

SPHERICAL SURFACES WITH CONSTANT MEAN CURVATURE IN HYPERBOLIC SPACE

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1. Introduction and basic definitions

Let H^3 be the three-dimensional hyperbolic space of constant sectional curvature $-K$, $K > 0$, and let G be the group of isometries of H^3 that leave a geodesic g pointwise fixed. We will call G the spherical group of H^3 ([dCD]), and g its rotation axis. A surface in H^3 invariant by G is called a spherical surface.

The hyperbolic space H^3 has a natural compactification \bar{H}^3 , obtained by adjoining to it the sphere at infinity $S^2(\infty)$. If M is a complete surface in H^3 , the asymptotic boundary $\partial_\infty(M)$ of M is the intersection of the closure \bar{M} of M in \bar{H}^3 with the sphere at infinity $S^2(\infty)$.

In this paper we will study spherical surfaces with constant mean curvature in hyperbolic space. These surfaces were studied by Mori [Mo], do Carmo-Dajczer [dCD], and Hsiang [H]; the description we will give here is more detailed, and we will give special emphasis on the asymptotic boundary behavior of such surfaces. This more complete description made it possible to obtain some new results about surfaces of constant mean curvature H in hyperbolic space (see [dCGT] and [Go]); also some new examples of such surfaces were obtained (see e.g. Theorem 3.7 that answers a question posed in [LR]). For $H > 1/2$ we obtain a family analogous to the Delaunay's family in R^3 [D].

A simple way to visualize the action of the spherical group in hyperbolic 3-space is obtained by using the ball model for H^3 , curve. Now we will prove (b). We may suppose that the y -coordinate

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2. Final remarks. The existence of a minimal surface in H^3 with boundary $\partial_\infty(M)$ is guaranteed by the existence of a minimal surface in H^3 with boundary $\partial_\infty(M)$.

1. If in (3.1) we consider the stereographic projection relative to the south pole of the unitary euclidean sphere, the identification of H^3 with R^3 is made through the correspondence $\sigma: H^3 \rightarrow R^3$ given by $\sigma(x, y, z) = (x, y, z/(1-z))$, where x, y, z are homogeneous coordinates on S^3 .

Then instead of (3.14) we have $\sigma^{-1}(x, y, z) = (x, y, z/(1-z))$. Besides the conditions that are $0 < z < 1$, the conditions $x^2 + y^2 + z^2 = 1$ follow from the straightforward calculation and the conclusions are similar.

5. If we set $\tilde{z} = z/(1-z)$, on the case $n = 4$, we recognize the beginning of the theory of minimal surfaces in H^3 . The identification of H^3 with R^3 is made through the correspondence $\sigma: H^3 \rightarrow R^3$ given by $\sigma(x, y, z) = (x, y, z/(1-z))$.

References

[G-2] C. Gomes and P. Simoes, Some remarks on minimal immersions in the hyperbolic space, Bol. Soc. Bras. Mat., vol. 18, no. 2 (1987), 55-65.

[H-0.1] D.A. Hoffman and R. Osserman, The Geometry of the Generalized Gauss map, AMS Memoirs 336 (1980).

[K] K. Kenmotsu, Weierstrass formula for surfaces of prescribed mean curvature, Math. Ann 285 (1980), 89-99.

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and taking for the rotation axis the geodesic $z = y = 0$ passing through the center of the ball. Each principal orbit is a circle orthogonal to the disk

$$B^2 = \{(x, y, z) \in B^3; z = 0\},$$

and intersects this disk in exactly two points symmetric with respect to the axis of rotation g . The orbit space can thus be represented by the half-disk

$$B_+ = \{(x, y) \in B^2; y > 0\}.$$

Since the orbits are orthogonal to B^2 , The orbital distance metric ($[BCH]$) is the hyperbolic metric.

In our study we will use methods from equivariant geometry as introduced by Wu-Yi Hsiang. Namely, given a surface M in H^3 , invariant by the spherical group, it intersects the orbit space B_+ in a curve f , called the *generating curve* of M . If M has constant mean curvature, we obtain an ordinary differential equation for the curve f (equation 2.4 in section 2). We will use both qualitative and quantitative methods in order to study this equation.

We will use Alexandrov's maximum principle ($[A]$, $[A1]$), as state for instance in $[LR]$, to be called here the *tangency principle*.

All results in this paper holds for spherical hypersurfaces in hyperbolic n -space; for simplicity we only consider the case $n = 3$.

This work is part of my Doctor thesis at IMPA. I am grateful to M.P. do Carmo, my advisor, for some valuable conversations and his constant interest while I was doing this work. Thanks are also due to Wu-Yi Hsiang for his encouragement and helpful conversations while I was studying equivariant geometry,

2. Differential Equation of the Generating Curve

First we will establish a general Lemma concerning isometric immersions invariant under some transformation groups of isometries of a space form. In spite of being simple, it is very useful in many situations.

Let G be a cohomogeneity-two group of isometries of the hyperbolic space H^3 , and H^3/G the orbit space, represented as a two-dimensional, totally geodesic submanifold of H^3 . If (x, y) is a coordinate system in H^3/G , we call the curves $x = \text{constant}$, y -coordinate curves. If a curve γ is symmetric with respect to some y -coordinate curve, we say that γ has an y -symmetry. In this case the surface M generated by γ under the action of the group G is symmetric with respect to the submanifold generated by the y -coordinate curve, and we also say that M has an y -symmetry.

Lemma 2.1

Under the conditions of the above paragraph, if there exists a coordinate system (x, y) in H^3/G such that the y -coordinate curves are generating curves of totally geodesic submanifolds of H^3 , then the following assertions hold

- If γ is the generating curve of some minimal submanifold N then γ is a graph hence N is embedded
- If the generating curve γ of a submanifold N with constant mean curvature is orthogonal to some y -coordinate curve, then γ has an y -symmetry with respect to it.

Proof: (a) is an immediate consequence of the tangency principle, because if f is not a graph it must be tangent to some y -coordinate curve. Now we will prove (b). We may suppose that the y -coordinate

curve is given by $x = 0$, and $f(0) = p$ is the point where f is orthogonal to it. Of course f is a graph in some neighbourhood of zero. Reflecting f on the coordinate curve we obtain two functions $y = f(x)$ and $y = g(x)$, where $g(x) = f(-x)$, defined in a neighborhood $[0, \epsilon)$ of the origin. In order to use the tangency principle for points in the boundary we must show that $f(x) \leq g(x)$, or $f(x) \geq g(x)$ in some neighbourhood of the origin. If this is not the case, there exists a sequence $x_n \rightarrow 0$ such that $f(x_n) = g(x_n) = 0$, and since has constant mean curvature, f and g are analytic, hence $f = g$. (q.e.d.)

We will consider in the orbit space geodesic coordinates, to be defined as follows: we parametrize the geodesic g (the rotation axis) by the arc length in such a way that $g(0)$ is the center of the ball B^2 ; for each point $p \in B_+^2$, let pq be the segment of geodesic whose length is equal to the distance from p to the geodesic g (such segment exists and is unique). The point p will have coordinates (x, y) , where $g(x) = q$ (i.e. x is the oriented arc length from the center to the point q), and y is the length of the segment. Note that this coordinate system satisfies the hypothesis of Lemma 2.1.

Now we will obtain the differential equation of the generating curve in the orbit space. For later purposes, we will work out the computations in the hyperbolic space of curvature $-K$, $K > 0$. These computations were given in [H], and we will present them here for the sake of completeness.

The orbital distance metric is given by

$$ds^2 = [(\cosh Ky)dx + dy]^2 / K^2.$$

The orbit of a point $p = (x, y)$ under the action of the spherical group is a circle of radius $\sinh(Ky)/K$, hence the volume function is given by

$$V(x, y) = c'(\sinh Ky), \quad 2.1$$

where $c' = c/K$, and c is the length of the unit circle in hyperbolic space.

We will denote the tangent vectors to the coordinate curves $x = cte$ and $y = cte$ by $\partial/\partial y$ and $\partial/\partial x$, respectively. The angle from the vector $\partial/\partial y$ to the velocity vector γ' of the generating curve γ will be denoted by α . We parametrize the curve γ by the hyperbolic arc length. The unit tangent vector to γ is given by

$$t = \sin \alpha e_x + \cos \alpha e_y,$$

where

$$e_x = (K/\cosh Ky)\partial/\partial x,$$

and

$$e_y = K\partial/\partial y,$$

are the unit vectors tangent to the coordinate curves.

The unit normal to the curve γ is given by

$$n = -K \sin \alpha \partial/\partial y + K(\cos \alpha / \cosh Ky)\partial/\partial x, \quad 2.2$$

and the following expression give the geodesic curvature of γ .

$$K_g = d\alpha/ds + K \sinh y dx/ds. \quad 2.3$$

Now using the equations 2.1, 2.2 and 2.3 above, and Proposition 7 from [BCH], we obtain the following differential equation

$$2H = d\alpha/ds + K \sinh y dx/ds + K \sin \alpha \cosh Ky / \sinh Ky \quad 2.4$$

for the generating curve γ of a spherical surface with constant mean curvature H .

Since $dx/ds = \sin \alpha / \cosh y$, and $dy/ds = \cos \alpha$, we obtain for $K = 1$, the one-parameter family of first order differential equations:

$$dx/ds = \sin \alpha / \cosh y;$$

$$dy/ds = \cos \alpha; \quad 2.5$$

$$d\alpha/ds = -\sin \alpha [\sinh y / \cosh y + \cosh y / \sin y] + 2H.$$

Note that the projection of the above field on the (α, y) plane is the one-parameter plane vector field

$$dy/ds = \cos \alpha \quad 2.6$$

$$d\alpha/ds = -\sin \alpha [\sinh y / \cosh y + \cosh y / \sinh y] + 2H.$$

Now we will study 2.6 in order to describe the generating curves of the spherical surfaces with constant mean curvature. Using the periodicity, we just have to study the above plane field in the region $U = \{[-\pi, \pi] \times (\alpha, y); y > 0\}$. Furthermore, since the sign of the mean curvature depends only on the orientation, we may suppose that $H \geq 0$. Multiplying the field in 2.5 by $\sinh y \cosh y$ (to eliminate the poles) we obtain

$$ds/ds = \cos \alpha \sin y \cosh y; \quad 2.7$$

$$d\alpha/ds = -\sin \alpha (\sinh^2 y + \cosh^2 y) + 2H \sinh y \cosh y.$$

It is easy to see that the phase portrait of the above field, for any value of the parameter H , is symmetric with respect to the vertical line $\alpha = \pi/2$.

For $H = 0$ the only singularities of the above field in the region U occurs at the points $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$ and these singularities are hyperbolic (see [PM]). The halflines $\alpha = -\pi$ and $\alpha = \pi$ are unstable manifolds of the singularities $(-\pi, 0)$ and $(\pi, 0)$, respectively, and the halflines $\alpha = 0$ is the stable manifold of the singularity at the origin. The phase portrait of the field in 2.6 in the region U is shown in the picture below.

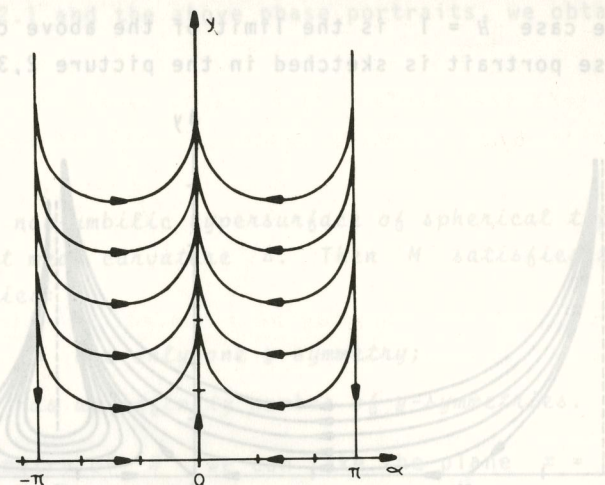


Figure 2.1

For $H \in (0, 1)$ it is easy to see that the singularities remain the same, and also are hyperbolic. In this case there exists an angle $\alpha_0 \in (0, \pi/2)$ such that the unstable manifold of the singularity $(0, 0)$ is asymptotic to the line $\alpha = \alpha_0$, and the stable manifold of the singularity $(\pi, 0)$ is asymptotic to the line $\pi/2 + \alpha_0$. The phase portrait is shown in the picture 2.2.

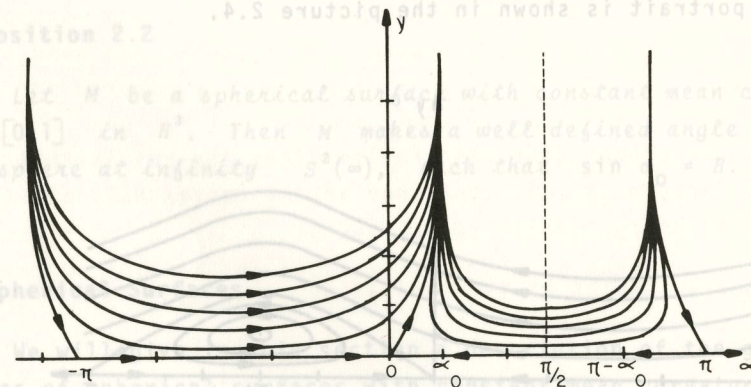


Figure 2.2

The case $H = 1$ is the limit of the above case when $\alpha_0 = \pi/2$. The phase portrait is sketched in the picture 2.3,

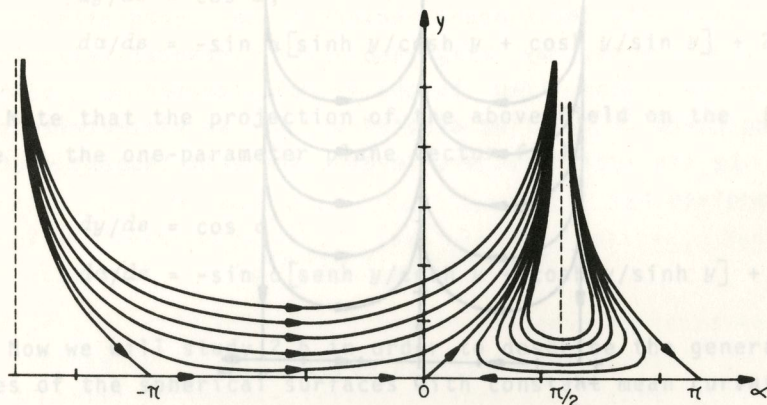


Figure 2.3

When $H > 1$, besides the singularities $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$, all hyperbolic, the field in 2.7 also has a singularity at the point with coordinates

$$\left(\frac{\pi}{2}, H + \sqrt{H^2 - 1}\right).$$

From the symmetry of the phase portrait with respect to the line $\alpha = \pi/2$ it is easy to see that this singularity is a center. The phase portrait is shown in the picture 2.4,

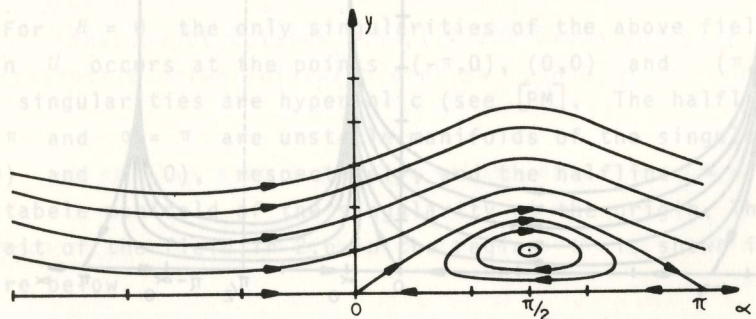


Figure 2.4

Using Lemma 2.1 and the above phase portraits, we obtain the following

Symmetry Lemma

Let M be a non-umbilic hypersurface of spherical type in H^3 with constant mean curvature H . Then M satisfies the following properties:

- If $H \in [0, 1]$ M has only one y -symmetry;
- If $H > 1$ M has an infinity number of y -symmetries.

Using an isometry of H we can take the plane $x = 0$ as an y -symmetry plane. We will always assume this to be the case in what follows.

The spherical surface corresponding to the separatrices in each of the above phase portraits has a fixed point, hence it is an umbilic surface (see Corollary to Theorem 1 in [H]). Also, since for $H \in [0, 1]$ the orbits in the above phase portraits are always asymptotic to some line $\alpha = \alpha_0$, we have the following result about the behaviour at infinity of a spherical surface, proved in a different manner in [H].

Proposition 2.2

Let M be a spherical surface with constant mean curvature $H \in [0, 1]$ in H^3 . Then M makes a well defined angle α_0 with the sphere at infinity $S^2(\infty)$, such that $\sin \alpha_0 = H$.

3. Spherical Surfaces

We will give in this section a description of the generating curves of spherical surfaces with constant mean curvature H for each of the three cases: $H = 0$, $H \in (0, 1]$, and $H > 1$,

First case: $H = 0$ (minimal spherical surfaces).

From Lemma 2.1 it follows that the generating curve of a minimal spherical surface is a graph, hence it is embedded. Moreover from the Symmetry Lemma, f has one y -symmetry.

Proposition 3.1

The asymptotic boundary of a minimal spherical surface in H consists of two circles.

Proof. Since the generating curve of M is a graph, and has a vertical symmetry, it follows from Lemma 2.2 that we have two possibilities for the asymptotic boundary: two disjoint circles, or two points. We must show that the last possibility does not occur. For this consider the family G_λ , $\lambda \geq 0$, of totally geodesic surfaces that intersect orthogonally the y -axis at a point with y -coordinate equal to λ . If the asymptotic boundary consists of two points (the points at infinity of the rotation axis), then the asymptotic boundary of any surface in the family G_λ does not intersect $\partial_\infty(M)$. Moreover, if λ is sufficiently large, G_λ does not intersect M . Since G_0 is the totally geodesic surface $y = 0$, we obtain a contradiction from the tangency principle. (q.e.d.)

Remark 3.1:

The above proposition also follows from the result that a minimal surface in H^3 is contained in the convex hull of its asymptotic boundary ($[An]$).

A computer sketch of the generating curve of a minimal surface is shown in figure 2.5.

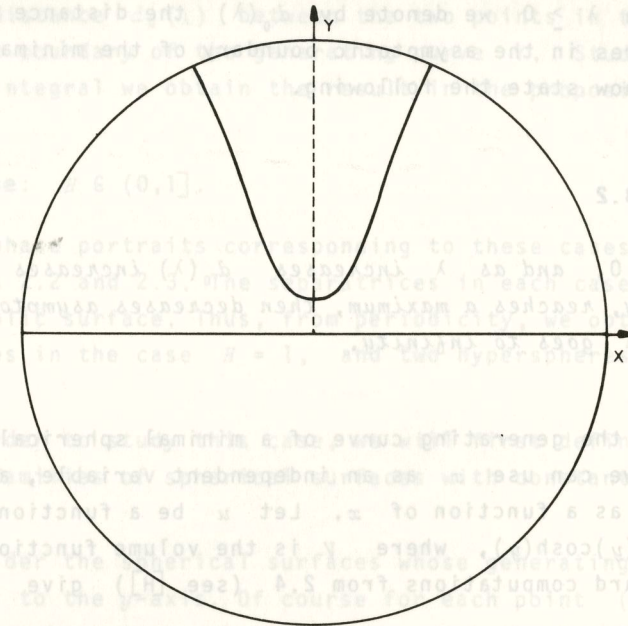


Figure 2.5

We will denote by M_λ , $\lambda \geq 0$, the family of minimal spherical surfaces that intersect the y -axis orthogonally at the point with coordinate λ (any spherical minimal surface is isometric to some surface in this family). An interesting and important question to be answered now is how the asymptotic boundary of the family M_λ varies when the parameter λ moves from 0 to ∞ . In order to state the result, we will give first a definition. Consider two points p and q in the asymptotic boundary of the orbit space B_+^2 , and take two y -coordinate curves $x = c_1$, $x = c_2$ whose asymptotic boundaries contain p and q respectively; the distance from p to q is defined by $[c_1 - c_2]$. The distance between two sets in the boundary of B_+^2 is defined in the usual way. If M is a spherical surface in H^3 , we define the distance between two components of its asymptotic boundary as the distance from the corresponding boundary components in the orbit space.

For each $\lambda \geq 0$ we denote by $d_0(\lambda)$ the distance between the two spheres in the asymptotic boundary of the minimal surface M_λ . We can now state the following.

Proposition 3.2

$d_0(0) = 0$, and as λ increases $d_0(\lambda)$ increases monotonically, reaches a maximum, then decreases asymptotically to zero as λ goes to infinity.

Proof. Since the generating curve of a minimal spherical surface is a graph, we can use x as an independent variable, and consider y as a function of x . Let u be a function such that $u' = v(y)\cosh(y)$, where v is the volume function in 2.1. Straightforward computations from 2.4 (see [H]) give

$$v \cosh^2 y / \sqrt{\cosh^2 y + (y')^2} - 2Hu = K \text{ (const.)} \quad 2.7$$

where K is a constant. If $y(0)$ is the intersection point of the generating curve with the y -axis, then $y'(0) = 0$, and from 2.7, with $H = 0$, we obtain

$$K = v(0)\cosh y(0).$$

Substituting this in 5.7, and solving for y' we obtain

$$(dy/dx) = \cosh y [v \cosh y / (v(0)\cosh y(0)) - 1],$$

hence

$$x(y) = \int_{y(0)}^y [\cosh y (v \cosh y / (v(0)\cosh y(0)) - 1)]^{-1/2} dy \quad 2.8$$

Note that $y(0)$ is exactly the parameter λ of the family M_λ .

If we let $y \rightarrow \infty$ in 2.8 then the value $x(\infty)$ obtained is

half the distance $d_0(\lambda)$ between the two points in the asymptotic boundary of the generating curve γ . Studying the improper integral we obtain the result in the proposition. (q.e.d.)

Second case: $H \in (0,1]$.

The phase portraits corresponding to these cases are given in figures 2.2 and 2.3. The separatrices in each case corresponds to an umbilic surface. Thus, from periodicity, we obtain two horospheres in the case $H = 1$, and two hyperspheres in the case $H \in (0,1)$.

In order to study this case, we will first define two standard families of spherical surfaces with constant mean curvature.

Consider the spherical surfaces whose generating curves are orthogonal to the y -axis. Of course for each point $(0,1)$ on this axis we have two such curves: one with the initial angle $\alpha = \pi/2$, and the other with the initial angle $\alpha = -\pi/2$ (when $H = 0$ the two families coincide). We will call first family the $(\pi/2)$ -family, and the second one the $(-\pi/2)$ -family. For each surface M_λ in any of these families, the parameter λ is the distance from the surface to the rotation axis.

Note that the separatrices divide the phase portraits in two regions each of these regions corresponds to one of the two above families defined above. We have the following.

Lemma 3.3.

Any nonumbilical spherical surface with constant mean curvature $H \in (0,1)$ is isometric to a unique surface in one of the two families defined in the previous paragraph.

Proof: It is clear that the surfaces in each family are distinct (i.e. not isometric). The result of the Lemma follows immediately from the phase portraits in figures 2.2 and 2.3. (q.e.d.)

First we will study the surfaces in the $(\pi/2)$ -family. In this case the angle α varies in the interval $(0, \pi)$, hence $\sin \alpha > 0$ and from the first equation in 2.5 we conclude that the generating curve is a graph. In particular it is embedded, and the same happens with the corresponding spherical surface M . Thus M divides the hyperbolic space in two components; one of them contains the rotation axis. If the mean curvature vector points to this component, we say that it points *towards the rotation axis*.

Proposition 3.4

Let M be a spherical surface with constant mean curvature in the $(\pi/2)$ -family. Then M satisfies the following conditions:

- The mean curvature vector points towards the rotation axis;
- If $H = 1$, the asymptotic boundary consists of two points
- If $H \in (0, 1)$ the asymptotic boundary consists of two disjoint circles.

Proof: a) By continuity of the family on the parameter H , we just have to prove the result for $H = 1$. For this we use the tangency principle by comparing the given surface with one of the two families of horospheres orthogonal to the y -axis.

Now we will prove b). Suppose that the asymptotic boundary does not consist of two points. Since M is symmetric with respect to the plane $x = 0$, and its generating curve is a graph, the asymptotic boundary consists of two spheres. Let H_ρ denote one of the two families of horospheres orthogonal to the rotation axis. It is easy to see that there exists a surface in this family tangent to M . Since M is embedded and its mean curvature vector points to the rotation axis (by a)), we obtain a contradiction from the tangency principle.

The proof of c) also uses the tangency principle but it is a little more subtle. Suppose that the asymptotic boundary of

M does not consist of two spheres, then as in the proof of the previous item, it must consist of two points (the points in the asymptotic boundary of the rotation axis). Consider the family S_ρ of hyperspheres in H^3 orthogonal to the y -axis, whose mean curvature vector points "downwards", and let S_{ρ_0} be the surface in this family whose asymptotic boundary is the equator of the sphere at infinity. From the tangency principle M is below S_{ρ_0} as indicated in the figure 2.6.

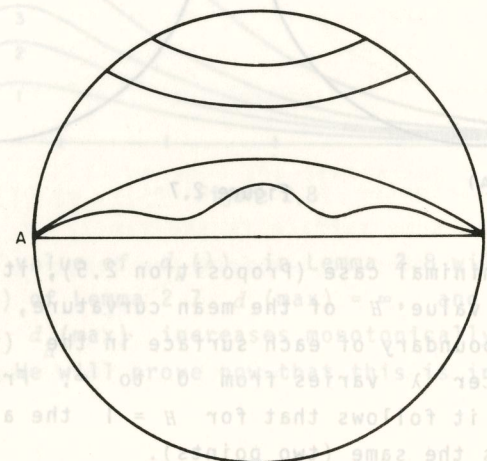


Figure 2.6

The above situation implies that M is at a finite distance from the rotation axis, and this is false as follows easily from the phase portrait in fig. 2.3. (q.e.d.)

In the pictures 2.7 (A) and (B) we show the computer drawing of the generating curve of a surface M for each case $H = 1$ (fig. 2.7-A) and $H \in (0, 1)$ (fig. 2.7-B).

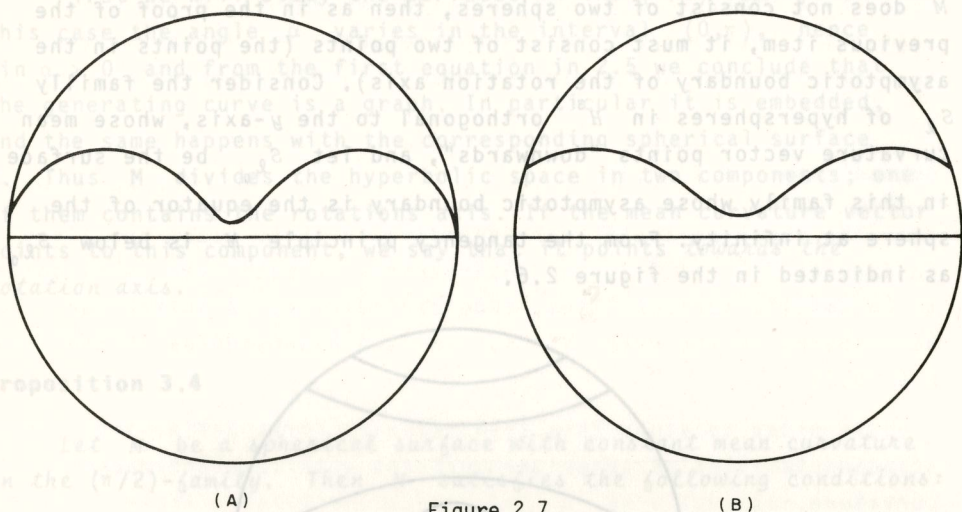


Figure 2.7

As in the minimal case (Proposition 2.5), it is important to study, for each value H of the mean curvature, the variation of the asymptotic boundary of each surface in the $(\pi/2)$ -family, when the parameter λ varies from 0 to ∞ . From b) of Proposition 2.7 it follows that for $H = 1$ the asymptotic boundary remains the same (two points).

For $H \in (0,1)$ we will denote by $d_H(\lambda)$ the distance between the two spheres in the asymptotic boundary of the surface M_H^λ in the family. We will prove the following.

Lemma 3.5

Using the above notations, when λ varies from 0 to ∞ , $d_H(\lambda)$ increases from a positive value $d_H(0)$ reaches a maximum, and then decreases monotonically to zero.

Proof: As in the proof of Proposition 3.2, we obtain from 2.7 an integral similar to 2.8, for any value of $H \in (0,1)$. The Lemma follows by making $y \rightarrow \infty$, and studying the resulting improper integral. (q.e.d.)

In the picture 2.8 obtained in the computer, we show the graphs of the function $d_H(\lambda)$ for different values of H .

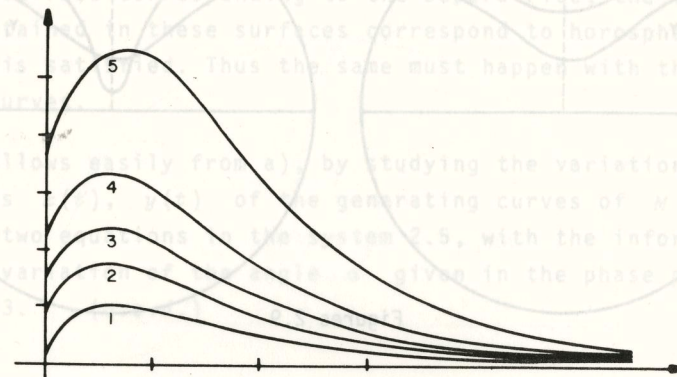


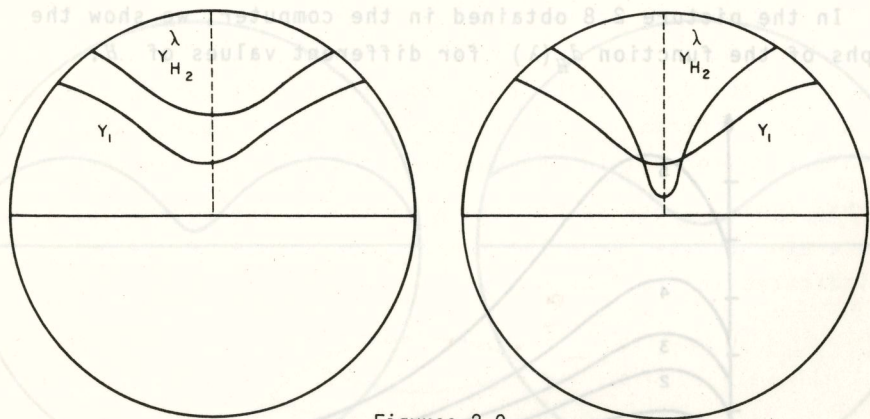
Figure 2.8

The maximum value of $d_H(\lambda)$ in Lemma 2.8 will be denoted by $d_H(\max)$. From b) of Lemma 2.7 $d_1(\max) = \infty$, and as the above graphs indicate, $d_H(\max)$ increases monotonically as H increases from 0 to 1. We will prove now that this is indeed true.

Lemma 3.5

$d_H(\max)$ is a monotone increasing function of H .

Proof: Let $H_1 < H_2$, and denote by γ_1 the generating curve of the surface M_{λ_1} in the family $M_{H_1}^\lambda$, such that $d_{H_1}(\lambda) = d_{H_1}(\max)$. Let γ_H^λ denote the family of generating curves of the surfaces in the family $M_{H_2}^\lambda$. If the result stated in the Lemma were false, then for each λ we would have $d_{H_2}(\lambda) \leq d_{H_1}(\max)$. We would have then one of the two possibilities in the figures 2.9,



Figures 2.9

Now varying the parameter λ conveniently, we obtain a tangency point of M_λ with some surface in the family $M_{H_2}(\lambda)$. Since $H_2 > H_1$ this is a contradiction from the tangency principle. (q.e.d.)

We will study now the spherical hypersurfaces in the $(-\pi/2)$ -family.

Proposition 3.6

Let M be a spherical surface in the $(-\pi/2)$ -family, with constant mean curvature 1. Then M has the following properties:

- a) The asymptotic boundary of M consists of two points;
- b) M self-intersects along one orbit.

Proof: a) Let $x(t)$ denote the abscissa of the generating curve of M . We must show that

$$\lim_{t \rightarrow \infty} x(t) = \infty. \quad 2.9$$

From the phase portrait in fig. 2.4, when $t \rightarrow \infty$ the integral curves corresponding to M are asymptotic to the invariant surface corresponding to the separatrix; the integral curves contained in these surfaces correspond to horospheres, hence 2.9 is satisfied. Thus the same must happen with the other integral curves.

b) Follows easily from a), by studying the variation of the coordinates $x(t)$, $y(t)$ of the generating curves of M using the first two equations in the system 2.5, with the information about the variation of the angle α given in the phase portrait in fig. 2.3. (q.e.d.)

In the picture 2.10 we show the generating curve of M obtained with the use of the computer

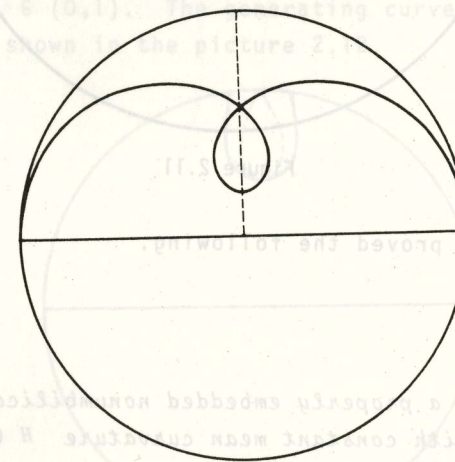


Figure 2.10

We proved above that the generating curve of a spherical surface with constant mean curvature $H = 0$ is a graph. When $H = 1$ it follows from the above proposition that the generating curve self-intersects in one point, and its asymptotic boundary consists of two points. By continuity, when H varies in the interval $(0,1)$ the asymptotic boundary varies from one situation

to another, hence for some value say $H_0 \in (0,1)$ of H , there exists a generating curve whose asymptotic boundary has just one point (see picture 2.11)

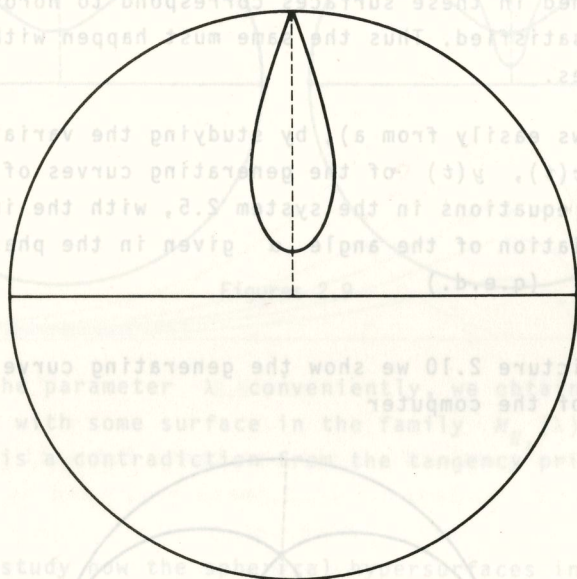


Figure 2.11

We have thus proved the following.

Theorem 3.7

There exists a properly embedded nonumbilical spherical surface in \mathbb{H}^3 with constant mean curvature $H \in (0,1)$, whose asymptotic boundary consists of one circle.

Note that the closure of the surface in the above theorem is a torus $S^1 \times S^1$. It answers a question posed in [LR].

Suppose now that M is a surface in the $(-\pi/2)$ -family, M_H , with constant mean curvature $H \in (0,1)$. Using the same idea of the proof of a) in proposition 2.10, it is possible to prove that

the asymptotic boundary of M consists of two circles. Also from the proof of Theorem 2.10 M may be embedded or not. We have the following

Proposition 3.8

There exists two real numbers H_0 and H_1 , $H_0 \leq H_1$, such that the following assertions hold:

- If $H > H_1$, no surface in the $(-\pi/2)$ -family M_H^λ is embedded
- If $H < H_0$, then each surface M_H in the $(-\pi/2)$ -family is embedded for large values of the parameter λ .

Proof: a) Let M be an embedded spherical surface with constant mean curvature $H \in (0,1)$. The generating curve of M is the non dashed curve shown in the picture 2.12.

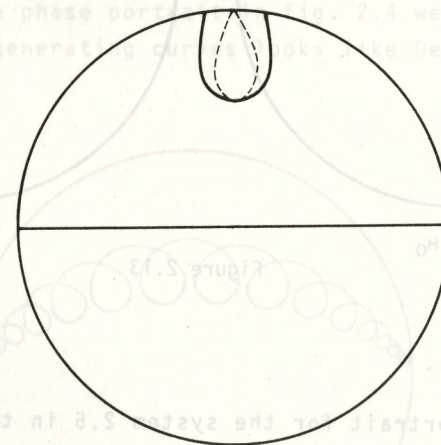


Figure 2.12

From the results in [GRR] there exists a parabolic surface N with constant mean curvature H_1 (approximately 0.65) tangent to M , contained entirely above M , and whose mean curvature vector

points in the same direction as that of M (we represent N by the dashed curve in the above picture). If the mean curvature of M is greater than H_1 , we obtain a contradiction from the tangency principle.

Item b) is proved in a similar manner using a minimal parabolic surface instead of N .

The above proposition is illustrated by the pictures below, obtained in the computer, where we plot five generating curves with different values of the parameter λ for $H < H_0$ (fig. 2.13-A) and $H > H_1$ (fig. 2.13-B).

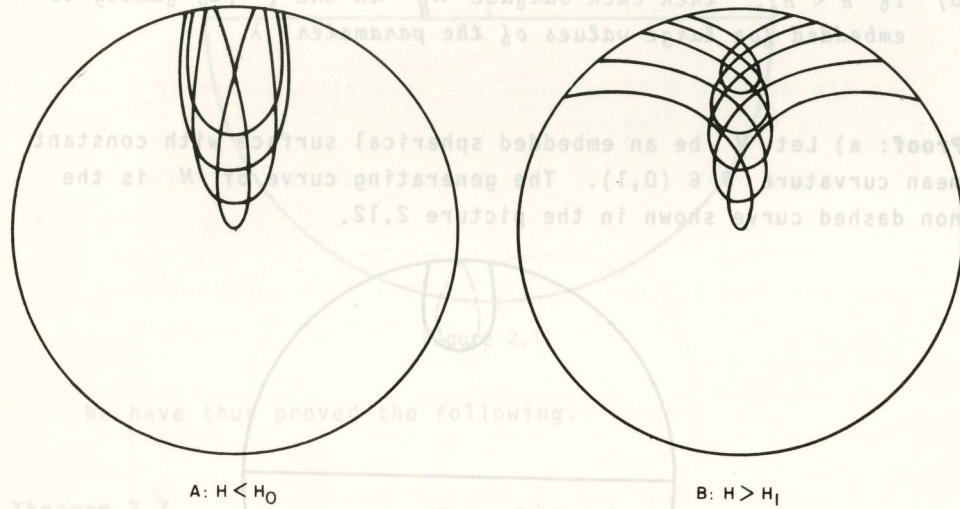


Figure 2.13

Third case: $H > 1$.

The phase portrait for the system 2.5 in this case is given in fig. 2.4. The center in this phase portrait corresponds to a cylinder: a surface equidistant from the rotation axis. From the first equation in 2.5 it follows that the generating curves corresponding to the closed orbits are embedded, and the variation of the angle α is bounded. Thus this curve is periodic and resembles Delaunay's unduloid ($[D]$), as is shown in the picture below.

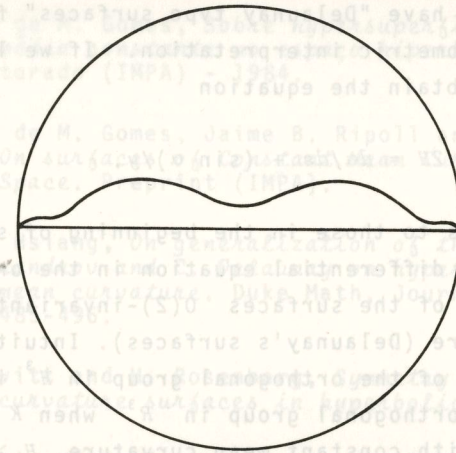


Figure 2.14

The surface corresponding to each separatrix in the phase portrait has a fixed point, hence it must be a sphere.

By studying the variation of the angle corresponding to the other orbits on the phase portrait in fig. 2.4 we conclude that the corresponding generating curves look like Delaunay's Nodoid (see fig. 2.15).

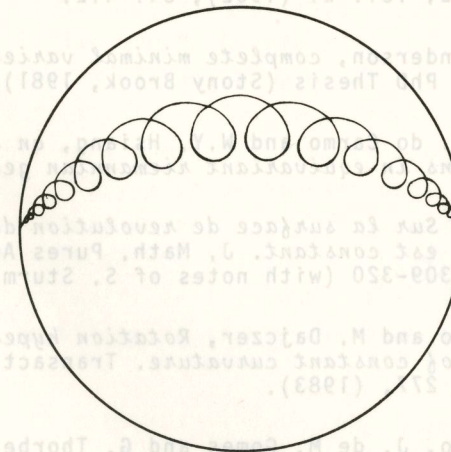


Figure 2.15

The fact that we have "Delaunay type surfaces" for $H > 1$ has an interesting geometric interpretation. If we let $H \rightarrow 0$ in equation 2.4, we obtain the equation

$$2H = d\alpha/ds + (\sin \alpha)/y. \quad 2.10$$

Computations analogous to those in the beginning of section 2 show that 2.10 is the differential equation in the orbit space of the generating curves of the surfaces $O(2)$ -invariant in R^3 with constant mean curvature (Delaunay's surfaces). Intuitively this means that the action of the orthogonal group in H^3 is approximated by the action of the orthogonal group in R^3 when $K \rightarrow 0$. Therefore the surfaces in H^3 with constant mean curvature $H > 1$ are approximated by Delaunay's surfaces in R^3 .

References

- [A] A. Alexandrov, *A characteristic property of spheres*. Ann. Mat. Pura Appl. 58 (1962), 303-315.
- [AI] A. Alexandrov, *Uniqueness theorem for surfaces in the large*. I,II,III,IV. 1956-1958, A.M.S. Translations, series 2, vol. 21 (1962), 341-412.
- [An] Michael T. Anderson, *complete minimal varieties in hyperbolic space*. PhD Thesis (Stony Brook, 1981).
- [BCH] A. Back, M.P. do Carmo and W.Y. Hsiang, *on some fundamental equations in equivariant riemannian geometry*. Preprint.
- [D] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constante*. J. Math. Pures Appl, Ser. 1,6 (1841) 309-320 (with notes of S. Sturm).
- [dCD] M.P. do Carmo and M. Dajczer, *Rotation hypersurfaces in spaces of constant curvature*. Transactions of the AMS (2) 277, (1983).
- [dCGT] M.P. do Carmo, J. de M. Gomes and G. Thorbergsson, *The influence of the boundary behaviour on hypersurfaces with constant mean curvature in hyperbolic space*. To appear in Comm. Math. Helv.

- [Go] Jonas de M. Gomes, *Sobre hypersuperfícies de curvatura média constante no espaço hiperbólico*. Tese de Doutorado (IMPA) - 1984.
- [GRR] Jonas de M. Gomes, Jaime B. Ripoll and Lucio L. Rodriguez, *On surfaces of Constant Mean Curvature in Hyperbolic Space*. Preprint (IMPA).
- [H] Wu-Yi Hsiang, *On generalization of theorems of A.D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*. Duke Math. Journal 49(3) - 1982, 485-496.
- [LR] G. Levitt and H. Rosenberg, *Symmetry of constant mean curvature surfaces in hyperbolic space*. Preprint.
- [M] H. Mori, *Minimal surfaces of revolution in H^3 and their stability properties*. Indiana Math. J. 30(1981), 787-794.
- [PM] Jacob Palis Jr. and Wellington de Melo, *Geometric Theory of Dynamical Systems*. Springer-Verlag, 1982.

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