ON FINITE GROUPS ADMITTING CERTAIN SHARP CHARACTERS

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1. INTRODUCTION

Let G be a finite group and χ a faithful character of G of degree n. A pair (G,χ) is called *of type* L when a finite subset $L = \{\chi(x) \mid x \in G^{\#}\}$ of C is given. Recently P.J. Cameron and M. Kiyota showed that $\frac{1}{|G|}\prod_{\ell \in L}(n-\ell)$ is an integer for any (G,χ) of type L. A pair (G,χ) of type L is called *L-sharp* if $|G| = \prod_{\ell \in L}(n-\ell)$, and then χ is called a *sharp character* of G. The main problem is to determine completely L-sharp pairs when L is given. Many people have studied it for some L's when χ is a permutation character. For example, it was solved for $L = \{0,1,2,\ldots,r-1\}$ $(r \in N)$ by C. Jordan and H. Zassenhaus (cf. [9]), it was also studied for $L = \{\ell\}$ $(\ell \in N)$ by N. Iwahori, T. Kondo, and H. Yamaki, etc ([5], [6], [7], [8]), and for some other L's by T. Ito and M. Kiyota ([4]). In case when χ is a character in general, it has been studied for several L's by P.J. Cameron and M. Kiyota ([1]). In particular, especially $\{-1,1\}$-sharp pairs were completely determined by them together with T. Kataoka in [2]. They also posed some problems in [1], one of which was to find other L-sharp pairs; for example, $L = \{-2,1\}$ or $\{2,-1\}$. In this note we shall treat the cases $L = \{-2,1\}$ and $\{2,-1\}$. These sharp pairs will be characterized by the degrees of the irreducible constituents of its sharp characters under some assumptions.

Let (G,χ) be either a $\{-2,1\}$ or $\{2,-1\}$-sharp pair, and χ be *normalized* (i.e. $\langle\chi,1_G\rangle_G = 0$). It follows from Proposition

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1.3 in [1] that χ is decomposed into three irreducible characters χ_1, χ_2, χ_3 of G. Let c.d.$(\chi) = \{\chi_1(1), \chi_2(1), \chi_3(1)\}$. We have the following:

Theorem A. Let (G,χ) be $\{-2,1\}$-sharp and χ normalized.

(1) If c.d.$(\chi) = \{m,4,4\}$, then $G \simeq Z_3 \times A_5$, $m = 5$.

(2) If c.d.$(\chi) = \{m,m,8\}$, then $G \simeq Z_3 \times PSL(2,7)$, $m = 7$.

Theorem B. Let (G,χ) be $\{2,-1\}$-sharp and χ be normalized.

(1) If c.d.$(\chi) = \{m,5,5\}$, then $G \simeq Z_3 \times A_5$, $m = 4$.

(2) If c.d.$(\chi) = \{m,8,8\}$, then $G \simeq Z_3 \times PSL(2,7)$, $m = 7$.

Our notation is largely standard (c.f. [3]).

We shall use frequently the following result to prove our theorems:

Ito's Theorem. (c.f. [3], Theorem (6.15)). Let A be a normal abelian subgroup of G. Then $x(1)$ divides $(G:A)$ for all $\chi \in Irr(G)$.

2. Proof of Theorem A

Proof of Part (1). Since $\chi_i(1) \mid |G| = (m+7)(m+10)$ for $i=1,2,3$, we have $|G| = 2^3 \cdot 11$ $(m=1)$, $2^2 \cdot 3^3$ $(m=2)$, $2^2 \cdot 3^2 \cdot 5$ $(m=5)$, $2^2 \cdot 5 \cdot 17$ $(m=10)$, $2^3 \cdot 3^2 \cdot 7$ $(m=17)$, or $2^4 \cdot 5 \cdot 7 \cdot 11$ $(m=70)$.

Case 1. $|G| = 2^2 \cdot 5 \cdot 17$, c.d.$(\chi) = \{10,4,4\}$.

A Sylow 5-subgroup of G is normal; a contradiction follows by Ito'Theorem.

Case 2. $|G| = 2^3 \cdot 11$, c.d.$(\chi) = \{1,4,4\}$.

Let $P \in Syl_2(G)$, $Q \in Syl_{11}(G)$. Then, $Q \lhd G$. Since $G \not\simeq P \times Q$, $C_G(Q) = S \times Q$ for some subgroup S of G of order 2^2, hence $S \lhd G$; a contradiction by Ito.

Case 3. $|G| = 2^2 \cdot 3^3$, c.d.$(\chi) = \{2,4,4\}$.

Let $P \in Syl_2(G)$, $Q \in Syl_3(G)$, and define the subgroups of G $Z(\chi_i) = \{x \in G \mid |\chi_i(x)| = \chi_i(1)$ $(i=1,2,3)$. As P is abelian, $P \leq C_G(s)$ for any $s \in P$. We denote by $cl_G(x)$ the conjugacy class of x in G. It follows from Theorem of Burnside (c.f. Theorem (3.8) in [3]) that $s \in Z(x)$ or $\chi_i(s) = 0$ for each $i = 1,2,3$; this is so since $(|cl_G(s)|, \chi_i(1)) = 1$ for any $i=1,2,3$. We shall show that there exists an $s \in P$ such that $\chi_i(s) = 0$ for all $i=1,2,3$. In fact, on supposing the contrary (i.e., for any $s \in P$, there is some $i = 1,2,3$ such that $s \in Z(\chi_i)$), we have $P \leq Z(\chi_1)$ since $\chi_2(1) = \chi_3(1) = 4$ and $\chi_i(1) \mid (G:Z(\chi_i))$ $(i=1,2,3)$ (c.f. Theorem (3.12) in [3]), and so $\chi_1(1) \mid (G:Z(\chi_1(1)) \mid (G:P)$; a contradiction. Thus $\chi(s) = 0$ for some $s \in P$, which conflicts with $L = \{-2,1\}$.

Case 4. $|G| = 2^2 3^2 \cdot 5$, c.d.$(\chi) = \{4,4,5\}$.

It is sufficient to prove the following lemma.

Lemma C. Let (G,χ) be either $\{-2,1\}$- or $\{2,-1\}$-sharp, and $|G| = 2^2 \cdot 3^2 \cdot 5$. If c.d. $(\chi) \supset \{4,5\}$, then $G \simeq Z_3 \times A_5$.

Proof of Lemma C. Let $P \in Syl_2(G)$, $Q \in Syl_3(G)$, $R \in Syl_5(G)$ and let N be a minimal normal subgroup of G. Since neither P nor Q are normal, $(G:N_G(Q)) = 2 \cdot 5$. By Ito's, $N \simeq Z_3$ or A_5. In case $N \simeq Z_3$, $|C_G(N)| = |G|$ or $2 \cdot 3^2 \cdot 5$. If $C_G(N) \simeq Z_3$, then $Q \lhd C_G(N)$, since $C_G(N)/N$ has a minimal normal subgroup isomorphic with Z_3. Hence $Q \lhd G$; a contradiction. If $C_G(N) = G$, then G/N is non-solvable; in fact, if not, G/N has a minimal normal subgroup $\cong Z_3$, by Ito. Thus $Q \lhd G$ and a contradiction is reached. As $\mathrm{Mult}(A_5) = 2$, $G \simeq Z_3 \times A_5$. In case $N \simeq A_5$, $C_G(N) \simeq Z_3$, so $G \simeq Z_3 \times A_5$, which satisfies the hypothesis by looking at its character table. This completes our proof of Lemma C.

Case 5. $|G| = 2 \cdot 5 \cdot 7 \cdot 11$, c.d.$(X) = \{70,4,4\}$.

It is sufficient to prove the following lemma.

Lemma D. Let (G,X) be $\{-2,1\}$-sharp. If $|G| = 2^4 \cdot 5 \cdot 7 \cdot 11$, then c.d.$(X) \not\supset \{70,4\}$.

Proof of Lemma D. Suppose c.d.$(X) \supset \{70,4\}$. Let $P \in \mathrm{Syl}_2(G)$, $Q \in \mathrm{Syl}_5(G)$, $R \in \mathrm{Syl}_7(G)$, $S \in \mathrm{Syl}_{11}(G)$ and let N be a minimal normal subgroup of G. We have the following two cases: $(G{:}N_G(S)) = 1$, $2^3 \cdot 7$. Note that $[Q,R] = 1$ in both cases.

Subcase. (a): $(G{:}N_G(S)) = 2 \cdot 7$. By Ito, $N \simeq Z_2$ or $Z_2 \times Z_2$, and so $|O_2(G)| \geq 4$. As $N_G(S) \trianglerighteq O_2(G)$, a contradiction, is reached.

Subcase. (b): $(G{:}N_G(S)) = 1$. Put $C_G(S) = S \times K$, $K \lhd G$; $|K| = 2^4 \cdot 5 \cdot 7$, $2^3 \cdot 5 \cdot 7$, $2^4 \cdot 7$ or $2^3 \cdot 7$. If $|K| = 2^3 \cdot 5 \cdot 7$, $Q \lhd K$ follows, since $[Q,R] = 1$; thus, $Q \lhd G$ which contradict Ito's Theorem.

If $|K| = 2^3 \cdot 7$ or $2^4 \cdot 7$, then some normal subgroup of G is isomorphic to Z_2 or $Z_2 \times Z_2$, and so $|O_2(G)| = 4$ by Ito. Thus, $O_2(G) \lhd N_G(R)$; a contradiction, since $(G{:}N_G(R)) = 2^3$.

If $|K| = 2^4 \cdot 5 \cdot 7$, then K has a minimal normal subgroup $M \simeq Z_2$,

$Z_2 \times Z_2$ or $Z_2 \times Z_2 \times Z_2$. As $Q \times M \lhd K$, it follows that $Q \lhd G$; a contradiction by Ito. This completes our proof of Lemma D.

Case 6. $|G| = 2^3 \cdot 3^2 \cdot 7$, c.d.$(X) = \{14,4,4\}$. It is sufficient to show the following lemma.

Lemma E. Let (G,X) be either $\{-2,1\}$- or $\{2,-1\}$-sharp, and $|G| = 2^3 \cdot 3^2 \cdot 7$. Then

(1) c.d.$(X) \not\supset \{14,4\}$.

(2) If c.d.$(X) \supset \{7,8\}$, $G \simeq Z_3 \times \mathrm{PSL}(2,7)$.

Proof of Lemma E. Let $P \in \mathrm{Syl}_2(G)$, $Q \in \mathrm{Syl}_3(G)$, $R \in \mathrm{Syl}_7(G)$, and let N be a minimal normal subgroup of G. Suppose either c.d.$(X) \supset \{14,4\}$, or $\{7,8\}$. By Ito, $N \simeq Z_2$, Z_3, $Z_3 \times Z_3$, or $\mathrm{PSL}(2,7)$. Note $[N,R] = 1$ for each of the N's above.

Subcase (a): $N \simeq Z_3 \times Z_3$. Then, $N = Q$, $C_Q(Q) = Q \times K \lhd G$ for some subgroup K of G containing R. Now, $|K| \lneq 2^3 \cdot 7$; in fact if not, $K = P \cdot R \lhd G$, and so, $\mathrm{cd}(G) = \mathrm{cd}(K)$ wich contain neither $\{14,4\}$ nor $\{7,8\}$. Thus, $R \lhd K$, $R \lhd G$; a contradiction is reached by Ito.

Subcase (b): $N \simeq Z_2$, $|C_G(N)| = |G|$ or $2^2 \cdot 3^2 \cdot 7$. If $|C_G(N)| = 2^2 \cdot 3^2 \cdot 7$, then $R \times N \lhd C_G(N) \lhd G$, and so, $R \lhd G$; a contradiction by Ito. If $C_G(N) = G$, then G/N has a minimal normal subgroup $\bar{M} \simeq Z_2$, $Z_2 \times Z_2$, Z_3 or $Z_3 \times Z_3$ (where $\bar{M} = M \mod N$, $M \leq G$). Now, $M \times N \lhd G$, and $\bar{R} \lhd G/M \times N$. As $[R,M] = [R,N] = 1$, $R \lhd G$; a contradiction by Ito.

Subcase (c): $N \simeq Z_3$. $|C_G(N)| = |G|$ or $2^2 \cdot 3^2 \cdot 7$. If $|C_G(N)| = 2^2 \cdot 3^2 \cdot 7$, then $R \times N \lhd C_G(N) \lhd G$, $R \lhd G$; a contradiction by Ito. In case $C_G(N) = G$, G/N is simple; in fact, if not, G/N

has a minimal normal subgroup $\bar{M} \simeq Z_2$, $Z_2\times Z_2$, $Z_2\times Z_2\times Z_2$, Q, or R (where $\bar{M} = M \bmod N$, $M \le G$). By Ito, G contains a normal subroup M, $M \simeq Z_2\times N$, or Q. If $M \simeq Q$, then $C_G(Q) \lhd G$ as in Subcase (a). Thus, $R \lhd G$; a contradiction by Ito. If $M \simeq Z_2\times N$, then $\bar{R} \lhd G/N$, so $M\times R \lhd G$, $R \lhd G$; a contradiction by Ito. Thus, $G/N \simeq \mathrm{PSL}(2,7)$. As Mult(PSL(2,7)) = 2, $G \simeq Z_3\times \mathrm{PSL}(2,7)$. So, c.d.$(\chi) \not\supset \{14,4\}$, since $14 \notin cd(G) = cd(\mathrm{PSL}(2,7))$.

Subcase (d): $N \simeq \mathrm{PSL}(2,7)$. Since $\mathrm{Out}(\mathrm{PSL}(2,7)) \simeq Z_2$, $C_G(N) \neq 1$; if not, $N \simeq \mathrm{Inn}(N) \lhd G \lhd \mathrm{Aut}(N)$, and so, $3 = (G:\mathrm{Inn}(N)) \mid |\mathrm{Out}(\mathrm{PSL}(2,7))|$; a contradiction. Hence, $C_G(N) \cap N = 1$, $G \simeq Z_3\times \mathrm{PSL}(2,7)$, $14 \notin cd(G)$. Actually, $G \simeq Z\times \mathrm{PSL}(2,7)$ satisfies the hypotesis and the corresponding character is unique with respect to $L = \{-2,1\}$, or $\{2,-1\}$, by looking at its character table. This completes our proof of Lemma E.

3. Proof of Theorem B

Proof of Part (1). Since $\chi_i(1) \mid |G| = (m+8)(m+11)$ for all $i=1,2,3$, we have $|G| = 2.5.13$ $(m=2)$, $2^2.3^2.5$ $(m=4)$, $2\cdot3^2\cdot5\cdot11$ $(m = 22)$ or $2^2\cdot5\cdot11\cdot13$ $(m = 44)$. If $|G| = 2^2\cdot5\cdot11\cdot13$, a Sylow 11-subgroup of G is normal; a contradiction by Ito. If $|G| = 2\cdot5\cdot13$, then a Sylow 5-subgroup of G is normal since $Z_{13}\times Z_5 \lhd G$; a contradiction by Ito. If $|G| = 2\cdot3^2\cdot5\cdot11$, a minimal normal subgroup N of G is either Z_3 or $Z_3\times Z_3$. Let $S \in \mathrm{Syl}_{11}(G)$; $S \lhd C_G(N) \lhd G$, so $S \lhd G$, a contradiction by Ito. If $|G| = 2^2\cdot3^2\cdot5$, then we have $G \simeq Z_3 \times A_5$ by Lemma C. This completes our proof of Part (1).

Proof of Part (2): Since $\chi_i(1) \mid |G| = (m+14)(m+17)$ for all $i=1,2,3$, we have $|G| = 2^4\cdot19$ $(m = 2)$, $2^3\cdot3^2\cdot7$ $(m = 7)$,

$2^4\cdot3^2\cdot17$ $(m = 34)$, or $2^3\cdot7\cdot17\cdot19$ $(m = 119)$. If $|G| = 2^3\cdot7\cdot17\cdot19$, then a Sylow 17-subgroup of G is normal; a contradiction by Ito. In case $|G| = 2^4\cdot19$, let $P \in \mathrm{Syl}_2(G)$, $Q \in \mathrm{Syl}_{19}(G)$, then $C_G(Q) = Q\times S \lhd G$ for some subgroup S of G of order 2^3, since $G \not\simeq P\times Q$. There exists $Z_4 \lhd S \lhd G$, and so $Z_4 \lhd G$; a contradiction by Ito. In case $|G| = 2^4\cdot3^2\cdot17$, as $cd(G) \supset \{34,8\}$, $G \not\simeq \mathrm{PSL}(2,17)$; so, G is not simple. Let N be a minimal normal subgroup of G, and $R \in \mathrm{Syl}_{17}(G)$. Since $(G:N_G(R)) = 2\cdot3^2$, $N \simeq Z_2$. As $\bar{R} \lhd G/N$, so $R \lhd G$, a contradiction. If $|G| = 2^4 3^2\cdot7$, then $G \simeq Z_3\times\mathrm{PSL}(2,7)$ by Lemma E. This completes our proof of Part (2).

Remark. Let (G,χ) be $\{2,-1\}$-sharp. In a manner similar to the above we have the following:

(1) If c.d.$(\chi) = \{m,m,4\}$, then $G \simeq Z_3 \times A_5$, $m = 5$.

(2) If c.d.$(\chi) = \{m,m,7\}$, then $G \simeq Z_3\times \mathrm{PSL}(2,7)$, $m = 7$.

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References

[1] P.J. Cameron and M. Kiyota, *Sharp characters of finite groups*, Preprint (1986).

[2] P.J. Cameron, T. Kataoka and M. Kiyota, *Sharp characters of finite groups of type* -1,1 . In preparation.

[3] I.M. Issacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[4] T. Ito and M. Kiyota, *Sharp permutation groups*, J. Math. Japan, 33 (1981), 435-444.

[5] N. Iwahori, *On a property of finite groups*, J. Fac. Sci. Univ. Tokyo, 11 (1964), 47-64.

[6] N. Iwahori and T. Kondo, *On finite groups admitting a permutation representation* P *such that* $\mathrm{Tr}\ P(\sigma) = 3$ *for all* $\sigma \neq 1$, J. Fac. Sci. Univ. Tokyo, 11 (1964), 113-144.

[7] N. Iwahori and T. Kondo, *A criterion for the existence of a nontrivial partition of a finite group with applications to refrection groups*, Math. Soc. Japan, 17 (1965), 207-215.

[8] M. Kanazawa and H. Yamaki, *On finite solvable groups with* $t(G) = 4$ *or* 5, J. Fac. Sci. Univ. Tokyo, 13 (1966), 207-215.

[9] H. Wielandt, *Finite Permutation Groups*, Academic Press, New York, 1964.

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