp e_b$. Then the second formula in p. 352 of [2] becomes

$$(1.1) \qquad e_1(\Lambda^{\ b}_{ii}) = \left\{1 + \frac{2n(\lambda - h)(\lambda - \mu)^2}{m_3(2\lambda - \mu - \nu)(\nu - \lambda)(\nu - \mu)}\right\}\Lambda^{\ 1}_{ii}\Lambda^{\ b}_{ii}$$

Since $2\lambda \neq \mu + \nu$ everywhere on L^λ, and since $\Lambda^{\ b}_{ii}(p) = 0$, we get

$$\Lambda^{\ b}_{ii} = 0$$

along γ. Next, we will show that $\Lambda^{\ 1}_{ii} = 0$ along γ if $2\lambda \neq \mu+\nu$ everywhere on γ. In the relation

$$(1.2) \qquad c + \lambda\mu = R_{aiia} = e_a(\Lambda^{\ a}_{ii}) + e_i(\Lambda^{\ i}_{aa}) - \sum_d \Lambda^{\ d}_{ii}\Lambda^{\ d}_{aa} - \sum_j \Lambda^{\ j}_{ii}\Lambda^{\ j}_{aa}$$

$$- (\Lambda^{\ a}_{ii})^2 - (\Lambda^{\ i}_{aa})^2 - \sum_r \{\Lambda^{\ r}_{ii}\Lambda^{\ r}_{aa} + 2\Lambda^{\ r}_{ai}\Lambda^{\ a}_{ir}\},$$

where $1 \leq a,d \leq m_1$, put

$$A = c + \lambda\mu - e_i(\Lambda^{\ i}_{aa}) + (\Lambda^{\ i}_{aa})^2 + \sum_j \Lambda^{\ j}_{ii}\Lambda^{\ j}_{aa} + \sum_r \Lambda^{\ r}_{ii}\Lambda^{\ r}_{aa}.$$

Then A is independent of choice of a; $1 \leq a \leq m_1$. On the other hand, we get from (1.2),

$$(1.3) \qquad A = e_a(\Lambda^{\ a}_{ii}) - \sum_d \Lambda^{\ d}_{ii}\Lambda^{\ d}_{aa} - (\Lambda^{\ a}_{ii})^2 - 2\sum_r \Lambda^{\ r}_{ai}\Lambda^{\ a}_{ir}.$$

If we extend e_b differentiably in a neighbourhood of γ so that e_b is tangent to every hypersphere of L^λ centered at p, we have easily

$$e_b(\Lambda_{ii}^{\ b}) = 0.$$

Therefore if we put $a = 1$ and $a = b$ in (1.3), it follows

$$e_1(\Lambda_{ii}^{\ 1}) - (\Lambda_{ii}^{\ 1})^2 - 2\sum_r \Lambda_{1i}^{\ r}\Lambda_{ir}^{\ 1} = -\Lambda_{ii}^{\ 1}\Lambda_{bb}^{\ 1} - 2\sum_r \Lambda_{bi}^{\ r}\Lambda_{ir}^{\ b},$$

or equivalently

(1.4) $$e_1(\Lambda_{ii}^{\ 1}) = (\Lambda_{ii}^{\ 1})^2 - \Lambda_{ii}^{\ 1}\Lambda_{bb}^{\ 1} + 2\sum_r (\Lambda_{1i}^{\ r}\Lambda_{ir}^{\ 1} - \Lambda_{bi}^{\ r}\Lambda_{ir}^{\ b})$$

along γ. Similarly we get from $c + \lambda\nu = R_{arra}$,

(1.5) $$e_1(\Lambda_{rr}^{\ 1}) = (\Lambda_{rr}^{\ 1})^2 - \Lambda_{rr}^{\ 1}\Lambda_{bb}^{\ 1} + 2\sum_i (\Lambda_{1i}^{\ r}\Lambda_{r1}^{\ i} - \Lambda_{bi}^{\ r}\Lambda_{rb}^{\ i})$$

along γ. Put $B = \sum_r (\Lambda_{1i}^{\ r}\Lambda_{ir}^{\ 1} - \Lambda_{bi}^{\ r}\Lambda_{ir}^{\ b})$ and $C = \sum_i (\Lambda_{1i}^{\ r}\Lambda_{r1}^{\ i} - \Lambda_{bi}^{\ r}\Lambda_{rb}^{\ i})$ for simplicity. Recall that first and second derivatives of $m_1\lambda + m_2\mu + m_3\nu = nh$ in the direction e_1 are

(1.6) $$m_2\Lambda_{ii}^{\ 1}(\mu - \lambda) + m_3\Lambda_{rr}^{\ 1}(\nu - \lambda) = 0$$

and

(1.7) $$m_2\{e_1(\Lambda_{ii}^{\ 1}) + (\Lambda_{ii}^{\ 1})^2\}(\mu - \lambda) + m_3\{e_1(\Lambda_{rr}^{\ 1}) + (\Lambda_{rr}^{\ 1})^2\}(\nu - \lambda) = 0.$$

Substitute (1.4) and (1.5) into (1.7), and use (1.6) to get

(1.8) $$m_2\{(\Lambda_{ii}^{\ 1})^2 + B\}(\mu - \lambda) + m_3\{(\Lambda_{rr}^{\ 1})^2 + C\}(\nu - \lambda) = 0.$$

Here, we know from (1.4) and (1.5) that B (C, resp.) is independent of i (r, resp.). So by the same argument as in [2], we know

$$B = \frac{m_3(\lambda - \mu)}{m_2(\nu - \lambda)}C.$$

Thus we obtain from (1.8), using (1.6),

$$n(\lambda - h)(\lambda - \mu)^2(\Lambda_{ii}^{\ 1})^2 - m_3(2\lambda - \mu - \nu)(\nu - \lambda)(\nu - \mu)B = 0.$$

Since $2\lambda \neq \mu + \nu$ everywhere on γ, we can substitute

$$B = \frac{n(\lambda - h)(\lambda - \mu)^2}{m_3(2\lambda - \mu - \nu)(\nu - \lambda)(\nu - \mu)}(\Lambda_{ii}^{\ 1})^2$$

into (1.4) to get

(1.9) $$e_1(\Lambda_{ii}^{\ 1}) = \left\{1 + \frac{2n(\lambda - h)(\lambda - \mu)^2}{m_3(2\lambda - \mu - \nu)(\nu - \lambda)(\nu - \mu)}\right\}(\Lambda_{ii}^{\ 1})^2 - \Lambda_{bb}^{\ 1}\Lambda_{ii}^{\ 1}$$

along γ. From $\Lambda_{ii}^{\ 1}(p) = 0$, Lemma 1.1 follows.

Lemma 1.2. Under the same situation as Lemma 1.1, let L^λ be a compact $\mathcal{D}(\lambda)$-leaf. If $2\lambda \neq \mu+\nu$ everywhere on L^λ and if $\Lambda_{ai}^{\ r} \neq 0$ for some a, i and r at a point of L^λ, then m_2, $m_3 \geq m_1$.

Proof. Let p, γ, e_1 and e_b be as in Lemma 1.1. Let B_{ai} be the $\mathcal{D}(\nu)$-component of $\nabla_{e_a}e_i$. Immediately from $B \equiv 0$, we get

(1.10) $$\|B_{1i}\| = \|B_{bi}\|.$$

From Lemma 1.1, (3.7) and (3.8) in [2], we obtain

$$\langle B_{1i}, B_{bi}\rangle = 0.$$

For e_a such that $e_a \perp e$ and $e_a \perp e_b$, it also follows from (3.7) and (3.8) in [2],

$$<B_{bi}, B_{ai}> = 0.$$

Thus $m_3 \geq m_1$ if $B_{ai} \neq 0$. $m_2 \geq m_1$ is similarly shown.

2. Proof of Theorem I

Let $\mu < \lambda < \nu$. Then $2\mu \neq \lambda + \nu$ and $2\nu \neq \lambda + \mu$ hold all over M. In particular, $\nu \neq 0$ since h is non-negative. So by Lemma 1.1, we get

$$(2.1) \qquad \Lambda_{aa}^{\ r} = \Lambda_{ii}^{\ r} = 0$$

everywhere on M. When $c > 0$, we have also

$$(2.2) \qquad \Lambda_{aa}^{\ i} = \Lambda_{rr}^{\ i} = 0$$

everywhere on M. When $c = 0$, if μ is identically zero on M, the theorem follows from Proposition 2.1 in [2]. If μ is not identically zero on M, then since $\{p \in M \,|\, \mu(p) \neq 0\}$ is open and dense in M by analiticity, we get (2.2) again from Lemma 1.1.

Claim. $m_2 = m_3$.

We have nothing to prove if $m_2 = m_3 = 1$. When m_2 or $m_3 \geq 2$, Claim follows from Lemma 1.2 if we show that M does not satisfy [I.C.]. Suppose [I.C.] were satisfied. Then if we choose e_a as a unit tangent vector field along a geodesic γ of L^λ, (1.2) could be written as

$$c + \lambda\mu = e_a(\Lambda_{ii}^{\ a}) - (\Lambda_{ii}^{\ a})^2.$$

Similarly, we should have

$$c + \lambda\nu = e_a(\Lambda_{rr}^{\ a}) - (\Lambda_{rr}^{\ a})^2.$$

Since $\mu < \lambda < \nu$ and λ is a non-zero constant, μ or ν has the same sign as λ. Then, $e_a(\Lambda_{ii}^{\ a})$ or $e_a(\Lambda_{rr}^{\ a})$ should be positive along γ, a contradiction.

Now, note that

$$n(\lambda - h) = m_2(2\lambda - \mu - \nu).$$

Thus (3.8) in [2] becomes

$$(2.3) \qquad (2\lambda - \mu - \nu)\left\{(\mu-\lambda)\Lambda_{ii}^{\ a}\Lambda_{ii}^{\ b} + (\nu-\mu)\sum_i \Lambda_{ai}^{\ r}\Lambda_{rb}^{\ i}\right\} = 0.$$

When $2\lambda \equiv \mu + \nu$ (and automatically $\lambda \equiv h$), the proof of Claim (i) in p. 350 of [2] shows that M is isoparametric. If $2\lambda \not\equiv \mu+\nu$ on M, then

$$(\mu - \lambda)\Lambda_{ii}^{\ a}\Lambda_{ii}^{\ b} + (\nu - \mu)\sum_i \Lambda_{ai}^{\ r}\Lambda_{rb}^{\ i} = 0$$

holds on any open set of M where $2\lambda \neq \mu + \nu$, therefore all over M, for arbitrally chosen e_a and e_b. Now the same argument as in the proof of Lemma 1.1 is applicable to $\Lambda_{ii}^{\ a}$ by canceling $\frac{\lambda-h}{2\lambda-\mu-\nu}$ in (1.1) and (1.9), and we get $\Lambda_{ii}^{\ a} = 0$ on M for any i and a.

3. Proof of Theorem II

Note that the essential conditions of Lemma 1.2 are that L^λ is a compact m_1-sphere of $\bar{M}(c)$ and that $2\lambda \neq \mu+\nu$ everywhere on L^λ. Under the situation of Theorem II, the first condition is satisfied for any leaf of $\mathcal{D}(\lambda)$ and $\mathcal{D}(\mu)$ since $c = 1$, where we assume $m_1, m_2 \geq 2$ and $m_3 = 1$. Moreover, at least one of λ and μ is equal to $\min\{\lambda, \mu, \nu\}$ or $\max\{\lambda, \mu, \nu\}$ so the second condition is satisfied for L^λ or L^μ. Anyway, we get from Lemma 1.2 that $m_3 \geq 2$ if [I.C.] fails.

Next, let $m_1 \geq 2$ and $m_2 = m_3 = 1$. When $\lambda = \min\{\lambda, \mu, \nu\}$ or $\max\{\lambda, \mu, \nu\}$, we can show similarly that m_2 and $m_3 \geq 2$ if

[I.C.] fails. Let $\mu < \lambda < \nu$. Then on L^{λ}, the argument in the end of §2 is valid because $m_2 = m_3$ is essential to elliminate the term $(2\lambda - \mu - \nu)$. Thus we get $m_2, m_3 \geq 2$ if [I.C.] fails.

4. Another proof of Theorem I in compact case

In this section, we prove Theorem III. This gives another proof of Theorem I in compact embedded case. We refer the readers to [3] for the details of Dupin hypersurfaces.

Let M satisfy conditions of Theorem III. Let $\lambda < \mu < \nu$. Then there is a point p of M such that

$$\nabla \log \frac{\mu-\lambda}{\nu-\mu} = 0$$

at p. This is equivalent to

$$\Lambda^{a}_{ii}(p) = \Lambda^{a}_{rr}(p),\ \Lambda^{i}_{aa}(p) = \Lambda^{i}_{rr}(p),\ \Lambda^{r}_{aa}(p) = \Lambda^{r}_{ii}(p)$$

(see p. 441 in [3]). Combining with

$$m_2\Lambda^{a}_{ii}(\mu - \lambda) + m_3\Lambda^{a}_{rr}(\nu - \lambda) = 0, \tag{1.6}$$

we get at p,

$$n(h - \lambda)\Lambda^{a}_{ii}(p) = 0.$$

From $\lambda < h$, we obtain

$$\Lambda^{a}_{ii}(p) = 0 = \Lambda^{a}_{rr}(p).$$

Similarly from $h < \nu$, we get

$$\Lambda^{r}_{aa}(p) = 0 = \Lambda^{r}_{ii}(p).$$

Now, consider the $\mathcal{D}(\mu)$-leaf L^{μ}_{p} at p. Since μ and $\sum_a (\Lambda^{a}_{ii})^2 + \sum_r (\Lambda^{r}_{ii})^2 + \mu^2 + 1$ are constant on L^{μ}_{p}, we get

$$\Lambda^{a}_{ii} = \Lambda^{r}_{ii} = 0 \quad \text{on} \quad L^{\mu}_{p}.$$

Moreover

$$\Lambda^{a}_{rr} = \Lambda^{r}_{aa} = 0$$

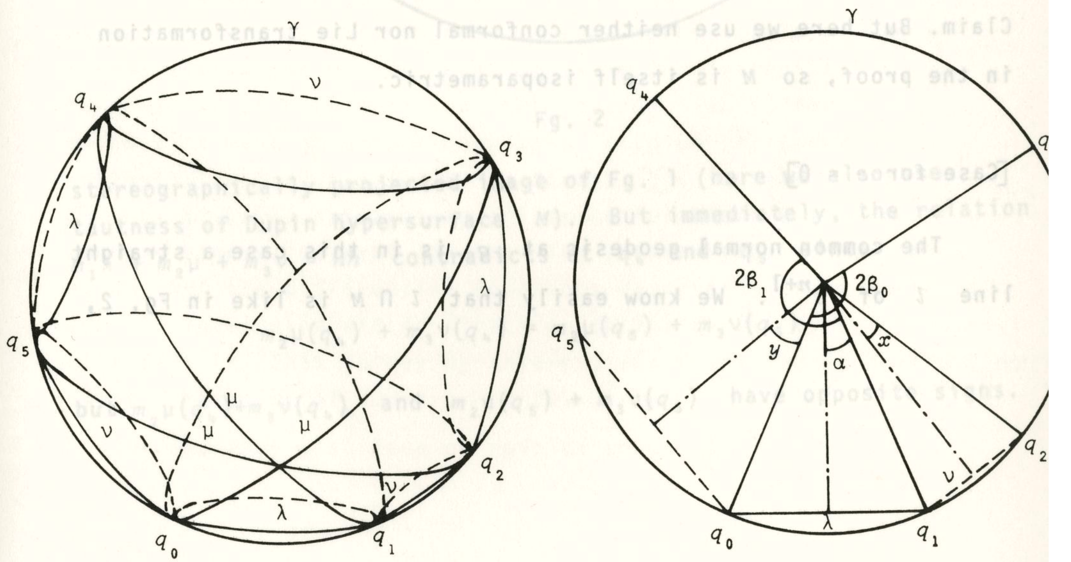
holds on L^{μ}_{p} by (1.5) and by $m_2\Lambda^{r}_{aa}(\lambda-\nu) + m_3\Lambda^{r}_{ii}(\mu-\nu) = 0$. Since L^{μ}_{p} is compact, there exists a point q of L^{μ}_{p} such that

$$\Lambda^{i}_{aa}(q) = 0 \qquad m_1 < i \leq m_1 + m_2.$$

Finally at q, the normal geodesic γ is "common" in the sense of [3].

[Case for $c > 0$].

By the argument in pp. 441-443 of [3] (here, we use tautness of Dupin hypersurfaces), $\gamma \cap M$ appears as in Fg. 1. Let $\gamma \cap M = \{q_0, q_1, \ldots, q_5\}$ as Fg. 1.



Fg. 1

Claim. $\lambda(q_0) = \lambda(q_2) = \lambda(q_4)$, $\nu(q_0) = \nu(q_1) = \nu(q_3)$.

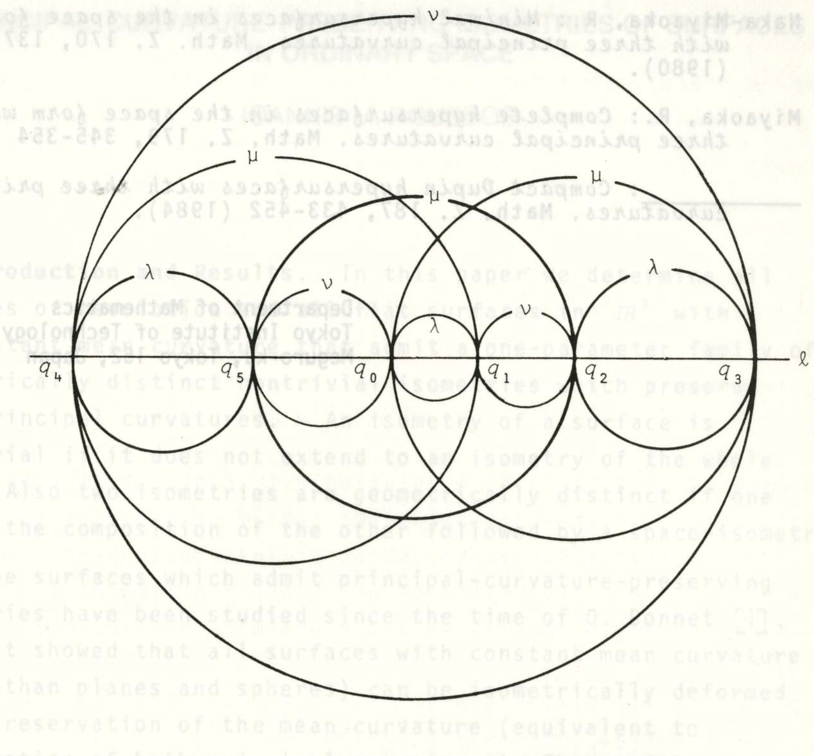
Let $\alpha = \cot^{-1}\lambda(q_0)$, $x = \cot^{-1}(-\nu(q_1))$, $y = \cot^{-1}(-\nu(q_0))$, $\beta_0 = \cot^{-1}\mu(q_0)$, $\beta_1 = \cot^{-1}\mu(q_1)$ where $0 < \alpha, x, y, \beta_0, \beta_1 < \pi$. Suppose that $x < y$ (i.e. $\nu(q_1) < \nu(q_0)$). Then $\mu(q_1) > \mu(q_0)$ (i.e. $\beta_1 < \beta_0$... (*)). On the other hand, we have $\lambda(q_3) = \cot(\beta_0-\alpha-x)$ and $\lambda(q_4) = \cot(\beta_1-\alpha-y)$. Since $\beta_0-\alpha-x > \beta_1-\alpha-y$, we get $\mu(q_3) > \mu(q_4)$, i.e. $\beta_0 < \beta_1$, contradicts (*). Thus we have $x = y$ and $\beta_0 = \beta_1$ at the same time. This argument holds if we replace q_0, q_1, by q_i, q_{i+1} where $i = 1,2,...,5$ (mod 6).

By virtue of Claim, using the argument in p. 451 of [3], we know that M is isoparametric.

[Remark] In [3], we need some conformal transformation to get a common normal geodesic γ, and some Lie transformation to get Claim. But here we use neither conformal nor Lie transformation in the proof, so M is itself isoparametric.

[Case for $c = 0$]

The common normal geodesic at q is in this case a straight line l of E^{n+1}. We know easily that $l \cap M$ is like in Fg. 2,



Fg. 2

stereographically projected image of Fg. 1 (here we also use tautness of Dupin hypersurface M). But immediately, the relation $m_1\lambda + m_2\mu + m_3\nu = nh$ contradicts at q_4 and q_5 since

$$m_2\mu(q_4) + m_3\nu(q_4) = m_2\mu(q_5) + m_3\nu(q_5)$$

but $m_2\mu(q_4)+m_3\nu(q_4)$ and $m_2\mu(q_5) + m_3\nu(q_5)$ have opposite signs.

References

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