A NOTE ON THE HILBERT SCHEME OF TWISTED CUBICS

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Abstract. We show the component of the Hilbert scheme of twisted cubic curves is the blowup of the variety of determinantal nets along the subvariety of nets with a fixed plane.

Introduction. Let X denote the subvariety of the grassmannian of nets of quadrics consisting of nets spanned by the minors of a 3x2 matrix of linear forms. Let I be the subvariety of X of nets with a fixed component. Denote by H the Hilbert scheme of twisted cubic curves. There is a natural map $h: H \to X$ defined by assigning to a (possibly degenerate) twisted cubic the quadratic part of its homogeneous ideal. We prove the following.

Theorem. $h: H \to X$ is the blowup of X along I.

This fact, stated as a "strong belief" in the paper of Ellingsrud, Piene and Stromme [EPS], remained sort of a nasty technical point midway to the obtention of the Chow ring of H. In fact, in [EPS] it is shown that X is a smooth projective variety and its Chow ring is calculated implicitly. We refer to [PS], [K] and [KSX] for further motivation and historical accounts.

Our proof goes roughly as follows. First notice H embeds in the grassmannian $G(10,20)$ of 10-dimensional subspaces of the space of cubic forms. Therefore, the rational map $h^{-1}: X \to H$ induces a rational map of X into $G(10,20)$. This map is defined by a natural homomorphism m of bundles over X with generic rank 10 and which drops rank precisely on I. We show the ideal

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of I in X is in effect generated locally by the 10×10 minors of a matrix representation of m. It follows that the blowup X' of X along I dominates H. The map $X'\to H$ is shown to be bijective, whence it is an isomorphism in view of Zariski's main theorem.

1. Notation and Preliminaries

We fix the following notation.

V = vector space of linear forms in the variables $x_0,\ldots,x_3$;

$G = G(3,S_2V)$, grassmannian of rank 3 subspaces of the space of quadratic forms

E = tautological rank 3 subbundle of $(S_2V)_G$.

1.1 Lemma. The Hilbert scheme H of twisted cubics embeds in the grassmannian $G(10,20)$ or rank 10 subspaces of the space S_3V of cubic forms.

Proof. We employ the argument of Mumford [M] p. 107. The main point is that, for any C in H, the ideal J_C is 3-regular (cf. [M] p. 99). To see this, we claim first that

$$H^1(J_C(2)) = 0.$$

This follows from the standard cohomology sequence of

$$0 \to J_C \to O_{P_3} \to O_C \to 0$$

together with the fact that

$$h^0(O_C(2)) = 7$$

(cf. [PS] top of p. 766).

Next, referring to [M], bottom of p. 102, it suffices to show that the intersection D of C with a general plane T is 3-regular. The natural sequence

$$0 \to J_D \to O_T \to O_D \to 0$$

yields

$$h^1(J_D(-1)) = h^0(O_D(-1)) = 3.$$

Choosing a line L in T disjoint from D we find

$$J_D \otimes O_L = O_L.$$

The latter is clearly 0-regular. Applying again the assertion at the bottom of p. 102 of [M] we find,

$$J_D \text{ is } (0+h^1(J_D(-1)))\text{-regular},$$

i.e., J_D is 3-regular as asserted, whence so is J_C.

Denote by J the ideal sheaf of the universal curve in $H\times P_3$ and write $p: H\times P_3 \to H$ the projection map. We have

$$R^1p_*(J(3)) = 0$$

and that $p_*(J(3))$ is a locally free, locally split subsheaf of $(S_3V)_G$ of rank 10 generated by its global sections. As in [M] p. 107 one concludes that H imbeds in $G(10,20)$.

Recall that each point in P gives rise to a unique net of linear forms. Thus, we have a natural map

$$P_3 \times P_3^* \to G$$

which sends a point-plane to a net of quadrics with fixed component. That's easily seen to be an embedding; call Y its image. Thus, we clearly have,

$$P_3 \times P_3^* \cong Y = \{\text{nets of quadrics with a fixed component}\}.$$

Notice Y consists of two orbits under the natural action of $GL(3)$; the open (resp. closed) orbit contains the net (x_0x_1, x_0x_2, x_0x_3) (resp. (x_0^2, x_0x_1, x_0x_2)).

1.2 Proposition. Let $m: E\otimes V \to (S_2V)_G$ be the homomorphism of bundles over G induced by

$$(S_2V) \otimes V \to S_3V.$$

Let W be the scheme of zeros of $\overset{10}{\wedge}\ m$, and let I be the scheme-theoretic intersection

$$I := W \cap X.$$

Then:

(1) Y is a smooth component of W;

(2) I is the closed orbit of Y and is isomorphic to the incidence subvariety of $P_3 \times P_3^*$;

(3) the restriction of m over $X-I$ is of constant rank 10.

1.3 Remark. Probably $Y = W$ but we won't need this. If the only closed orbits of $GL(V)$ in G were I and the orbit of (x_0^2, x_0x_1, x_1^2) the equality would follow.

Proof. Set $w = (x_0^2, x_0x_1, x_0x_2)$ and set

$$u = \begin{bmatrix} 0 & x_0 \\ x_0 & 0 \\ x_2 & x_1 \end{bmatrix}$$

Clearly w lies in $Y \cap X$. The image of m at the fibre over w is the span of

$$x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_0x_2^2, x_0^2x_3, x_0x_1x_3, x_0x_2x_3.$$

Hence w lies in W along with its orbit I. A similar calculation over (x_0x_1, x_0x_2, x_0x_3) shows that W contains Y.

Since the dimension of Y is 6 and that of G is 21, the assertion (1) will follow from the following

Claim. The ideal of W at w contains 15 functions with linearly independent differentials.

To prove the claim, let $G_0 \subseteq G$ be the standard open neighborhood of w in G isomorphic to affine space of 3×7 matrices with entries the coordinate functions a_{ijk} as written below. The restriction of the tautological bundle E_0 to G is trivial, with basis

$$(1.4)\quad \begin{cases} v_1 = x_0^2 + a_{103}x_0x_3 + a_{111}x_1^2 + a_{112}x_1x_2 + \dots + a_{133}x_3^2; \\ v_2 = x_0x_1 + a_{203}x_0x_3 + \dots \\ v_3 = x_0x_2 + a_{303}x_0x_3 + \dots \end{cases}$$

Notice the index $i(=1,2,3)$ gives the row, whereas jk put in lexicographical order gives the 7 columns.

The matrix of m with respect to the basis

$$v_i \otimes x_j \quad \text{and} \quad x_rx_sx_t$$

may be easily written down. Performing elementary row operations modulo the square of the maximal ideal $M = (a_{ijk})$, we achieve a matrix in echelon form,

$$\begin{bmatrix} I & * \\ 0 & A \end{bmatrix}$$

where I is the 9×9 identity matrix and the nonzero entries of A mod M^2 are listed below up to sign and repetitions:

$$(1.5)\quad \begin{matrix} a_{111}, a_{112}, a_{113}, a_{122}, a_{123}, a_{133}, a_{222}, a_{223}, a_{233} \\ a_{311}, a_{313}, a_{312}-a_{211}, a_{322}-a_{212}, a_{323}-a_{213}, a_{333}. \end{matrix}$$

This proves the claim.

To prove (2) and (3), pick a net x in X not in Y. We know it spans the homogeneous ideal of a curve. The space of cubic forms in that ideal is easily seen to be of rank 10. Thus x is not in W. This shows that $W \cap X$ is a subset of Y. Since $Y-I$ is disjoint from X (e.g., there is no C in H with homogeneous ideal containing (x_0x_1, x_0x_2, x_0x_3)) we see that $W \cap X = I$ holds as sets. To finish the proof of (2), it suffices to show that the tangent spaces of W and X at w intersect along a space of dimension 5 (=dim. I). Since (1.5) gives equations for T_wW in T_wG, it remains to obtain generators for T_wX.

For this, let U be the open subset of the affine space of 3×2 matrices of linear forms with independent 2×2 minors. By definition of X, we have a map $p: U \to X$ so that T_wX is the image of T_uU (cf. [EPS]). Write a tangent vector $L = (L_{ij})$ in T_uU, a matrix of linear forms,

$$L_{ij} = \sum L_{ijk}x_k.$$

The image of the "infinitesimal curve" $u+\varepsilon L$ in X is the ($k[\varepsilon]$ - valued) net spanned by

$$x_0^2 + \varepsilon x_0(L_{12}+L_{21}),$$
$$x_0x_1 + \varepsilon(x_0L_{32}+x_1L_{21} - x_2L_{22}),$$
$$x_0x_2 + \varepsilon(x_0L_{31} - x_1L_{11} + x_2L_{12}).$$

Performing row operations mod. ε^2, we find the coordinates of $dp(L)$ in standard form,

$$(1.6)\quad (a_{ijk}) = \left|\begin{matrix} \scriptstyle(03) & \scriptstyle(11) & \scriptstyle(12) & \scriptstyle(13) & \scriptstyle(22) & \scriptstyle(23) & \scriptstyle(33) \\ L_{123}+L_{213} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{323} & L_{211} & L_{212}-L_{221} & L_{213} & -L_{222} & -L_{223} & 0 \\ L_{313} & L_{111} & L_{121}-L_{112} & -L_{113} & L_{122} & L_{123} & 0 \end{matrix}\right|$$

Thus, T_wX is the subspace of T_wG (identified with 3×7 matrices) consisting of matrices of the above form. In view of (1.5), we see that $(T_wX) \cap (T_wW)$ is annihilated by

$$L_{222}, L_{223}, L_{111}, L_{121}-L_{112}-L_{211}, L_{122}-L_{212}+L_{221}, L_{123}-L_{213}, L_{113}.$$

This yields the 5-dimensional space,

$$(T_wX) \wedge (T_wW) = \left\{\begin{bmatrix} 2a & 0 & 0 & 0 & 0 & 0 & 0 \\ b & d & e & a & 0 & 0 & 0 \\ c & 0 & d & 0 & e & a & 0 \end{bmatrix} : a,b,c,d,e \text{ arbitrary}\right\},$$

as desired.

2. Proof of the theorem

Let $f: X' \to X$ be the blowingup of I in X and write I' for the exceptional divisor. Since I and X are smooth, we have that I' is the projective bundle of normal directions of I in X.

Consider now the restriction of the homomorphism m defined in (1.2). Since its restriction over X-I is of constant rank 10, we obtain a map

$$X\text{-}I \to G(10,20)$$

which factors through H. Let $h': X\text{-}I \to H$ be the induced map. Clearly, h' is the (birational) inverse of the natural map

$$h: H \to X.$$

Embedding $G(10,20)$ in P^N by Plucker, the induced rational map of X to P^N is defined by a linear system with basis locus (scheme-theoretically!) equal to I. Blowing up yields a map

$$X' \to P^N$$

(cf. Hartshorne [H] p. 168)). It factors through H; write

$$g: X' \to H$$

for the induced map.

We proceed to show g is an isomorphism. By construction, its restriction to X'-I' is an isomorphism onto $H\text{-}h^{-1}(I)$. Since H is smooth (cf. [PS]), Zariski's main theorem leaves us the task of showing the restriction of g to $I' = g^{-1}h^{-1}(I)$ is bijective. Set

$$P = f^{-1}(w), \quad P' = h^{-1}(w)_{red}$$

with w as in the proof of (1.3). Thus, P is the projective space of normal directions to I in X at w, whereas P' is the projective space of cubic curves in the plane $x_0 = 0$ singular at (0:0:0:1). Since P is reduced, the restriction of g factors through P'.

Each tangent vector v in $T_w X$ not in $T_w I$ gives a point $\tilde{v}$ in P. If we choose a curve $c:A^1 \to X$ such that $c'(0) = v$ and such that $c(t)$ is not in I for $t \neq 0$, it lifts uniquely to a curve $\tilde{c}:A^1 \to X'$ such that $\tilde{c}(0) = \tilde{v}$. Moreover, we have

$$g(\tilde{v}) = \lim_{t\to 0} g\tilde{c}(t).$$

We apply these considerations to tangent vectors represented by 3×7 matrices as in (1.6) with all entries equal zero except for

$$L_{111} = A \quad \text{and} \quad L_{113} = B, \quad (A:B) \quad \text{in } P^1.$$

Define the curve

$$c(t) = (x_0^2, x_0x_1, x_0x_2 - tx_1(Ax_1 + Bx_3)).$$

One sees readilly (as in [H] p. 260) that we have,

$$\lim_{t\to 0} g\tilde{c}(t) = (x_0^2, x_0x_1, x_0x_2, x_1^2(Ax_1 + Bx_3)).$$

Therefore, there are projective lines $L \subset P$, $L' \subset P'$ such that g maps L isomorphically onto L'. This implies easily that $P \to P'$ is an isomorphism.

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References

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Note added in Proof. After submission of this note I received a letter from R. Piene (7/8/86, enclosed with a preprint) stating: "As you can see from the introduction we finally managed to show that H *is* the blow up of X along I, but we haven't started to write it up yet (we must include the Chow ring of H so it takes a little work). The idea is roughly as this: we choose a (particular) family

$$T = A^1 \to X$$

passing through a "worst" point of I ($0 \in I$, $T-\{0\}$ $(X-I)$. Then one expresses I as the support of the scheme defined by some Fitting ideal on X, pull this back to T and see it is reduced; then lift $T \to H$ and obtain (this needs a little tor_1- argument) that $(T\cdot D) = 1$ at the point 0, where $D = f^{-1}(I)$. This shows that D can't have embedded components, and we're done. You'll get the details later!"

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