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CHAOS AFTER BIFURCATION OF A MORSE-SMALE DIFFEOMORPHISM THROUGH A ONE-CYCLE SADDLE-NODE AND INTERATIONS OF MULTIVALUED MAPPINGS OF AN INTERVAL AND A CIRCLE

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1. Introduction

This paper contributes to a study of an amazingly complicated dynamics which can arise right after crossing a boundary of a space of simple dynamical systems. We continue investigations started by Newhouse, Palis and Takens in [N-P-T]. Let us consider a generic 1-parameter family of diffeomorphisms f_μ on a compact manifold which begins inside the space of Morse-Smale diffeomorphisms an crosses the boundary with a diffeomorphism f_b which still has finite ω-and α-limit sets and for which a pair of saddles is glued together to a saddle-node P. The whole paper is connected with the following

Conjecture A (homoclinic tangency). If $W^s(P) \cap W^u(P)\setminus\{P\} \neq \emptyset$ (existence of a cycle) then there are parameters beyond b, arbitrarily close to b, for each of which a homoclinic tangency happens (i.e. for a hyperbolic periodic point q, $W^s(q)$ has a point of a nontransversal intersection with $W^u(q)$), except the case of a non-critical 1-cycle. (all the definitions will be reminded later).

This Conjecture was answered in positive in [N-P-T] except the bi-critical 1-cycle case. This remaining case will be the

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* Most of this paper was done during my stay at IMPA in 1979/1980. I want to express my gratitude for inviting me there and care during my stay. I miss the beautiful Brasil sometimes.

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topic of our paper. We give a positive answer for several special cases. We deal also with several weaker conjectures (for generic families!): failure of mild stability, the existence of transversal homoclinic intersections (so the existence of Smale horseshoes), heteroclinic tangencies.

In the case dim $W^s(P)$ = dim M (or dim $W^u(P)$ = dim M) for parameters beyond b we have an attractor reminding the Henon one. (see Fig. 2, compose the dynamics drawn there with a small counterclockwise rotation). In the opposite case we look for tangences in an Henon-like, saddle-like invariant set.

The existence of a homoclinic tangency implies the existence of wild hyperbolic sets (i.e. with an analogous tangency property between the stable and ustable foliations) for an abundant set of parameters B (including intervals). In particular it implies failure of hyperbolicity on the set of the nonwandering points for these parameters. The set B and its complement was studied by Newhouse, Palis and Takens in a number of papers. In particular it was studied the case when all periodic orbits of f_b were hyperbolic but there happened a homoclinic tangency for f_b. In this situation a *homoclinic explosion* happened (see for example [P-T]). Here we deal with a *cycle saddle-node explosion*. I hope to study the set B in this situation in a subsequent paper. (This discussion is continued at the end of § 5.)

The main observation to verify Conjecture A and related conjectures is reduction to studying *1*-dimensional multivalued maps. The case when graphs of these maps are given by closed curves in $\mathbb{R}^2$ is studied in §.4. The case of curves $\gamma \subset \mathbb{R}^2$ with $\gamma + (m,m) = \gamma$ for an integer m (multivalued map of a circle) is studied in §.5. (These curves will be obtained with the use of the curves $W^s(P) \cap W^u(P) \setminus \{P\}$). This study of *1*-dimensional dynamics is a kernel and a main part of the paper. Sections 1 and 2 have introductory character. The reduction theorem is proved in Section 3.

The paper finishes with §.6 which is only roughly connected with the rest of the paper. We consider there some naturally arising examples of multivalued mappings, not connected with the

saddle-node bifurcations (§.6. is merely outlined, I hope to develop it in a subsequent paper).

Now we pass to a more precise and formal description:

Let us consider a smooth one-parameter family of diffeomorphisms f_μ, $\mu \in \langle 0,1\rangle$ on a compact manifold M, satisfying the following properties:

(*1) There exists $b \in (0,1)$ such that f_μ is Morse-Smale for every $\mu < b$ but not for b itself. The sets of the ω-and α-limit points are finite for $\mu \leq b$

(*2) All periodic orbits for f_b are hyperbolic except exactly one saddle-node periodic orbit P;

(*3) Let $W^u(Q)$ $(W^s(Q))$ denote the unstable (stable) manifold for any hyperbolic or saddle-node periodic orbit Q. Then $W^u(P) \cap W^s(P) \setminus \{P\} \neq \emptyset$ and there is no periodic orbit $Q \neq P$ such that $W^u(Q) \cap W^s(P) \neq \emptyset$ and $W^u(P) \cap W^s(Q) \neq \emptyset$ (i.e. there is a cycle of length *1* and no longer cycle).

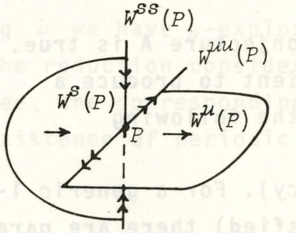


Fig.1. saddle-node P


Fig. 2. a 2-dimensional example

A bifurcation satisfying (*1)-(*3) is called a *1-cycle saddle-node bifurcation*.

To simplify the paper we shall assume that P is a fixed point for f_b, unlike on Fig. 2. All the results of the paper stay true without this assumption with the proofs slightly changed.

Throughout the paper we will need to use several genericity assumptions for our arcs. So we assume

(*4) Genericity. In particular all stable and unstable manifolds meet transversally for f_b. The bifurcation unfolds generically. The hyperbolic part of the spectrum of $Df_b(P)$ has no resonances. The family of curves $W^u(P) \cap W^s(P)$ satisfies some genericity conditions (they will be listed in the following parts of the paper).

In the paper [N-P-T] the leading importance is given to the notions of stability and mild stability. The family f_μ satisfying (*1) is called *mildly stable* if for every C^∞-close family g_μ there exist a homeomorphism $\alpha:<0,1> \to <0,1>$, $\varepsilon > 0$ and a set of homeomorphisms $h_\mu : M \to M$ such that for every $\mu \in <0,b+\varepsilon>$ h_μ conjugates f_μ with $g_{\alpha(\mu)}$.
(Continuity of h with respect to μ is not demanded. If it holds additionally, then f_μ is called *stable*).

The full characterization of generic arcs satisfying the property (*1) which are stable was done in [N-P-T]. The full characterization was done there also for mild stability, except for the arcs satisfying the conditions (*1)-(*4). In connection with that, the following conjecture is stated in [N-P-T]:

Conjecture B. (failure of mild stability). No cycle saddle-node bifurcation is mildly stable.

This Conjecture is true whenever Conjecture A is true. In fact to prove Conjecture B it is sufficient to produce a heteroclinic tangency, namely to prove the following

Conjecture C. (heteroclinic tangency). For a generic 1-cycle saddle-node bifurcation ((*1)-(*4) satisfied) there are parameters beyond b, arbitrarily close to b, for each of which there exist hyperbolic points p,q such that $W^u(p)$ has a point of a nontransveral (nondegenerated) intersection with $W^s(q)$.

The implication $C \Longrightarrow B$ bases on the Palis invariant $\log\lambda / -\log\eta$ in the plane case. See Fig. 3.

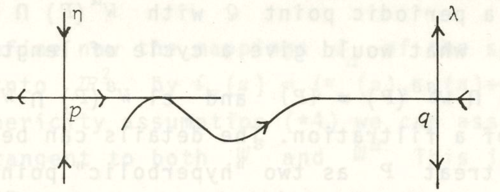


Fig. 3

Homoclinic tangency implies, after an arbitrarily small change of the parameter, the existence of a transversal homoclinic orbit. However in a number of cases, whereas we do not know whether Conjecture A holds we are able to prove

Conjecture D. (transversal homoclinic intersection). For a generic 1-cycle saddle-node bifurcation, for parameters arbitrarily close to b there exist transversal homoclinic orbits.

It turns out that Conjectures A,C,D reduce to the analogous conjectures about 1-dimensional multivalued dynamics.

At least it is a fact, not merely conjecture, that after passing b we have Ω-explosion, even Per-explosion. It will follow from the reduction consideration (shadowing) since, as the reader will see, the corresponding fact about 1-dimensional dynamics (the existence of periodic points) is trivial.

2. Reduction to multivalued mappings

It is known (see [T]) that in the generic case there exist unique, smooth f_b-invariant, codimension 1 foliations $\mathcal{W}^s$, $\mathcal{W}^u$ of $W^s(P)$, $W^u(P)$ respectively, which include the strong stable, unstable, manifolds $W^{ss}(P)$, $W^{uu}(P)$ as leaves. The saddle-node P is called *s-critical* (*u-critical*) if $W^s(P) \cap W^u(P) \setminus \{P\}$ is not transversal to $\mathcal{W}^s$ in $W^s(P)$ (not transversal to $\mathcal{W}^u$ in $W^u(P)$). P is called *bi-critical* if it is both s-critical and u-critical.

$W^s(P) \cap W^u(P)$ is closed in $W^s(P)$ and in $W^u(P)$. Otherwise there would exist a periodic point Q with $W^u(P) \cap W^s(Q) \neq \emptyset$ and $W^u(Q) \cap W^s(P) \neq \emptyset$, what would give a cycle of length 2.

Also cl $W^u(P) \cap W^{ss}(P) = \{P\}$ and cl $W^s(P) \cap W^{uu}(P) = \{P\}$ by the existence of a filtration. The details can be found in [N-P-T] (One can treat P as two "hyperbolic" points. For one of them the role of the standard unstable and stable manifolds is played by $W^u(P)$ and $W^{ss}(P)$, for the other one by $W^{uu}(P)$ and $W^s(P)$ respectively.

Thus one concludes that $W^u(P) \cap W^s(P)$ consists of a finite number of closed curves $\gamma_1, \ldots, \gamma_k$ going through P and of a finite number of sequences of closed curves, $\delta_{1,n}, \ldots, \delta_{1,n}$, $n = \ldots -1, 0, 1, \ldots$, not containing P, such that $f(\delta_{i,n}) = \delta_{i,n+1}$, see Fig. 4.

It is known that there exists a smooth, f_b-invariant 1-dimensional central manifold W^c through P (not unique), transversal to $\mathcal{W}^s$ and $\mathcal{W}^u$ in W^s, W^u respectively, and that $f_b|W^c$ embeds into a smooth flow ϕ_t, $\phi_1 = f_b|W^c$.

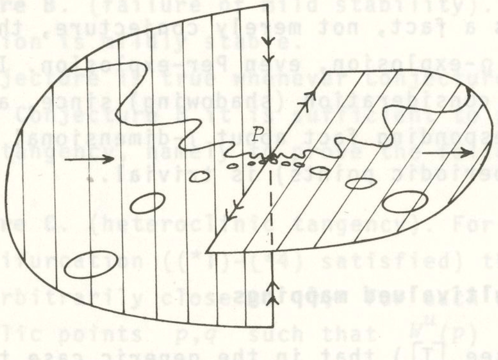


Fig. 4

Fix arbitrary leaves ℓ_s, ℓ_u of the foliations $\mathcal{W}^s, \mathcal{W}^u$ respectively. Let π_s be a projection of $W^s(P) \setminus W^{ss}(P)$ onto $\mathbb{R}$, along the leaves, with $\pi_s(\ell_s) = 0$ and on $\ell \in \mathcal{W}^s \setminus \{W^{ss}\}$

$$\pi_s(\ell) = t \quad \text{if} \quad \phi_t(\ell_s \cap W^c) \in \ell.$$

Similarly we define a projection π_u of $W^u(P) \setminus W^{uu}(P)$ onto $\mathbb{R}$.

Let us define now the mappings i_a of the set $G = W^s(P) \cap W^u(P) \setminus \{P\}$ into $\mathbb{R}^2$ by $i_a(z) = (\pi_u(z), \pi_s(z)+a)$. Denote $i = i_0$. Due to the Genericity assumption (*4) we can assume that at no point, G is tangent to both $\mathcal{W}^s$ and $\mathcal{W}^u$. This implies that is an immersion.

We shall consider the sets $i_a(G)$ in $\mathbb{R}^2$ as graphs of *multivalued functions* from $\mathbb{R}$ to $\mathbb{R}$. In fact in the case a point $z \in \mathbb{R}^2$ is a point of selfintersection of i_aG we shall consider z as a set of distinct points related to the points of the pre-image $i_a^{-1}(z)$. In other words we consider every point $z \in i_aG$ together with a smooth germ of i_aG containing z (provided the selfintersections are transversal) However we will not specify it in our notation.

For every $x \in \mathbb{R}$ denote $i_aG(x) = \{y : (x,y) \in i_aG)\}$. The word i_aG-*trajectory* will be used for any sequence $x_1, \ldots, x_k \in \mathbb{R}$ such that $(x_j, x_{j+1}) \in i_{a+\tau_j}G$ for some integers τ_j, $j = 1, \ldots, k$. τ_j's will be called the *translation integers*. Sometimes we shall use the word i_aG-trajectory to the sequence of point $z_j = (x_j, x_{j+1}) \in i_{a+\tau}G \subset \mathbb{R}^2$. Similarly one can consider the *inverse multivalued function* $(i_aG)^{-1}(y) = \{x : (x,y) \in i_aG\}$.

The definition of the derivative of i_aG at $x \in \mathbb{R}$, $y \in i_aG(x)$ is standard, one considers the branch of i_aG passing through (x,y). If the derivative is equal to 0 we call the pair (x,y) h-critical (horizontally critical) we call x an h-critical point and y and h-critical value. If the derivative is ∞ we use the terms *v-critical* (vertically critical). For a pair neither h-nor v-critical we use the term *non-critical*.

After projecting to the circle the definitions of a periodic trajectory, a sink, a source, stable and unstable sets are standard Observe that for a source x, for example, the stable set is just the set of all the pre-images of x under iterations. The unstable set is a family of arcs (Contrary to the case of a (1-valued) function this set need not be connected.

On $\mathbb{R}$ we call a finite i_aG-trajectory *periodic* either if it is periodic in a standard sense or if only its projection to the circle is periodic, (i.e. the translation integers need not be equal to 0). The stable, unstable sets for an i_aG-trajectory in $\mathbb{R}$ also have either standard meanings or they are the projection pre-images of the objects from the circle.

Below we shall formulate Conjectures a,c,d for iterations of i_aG and Reduction Theorem that these conjectures imply Conjectures A,C,D. Next Section will be devoted to an outline of the proof of Reduction Theorem. In [N-P-T] it is shown how to reduce the study of f_μ for $\mu > b$ to the study of i_aG in the case P is not u-critical, i.e. i_aG is the graph of a function (or in the case where P is not s-critical, where $(i_aG)^{-1}$ is the graph of a function). We shall adapt the [N-P-T] construction to our general, bi-critical case. Then in Section 4,5, one can forget about the diffeomorphisms f_μ at all and deal only with 1-dimensional multivalued dynamics. We shall give there proofs of Conjectures a, c and d in a number of cases under some assumptions about the shape of G. Sometimes we shall be able to reduce these conjectures to conjectures standard in 1-dimensional dynamics, concerning a possibility to perturb a mapping to create sinks.

Let us list all the situations which we call *tangencies* for iterations of i_aG

(i) There exist i_aG periodic sources x, x' and an h-critical point $c \in W^u(x)$ such that there exists an i_aG-trajectory c_j with the pair $c_0 = c$, c_1 h-critical and $c_r = x'$ for some $r > 0$.

(ii) There exist an i_aG-periodic sink x and an i_aG-periodic source x' such that $x' \in (i_aG)^r(x)$ for some positive integer r.

(iii) There exists an i_aG-trajectory y_j such that the pair (y_0, y_1) is h-critical, (y_{l-1}, y_l) is v-critical for some $l > 1$, $y_0 \in W^u(x)$, $y_l \in W^s(x')$ for a source x and a sink x'.

By (i'), (ii'), (iii') we denote the analogous situations for the inverse multivalued function $(i_aG)^{-1}$. (Observe however that only (i) provides a new case since every tangency of the types (ii) or (iii) yields the tangency of the types (ii') or (iii') along the same trajectory).

We call tangency: *homoclinic tangency* if $x = x'$.

We assume in all the cases above that there are no other critical pairs on the trajectories of the tangencies under discussion except the mentioned ones. (This is always so for generic G).

Conjecture a. For a generic G such that iG is not a graph of an invertible function, there exists $a \in \mathbb{R}$ such that for i_aG a homoclinic tangency happens.

Conjecture c. For a generic G such that iG is not a graph of an invertible function, there exists $a \in \mathbb{R}$ such that for i_aG a (heteroclinic) tangency happens.

If the assertion of this Conjecture is that just the case (i) or (i') happens, we call this: Conjecture c(i). Analogously we have Conjectures c(ii) and c(iii). (Of course Conjecture c(iii) is considered only for the saddle-node P bi-critical.)

Conjecture d. For a generic G such that iG is not a graph of an invertible function, there exists $a \in \mathbb{R}$ such that for i_aG a homoclinic orbit exists (Namely for some periodic source x for $i_a(G)$ (or $(i_a(G))^{-1}$) there exists an i_aG-trajectory (y_j) such that $y_0 = x$, $y_j \to \mathrm{Orb}\,(x)$ and the trajectory (y_j) is different from $\mathrm{Orb}(x)$ - the periodic orbit of x).

Due to the genericity, the assertion of Conjecture d means in fact *transversally homoclinic*, which means that there are no critical pairs in the trajectory (y_j).

Theorem (Reduction Theorem). Conjecture A, for 1-cycle saddle-node bifurcation, follows from Conjecture a. Analogously C follows from c and D follows from d.

Let us finish this Section with the following rather trivial but important observation:

Proposition 1. In the case where the exists a noncritical point of the self-intersection of $i(G)$ there exists $a \in \mathbb{R}$ such that i_aG has a homoclinic orbit (see Conjecture d).

Proof. It is enough to consider a parameter a such that the self-interection point v is an $i_a(G)$-fixed point. We can consider the point v together with each of the branches of $i_a(G)$ intersecting at v. So we can treat v as two different points $v^{(1)}$ and $v^{(2)}$. So in the case $v^{(1)}$ is a source the trajectory $\ldots v^{(1)}, v^{(2)}, v^{(1)} \ldots$ is homoclinic.

(Observe that if $v^{(1)}$ is a source and $v^{(2)}$ is a sink then even Conjecture c(ii') holds) □

3. Outline of the proof of Reduction Theorem

We shall prove here only one of the cases of Reduction Theorem, the case which states that Conjecture C follows from c(i). The other cases can be handled similarly.

Pay attention to the following phenomena: 1. The shadowing procedure demands considering the family of transformations of the form $a \to i_aG$ (translation) rather than more general 1-parameter families. 2. When $\mu \downarrow b$, the corresponding parameter a (mod 1) runs around the circle $\mathbb{R}/\mathbb{Z}$ infinitely many times with growing speed. This explains why we consider a running in big intervals.

Lemma 1. For every $a \in \mathbb{R}$ there exist a sequence $\mu_n \downarrow b$ and a sequence of integers $k_n \to \infty$ such that for every $p,q \in G$ satisfying

$$(1) \qquad \pi_s(p)+a = \pi_u(q)$$

there exist sequences $p^{(n)} \to p$, $q^{(n)} \to q$ such that $f_{\mu_n}^{k_n}(p^{(n)})=q^{(n)}$.

Proof. Extend W^u to an f_μ-invariant foliation with C^r-leaves which continuously depend on points in $M \times \mathbb{R}$ in C^r-topology (r is large) on a neighbourhood of (P,b) and then to an f-invariant foliation $\mathcal{W}_s^u$ on a neighbourhood of $W^s(P) \times \{b\}$ and to an f-invariant foliation $\mathcal{W}_u^u$ on a neighbourhood of $W^u(P) \times \{b\}$. Analogously extend W^s to $\mathcal{W}_u^s$ and $\mathcal{W}_s^s$. It is easy to find μ_n, k_n, $p^{(n)}$, $q^{(n)}$ if p, q satisfying (1) are already (arbitrarily) chosen.

We shall prove that the choice of μ_n, k_n is appropriate to any other pair of points satisfying (1). Indeed $f_{\mu_n}^{k_n}(\mathcal{W}_s^u(p^{(n)})) \to W^u(q)$ so if one wants to replace q by an arbitrary point $q' \in W^u(q)$ one just needs to replace the points $p^{(n)}$ by adequate points from $\mathcal{W}_s^u(p^{(n)})$. To replace p by $p' \in W^s(p)$ one can proceed similarly with use of $\mathcal{W}_u^s$.

Finally suppose that p is replaced by $\phi_t(p)$ and q by $\phi_t(q)$, for $p,q \in W^c$. An easy case is when ϕ_t can be extended to a flow $\tilde{\phi}_t$ on a neighbourhood of (P,b) in $M \times \mathbb{R}$ such that $\tilde{\phi}_1 = f$. Denote by $\tilde{\phi}_t^s$ the extension of $\tilde{\phi}_t$ to a neighbourhood of $W^s(P) \times \{b\}$, by $\tilde{\phi}_t^u$ the extension of $\tilde{\phi}_t$ to a neighbourhood of $W^u(P) \times \{b\}$, so that $\tilde{\phi}_1^s(u) = f$. Then

$$f_{\mu_n}^{k_n}(\tilde{\phi}_t^s(p^{(n)})) = \tilde{\phi}_t^u(f_{\mu_n}^{k_n}(p^{(n)})) = \tilde{\phi}_t^u(q^{(n)}) \to \phi_t(q)$$

In the general case one can use the following fact which can be deduced from the Takens standard form of a germ of dynamics at a partially hyperbolic fixed point (see [T]):

In the generic case (no resonances in the hyperbolic part of the spectrum of $Df_b(P)$) for arbitrarily large integer r there exists a C^r-change of coordinates in some neighbourhood of (P,b) (with the parameter coordinate μ unchanged) such that in the new coordinates

$$f_\mu(x, y_1, \ldots, y_s, z_1, \ldots, z_u) = \Big(g_\mu(x), \sum_{i=1}^{s} a_{i1}(x,\mu)y_i, \ldots$$
$$\ldots, \sum_{i=1}^{s} a_{is}(x,\mu)y_i, \sum_{i=1}^{u} b_{i1}(x,\mu)\, z_i, \ldots, \sum_{i=1}^{u} b_{iu}(x,\mu)z_i\Big).$$

a_{ij}, b_{ij} are smooth functions, $(a_{ij}(0,b))$, $(b_{ij}(0,b))^{-1}$ are contracting matrices, $g_\mu(x)$ is a 1-dimensional saddle-node bifurcation.

Denote the projection to the coordinates x, μ by π. We have $\pi p^{(n)} \to \pi p$, $g_{\mu_n}^{k_n}(\pi p^{(n)}) = \pi q^{(n)} \to \pi q$. There exists an adapted smooth flow ψ such that $\psi_1 = g$ for $\mu = b$ and $\psi_1 - g$ is C^r-flat at $\pi(P,b)$. (i.e. the r-jet is 0) (see [N-P-T]). Now by Lemma 3.7 from [N-P-T] we have

$$\lim_{n\to\infty} g_{\mu_n}^{k_n}(\psi_t(\pi p^{(n)})) = \lim_{n\to\infty} \psi_{t+k_n}(\pi p^{(n)}) =$$

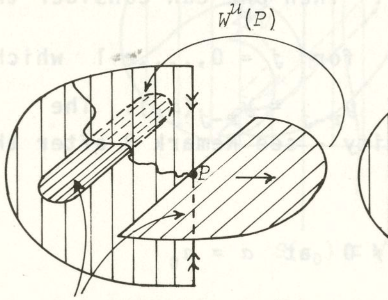
$$= \lim_{n\to\infty} \psi_t(g_{\mu_n}^{k_n}(\pi p^{(n)})) = \lim_{n\to\infty} \psi_t(\pi q^{(n)}) = \psi_t(\pi q).$$

Now one needs to find points $r^{(n)} \in \pi^{-1}(\psi_t(\pi p^{(n)}))$ such that $r^{(n)} \to \phi_t(P)$ and $f_{\mu_n}^{k_n}(r^{(n)}) \to \phi_t(q)$. This is easy with the help of the foliations $\mathcal{W}_{u(s)}^{u(s)}$. Proof of Lemma 1 is finished. $\square$

Every $i_a(G)$-trajectory (x_j) with the translation integers τ_j corresponds to a sequence of points $p_j \in G$ such that for every j $\pi_u(p_j) = x_j$, $\pi_s(p_j) + a + \tau_j = x_{j+1}$ (namely $p_j = (i_{a+\tau_j})^{-1}(x_j, x_{j+1})$). If (x_j) is a periodic source (or sink) trajectory of period m then by the above Lemma and considerations standard in the hyperbolic dynamics one can prove that the trajectory (p_j) can be shadowed by periodic f_{μ_n}-trajectories of periods $mk_n + \sum_{j=1}^m \tau_j$. These trajectories $(p_i^{(n)})$ are hyperbolic.

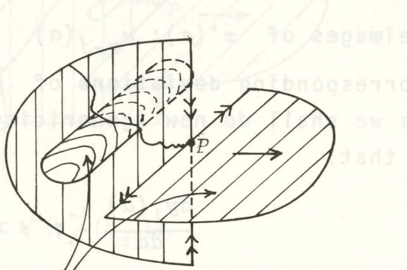
Observe now that a disc D_n in the unstable manifold $W^u(p_i^{(n)})$ for f_{μ_n}, is C^∞-close to the respective piece D of $W^u(P)$. D can be chosen independently of n, being a bounded part of $\pi_u^{-1}(d)$, where d is a small arc in the unstable set $W^u(x_j)$ centered at x_j.

[Remark 1: If the periodic source trajectory (x_j) contains no v-critical pair then $W^u(p_i^{(n)})$ contains a codimendion 1, very strong unstable invariant manifold at $p_i^{(n)}$, C^∞-close to the leaves of $\mathcal{W}_u^u$, see Fig. 5. In the case when some pairs are v-critical the unstable directions along W^s and $\mathcal{W}^u$ interchange positions roughly speaking, if $\dim W^u(P) = \dim W^s(P) = 2$. Then the expansion is very strong on the whole $W^u(p_i^{(n)})$, see Fig. 6.]



foliation $\mathcal{W}^u$ on $W^u(P)$

Fig. 5



foliation $\mathcal{W}^u$ on $W^u(P)$

Fig. 6

Assume (cond. (i)§.2) that $W^u(x_j)$ contains an h-critical point c. More exactly let (y_j) be an $i_a(G)$-trajectory, $j = 0,\dots,l$ such that $y_0 \in d$ and the pair $c = y_{l-1}, y_l$ is h-critical. Assume additionally (genericity) that

(2) no pair (y_j, y_{j+1}) is h-critical for $j = 0,\dots,l-2$.

It follows that for large n, $f^{k_n + \Sigma\tau_j}(D_n)$ contains a disc U_n C^∞-close to a disc U in $W^u(P)$ centered at the point $t = (i_{a+\tau_{l-1}})^{-1}(y_{l-1}, y_l)$, tangent at t to $\mathcal{W}^s$ (in a nondegenerate way, due to the Genericity Assumption).

Specify the above a as a_0. Assume (cond. (i)§.2) that the trajectory y_j can be continued for $j > l$ such that it hits a periodic source $x'(a_0)$ (i.e. $y_r = x'(a_0)$ for some $r \geq l$). Denote the corresponding shadowing f_{μ_n}-periodic points by $p'^{(n)}(a_0)$.

If the parameter a varies around a_0 one can consider smooth functions $x(a), x'(a)$ whose values are periodic sources for $i_a G$. The critical point c stays unchanged and it stays in $W^u(x(a))$.

Assume from now on (genericity) that no pair (y_j, y_{j+1}) for $j \geq 1$ is h-critical for $i_a G$. Then one can consider the $(i_a G)^j$-preimages of $x'(a)$: $y_{r-j}(a)$ for $j = 0,\dots,r-1$ which are the corresponding deviations of $y_{r-j} = y_{r-j}(a_0)$. The hypothesis we shall do now (genericity - see Remark 3 after the proof) is that:

$$\frac{dy_1(a)}{da} - 1 \neq 0 \quad \text{at} \quad a = a_0 \tag{3}$$

When a varies, $\mu_n(a)$ for which the Lemma holds can be chosen to depend on a continuously (k_n's do not depend on a). Then the $f_{\mu_n(a)}$-periodic points $p^{(n)}(a)$ and $p'^{(n)}(a)$, which correspond to the points $x(a)$ and $x'(a)$ respectively, change continuously. Then, for every n, $U_n(a)$ can be chosen to depend continuously on a in C^∞-topology. Consider a continuous family of discs $S(a)$ in $W^s(\pi_s^{-1}(z(a)))$ where $z(a) = y_1(a) - a - \tau_{1-1}$, centered close to the point t.

Hypothesis (3) implies

$$\frac{dz(a)}{da} \neq 0 \quad \text{at} \quad a = a_0 \tag{4}$$

For every large n consider a continuous family of discs $S_n(a)$, C^∞-close to $S(a)$, in the respective iteration of $f_{\mu_n(a)}$-preimage of $W^s(p'^{(n)}(a))$, see Fig. 7

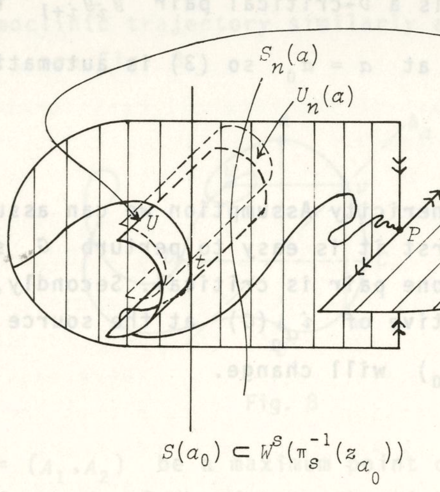


Fig. 7

In view of (4) for some $a_1 > a_0$ $U \cap S(a_1) = \emptyset$ and for some $a_2 < a_0$ $U \cap S(a_2) \neq \emptyset$ (or conversely) and the intersection is transversal. So the same holds for $U_n(a)$ and $S_n(a)$ if n is sufficiently large. Let a' be the largest a in the interval $\langle a_2, a_1 \rangle$ such that $U_n(a') \cap S_n(a') \neq 0$. Then $U_n(a')$ and $S_n(a')$ do not intersect transversally, otherwise the intersection would persist. This finishes the proof of the $c(i) \Longrightarrow C$ case of Reduction Theorem. $\square$

Remark 2. If there is more than one critical pair $y_j\ y_{j+1}$ for $j = 0,\dots,r-1$ on the $i_a G$-trajectory (y_j) the assumption which is sufficient (together with (3)) to prove a tangency for f_{μ_n} is that there is no triple of indices $i < j < k$ such that the pairs $y_i\ y_{i+1}, y_k\ y_{k+1}$ are h-critical and $y_j\ y_{j+1}$ is v-critical. Under this assumption one deals with $y_{1-1} y_1$ which is the last h-critical pair (The assumption (3) concerns then, of course, just the pair $y_{1-1} y_1$, to have sense).

(Observe that if there is a v-critical pair $y_i y_{i+1}$ for some $i \geq 1$ then $\frac{dy_1(a)}{da} = 0$ at $a = a_0$ so (3) is automatically satisfied.).

Remark 3. Due to the Genericity Assumption we can assume that (3) is always satisfied. First it is easy to perturb G so that along $y_j(a_0)$ exactly one pair is critical. Secondly, to have (3) true, change the derivative of $i_{a_0}(G)$ at the source x', if necessary. Then $\frac{dx'}{da}(a_0)$ will change.

4. Closed curves, multivalued mappings of an interval

We consider the case when $G = W^s(P) \cap W^u(P) \setminus \{P\}$ contains a closed curve δ.

Theorem 2. If G contains a closed curve δ then there exists $a \in \mathbb{R}$ such that for $i_a G$ a homoclinic intersection exists (see Conjecture d).

Proof. It is enough to find points $u = (u_1,u_2)$, $v = (v_1,v_2)$, $w = (w_1,w_2) \in i(\delta)$ such that

(1) $$u_1 = v_1,\ v_2 = w_2 \quad \text{and}$$

(2) $$v_2 - u_2 = v_1 - w_1.$$

If this is achieved one can take a such that v is $i_a(\delta)$-fixed. (Formally this point is (v_1,v_2+a) now rather than (v_1,v_2). So in order to avoid this complication of notation we shall assume that the diagonal Δ_a of the coordinate system in $\mathbb{R}$ moves with the change of a and $i_a G$ stays unmoved.) Then with use of v and the period two trajectory u,w one

finds the homoclinic trajectory similarly as in Remark 1§3: $v,v,u,w,v,\ldots$, see Fig. 8.

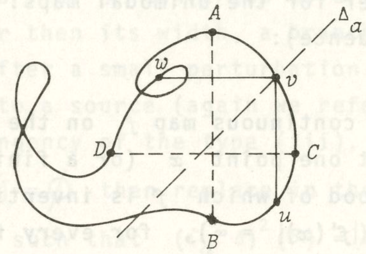


Fig. 8

Let $A = (A_1,A_2)$ be a maximum point of $i\delta$ (maximal A_2). Let us walk from A along $i\delta$ to the right. Denote by B the first point we meet (different from A) with $B_1 = A_1$. Let $C = (C_1,C_2)$ be a point belonging to the curve AB with maximal C_1 (right hand extremum in AB). Finally move from A along $i\delta$ to the left and denote by D the first point with $D_2 = C_2$. (We can assume by genericity of δ that C and A are strict extrema in the respective curves.

Observe that all the triples $(u,v,w) \in \mathbb{R}^6$ satisfying (1), such that $v \in AC$, $u \in CB$, $w \in DA$, form a family of smooth curves immersed in $\mathbb{R}^6$. These curves have two ends only: (B,A,A) and (C,C,D). Hence the family consists of one curve η with these ends and possibly some closed curves ("bubbles" – compare this with Figures 22-24 in Sec. 5) At (B,A,A) $v_2-u_2 > v_1-w_1 = 0$. At (C,C,D) $v_2-u_2 = 0$ and we can assume that $v_1-w_1 > 0$ (Otherwise, if $v_1-w_1 \leq 0$, δ would have a selfintersection point and we would have the case considered in Remark 1 §.3) By Darboux Theorem (that continuous functions take all intermediate values) there exists $(u,v,w) \in \eta$ satisfying property (2). $\square$

Remark 1. If one can find u,v,w which satisfy the properties (1) and (2) such that v is a sink and (u,w) is a source (or conversely) then clearly Conjecture c(ii) (hence Conjectures B,C) is true.

In the sequel we will show that Conjecture c(iii) reduces sometimes to the following variant of the well known conjecture that for any map on the interval one can produce sinks by a small perturbation (or stronger for the unimodal maps: that one can change the kneading sequence):

Conjecture e. For every continuous map f on the interval I, smooth except on at most one point x (or a finite number of points) in a neighbourhood of which f is invertible and the inverse map is smooth ($|f'(x)| = \infty$), for every finite number of points $y_1,\dots,y_m$ and every point z, there exists a C^∞ arbitrarily small perturbation g of f such that the sequence $g^n(z)$ converges to a sink and the 1-jets of f and g are the same at the points $x,y_1,\dots,y_m$ (C^∞-small perturbation at x means that the inverse functions are C^∞-close to each other).

Now we shall prove

Theorem 3. If G contains a closed curve δ such that i is an embedding on δ and $i(\delta)$ bounds a convex set in $\mathbb{R}^2$, then Conjecture e implies appearence of a tangency (iii) for some $a \in \mathbb{R}$ (see Conjecture c(iii)) for an arbitrarily small perturbation of the family f_μ, hence failure of mild stability for f_μ (see Conjecture B).

Proof. We can assume that the height of $i\delta$ is larger than the width (i.e. $\sup(i\delta)_2 - \inf(i\delta)_2 > \sup(i\delta)_1 - \inf(i\delta)_1$. Otherwise we would consider $(i\delta)^{-1}$.

Denote the right hand extremum, the maximum, the left hand extremum and the minimum points of $i\delta$ by A,B,C,D respectively (by the genericity we can assume that these are strict extrema). As in Proof of Theorem 2 instead of moving $i_a\delta$ with the changing a we consider the diagonal Δ_a moving. We shall use the convention that any arc KL in $i\delta$ goes from K to L counterclockwise.

We shall assume that for $a = 0$ $\Delta_0 \ni B$. Take $a = a_0 > 0$ such that $i_{a_0}(\delta)(B) = A$. If $i_{a_0}(\delta)(A) \neq \emptyset$ then, due to the convexity, the arc AC is a graph of a map of the interval C_1A_1 into itself. So, if Conjecture e holds, after a small perturbation a forward trajectory of A converges to a sink. Since the height of $i\delta$ is greater then its width, a branch of $(i_a\delta)^{-1}$ maps BD into itself, so after a small perturbation a backward trajectory of B converges to a source (again we refer to Conjecture e). So we obtain a tangency of the type (iii).

If $i_{a_0}(\delta)(A) = \emptyset$ then replace in the consideration a_0 by a_1, $0 < a_1 < a_0$, such that $(i_{a_0}\delta)^2(B) = C$. Then again refer to Conjecture e. (see Fig. 9.)

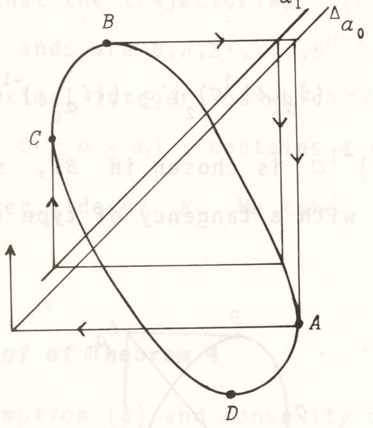


Fig. 9

Remark 2. The use of convexity was rather mild. Is it possible to omit it at all without significant change of the above proof?

Remark 3. The property of the convergence of the trajectories of A,B,C to sinks under iterations of $i\delta$ or $(i\delta)^{-1}$ in the above proof is persistent. So this property could be included into the genericity property. So relying on Conjecture e, assumed in the statement of Theorem 3 that G is generic we can assert that the Conjecture c(iii) and Conjecture C are true for the family f_μ itself

The rest of this section will be devoted mainly to proving appearence of a homoclinic tangency or a tangency of type (i) for $i_a G$ for some $a \in \mathbb{R}$ (see Conjectures a and c(i)), under the assumption that $i\delta$ is convex and some additional assumptions.

Theorem 4. If $i\delta$ bounds a convex set such that the height of $i\delta$ is greater than its width and Δ_{a_0} intersects BD at a point $E = (E_1, E_2)$ such that the derivative of $i_{a_0}(\delta)$ at E_1, for the value E_2, satisfies:

$$(3) \qquad \left|\frac{d(i_{a_0}\delta)}{dx}(E_1)\right| > 1$$

and such that

$$(4) \qquad |C_2 - (i_{a_0}\delta)^{-1}C)_2| \geq ((i_{a_0}\delta)^{-1}C)_1 - C_1$$

(the preimage $(i_{a_0}\delta)^{-1}C$ is chosen in BD, see Fig. 10), then there exists $a \in \mathbb{R}$ with a tangency of type (i) for $i_a G$ (see Conjecture c(i)).

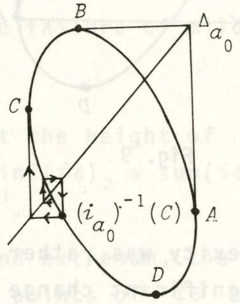


Fig. 10

Proof. Let K be the rectangle

$$K = \{(a,y): 0 \leq a \leq A_1 - B_1 = a_0, \quad D_2 \leq y \leq B_2\}$$

Claim $\bigcup_{n \geq 0} \{(a, ((i_a(\delta))^n(B))_2): 0 \leq a \leq a_0\}$ contains a continuous curve γ joining the upper side of K with the lower side of K.

Outline of Proof of the Claim (The proof in detail will be contained under assumptions weaker than connectivity, in the Proof of Lemma 2):

Let γ_1 be the curve $\{(a, i_a\delta(B)_2): 0 \leq a \leq a_0\}$ in K joining $(0, B_2)$ with $(0, B_2^1)$, where B^1 is the $i_0\delta$-image of B different from B. By induction define B^n as the $i_0\delta$-image of B^{n-1} lying in the curve DB, if such an image exists. Let $\gamma_n \subset \{(a, (i_a\delta)^n(B)_2): 0 \leq a \leq a_0\}$ be the curve joining $(0, B_2^{n-1})$ with $(0, B_2^n)$ such that the trajectories $(((i_a\delta)^k(B))_2)_{k=0}^n$ corresponding to its ends are $B, B, B^1, \ldots, B^{n-1}$ and $B, B^1, \ldots, B^n$. If $n = N$ is the maximal integer for which B^n exists, then $\{(a, ((i_a\delta)^{N+1}(B))_2): 0 \leq a \leq a_1\}$ contains a curve γ_{N+1} joining $(0, B_2^N)$ with the lower side of K. We take $\gamma = \bigcup_{k=1}^{N+1} \gamma_k$, see Fig. 11. □

We shall end now Proof of Theorem 4

Due to the assumption (3) and convexity of BE (cf. Remark 4 which will follow) for every a such that $0 \leq a \leq a_0$ there exists a unique $i_a\delta$-fixed point $z(a)$ in BD which is a source (a sink for $(i_a\delta)^{-1}$). The curve $\zeta = \{a, z(a)_2\}$ is continuous and joins the left and right sides of K. The conclusion is that ζ and γ have a point of intersection for some $a = a_1$. Observe finally that $B \in W^u(x)$ for a periodic source $x \in BD$ for $i_{a_1}\delta$. Indeed, if $z(a_1) \in BC$ then obviously $B \in W^u(z(a_1))$. If $z(a_1) \in CD$ then (4) for a_0 implies the similar inequality for a_1 (due to the convexity of CD, this is a simple exercise). But this implies for the branch BD of $(i_{a_1}\delta)^{-1}$, which we denote by F, that $F(CF(C)) \subset CF(C)$. Since F is monotone on $CF(C)$ forward iterations F^n on every point of BD except a finite number of periodic sources converge to an orbit of a sink. In particular $F^n(B)$ converge to an orbit of an F-sink x (maybe after a small perturbation of F). So $B \in W^u(x)$ for the $i_{a_1}\delta$-source x. $\square$

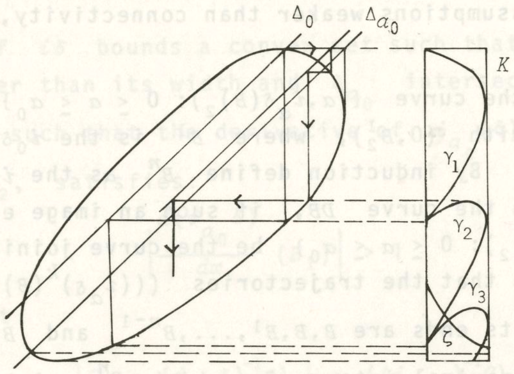


Fig. 11

Example. Let $i\delta$ be an ellipse. We shall prove that the assumptions of Theorem 4 are satisfied.

Denote the ellipse by ∂E and its extremal points by A,B,C,D as before (see Fig. 12)

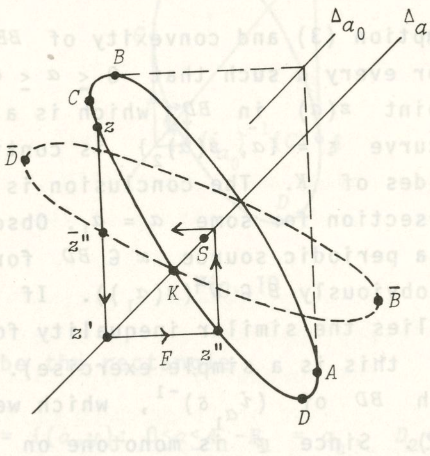


Fig. 12

Assume:

(5) the height is larger than the length $(B_2-D_2 > A_1-C_1)$

Let Δ_{a_2} be the diagonal passing through the centre S of ∂E. For every set X denote the set symmetric to X with respect to Δ_{a_2} by $\bar{X}$. Denote by E the closed region bounded by ∂E. Denote by K a point of intersection of Δ_{a_2} with the arc BD in ∂E.

We can assume that $K \in CD$, since if $K \in BC$ the assumptions of Theorem 4 are obviously satisfied. Due to (5) $\bar{D}_1 < C_1$, so $\bar{D} \notin E$. So for the arc $\bar{D}K$ in $\partial \bar{E}$, $\bar{D}K \cap E = \{K\}$. (Otherwise $\bar{D}K$ must intersect $BD \subset \partial E$ in at least one point $K^1 \neq K$. $\bar{K}^1 \in KD$ is also a point of the intersection of ∂E with $\partial \bar{E}$ and $\bar{K}^1 \neq K^1$, if not then we would have $K^1 = K$.

$K, K^1, \bar{K}^1 \in BD$ so the points symmetric to them with respect to S are different and also in $\partial E \cap \partial \bar{E}$. We have 6 points of intersection of the ellipse ∂E with $\partial \bar{E}$, but at most 4 are possible. We have a contradiction.)

Denote the branch BD of $(i_{a_2}\delta)^{-1}$ by $F(y)$. We have just proved that

$$\frac{dF}{dy}(K) \geq -1 \tag{6}$$

Let z be any point in CK and zz' be the vertical interval joining z with Δ_{a_2} and let $z'z''$ be the horizontal interval joining z' with BD. Since $\bar{z}'' \in zz'$ we have length $(zz') \geq$ length $(z'z'')$. Of course $z'' = F(z)$. The conclusion is that $F \circ F(z) \in zD \subset CD$.

The same holds for Δ_{a_0} since it is above Δ_{a_2} and because of the convexity of CD we have the strict inequality in (6) corresponding to Δ_{a_0}. So the assumptions of Theorem 4 hold, even better: $B \in W^u(E)$ for $i_{a_0}\delta$ so we have the homoclinic tangency phenomenon (Conjecture a). $\square$

Corollary 1. Let $f_{\xi,\mu}$ be a generic two-parameter family of diffeomorphisms satisfying (*1)-(*4) for every $\xi > \xi_0$, and satisfying (*1)-(*3) for $\xi = \xi_0$ with $W^u(P) \cap W^s(P) \setminus \{P\}$ consisting of exactly one orbit w (the intersection of $W^u(P)$ with $W^s(P)$ must be nontransversal there, see Fig. 13)

Then for every $\xi > \xi_0$ if ξ is sufficiently close to ξ_0 there exists $a \in \mathbb{R}$ such that for $i_a G$ with $G = G(\xi)$ given by $f_{\xi,b(\xi)}$ a homoclinic tangency happens (Conjectures a and A hold).

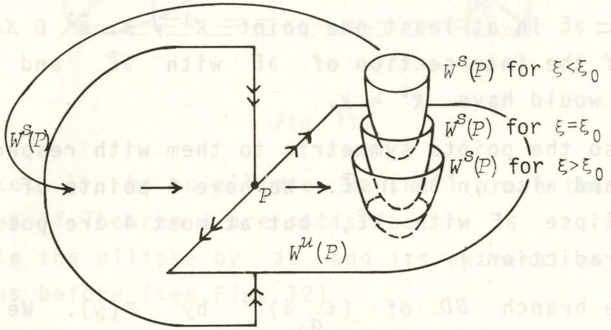


Fig. 13

Proof. For $\xi > \xi_0$, $\xi \approx \xi_0$. $W^u(P) \cap W^s(P) \setminus P$ is an orbit of a closed curve δ. $i\delta$ is C^∞-close to an ellipse if ξ is close to ξ_0. So we have the situation like in the Example above and we can refer to Theorem 4.

Remark 4. In Proof of Theorem 4 we need the convexity of BC in order to know that ς is continuous. Without the convexity assumption the situation drawn on Fig. 14 can happen

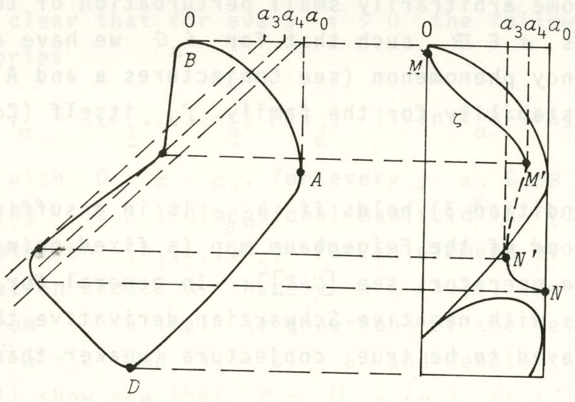


Fig. 14

The fixed point line MN on this Figure is a sink line between M' and N'.

To avoid such kind of troubles for iterations (for periodic, not only fixed, points) one can assume that the Schwarzian derivative of the branch BD of $(i\delta)^{-1}$ is negative. This will allow to omit the assumption (4) in Theorem 4 and almost omit the assumption (3).

Theorem 5. Assume that G contains a closed curve δ and the following assumptions are satisfied (we use the same notation as in Proof of Theorem 3):

1) The whole curve DB is below the diagonal Δ_0;

2) BD as a branch of $(i_a\delta)^{-1}$ is a graph of a function h_a with exactly one minimum and with negative Schwarzian derivative;

3) If there exists $a < a_0$ such that the kneading sequence for h_a is "Feigenbaum-like", then there exists $a_5 < a_0$ also with this property and such that there exists an arbitrarily small perturbation of BD which increases this kneading sequence.

Then, for some arbitrarily small perturbation of the family f_μ, there exists $a \in \mathbb{R}$ such that for $i_a G$ we have a homoclinic tangency phenomenon (see Conjectures a and A), hence failure of mild stability for the family f_μ itself (Conjecture B).

Remark 5. The condition 3) holds if h_{a_5} is in a sufficiently small neighbourhood of the Feigenbaum map (a fixed point for a renormalization operator, see [*C-E*]). In general for the unimodal mappings with negative Schwarzian derivative this is a well-known, believed to be true, conjecture (weaker than Conjecture e).

Remark 6. Again as in Remark 3 observe that if one knows that property 3) is always true, then for a generic G one can increase the kneading sequence of h_{a_5} just by an arbitrarily small increase of a_5. So relying on 3) one can assert in Theorem 5 that homoclinic tangency happens for the family f_μ itself.

To prove Theorem 5 we will follow the same ways as in Proof of Theorem 4. First, under the assumption 1) we shall prove

Lemma 2. $\bigcup_{n \geq 0} \{(a, (i_a \delta)^n (B)_2) : 0 \leq a \leq a_0\}$ contains a continuous curve γ joining the upper and lower sides of K (We do not assume convexity of DB while in the Claim we did!)

Proof. Assume for the simplicity of the notation that DB in $i\delta$ has no selfintersection points. We shall consider curves in $(\mathbb{R}^2)^n$ for positive integers n and the same curves in $(\mathbb{R}^2)^\infty = \{(\ldots z_{-\underline{n}}, \ldots, z_{\underline{0}})\} : z_{\underline{i}} \in \mathbb{R}^2$ where $(\mathbb{R}^2)^n$ is embedded into $(\mathbb{R}^2)^\infty$ by $\varepsilon_n(z_{\underline{1}}, \ldots, z_{\underline{n}}) = (\ldots, B, B, z_{\underline{1}}, \ldots, z_{\underline{n}})$. (We underline the indices to make a distinction between the indices 1 or 2 which denote the first, or the second coordinate of a point.) Denote by $\pi_{i_1, i_2, \ldots, i_k}$ the projection of $(\mathbb{R}^2)^n$ to the coordinates $i_1, \ldots, i_k$ $(1 \leq i_1, \ldots, i_k \leq n)$.

It is clear that for every $n > 0$ the following set of i_a-trajectories

$$\tilde{\gamma}_n = \{(z_{\underline{1}}, \ldots, z_{\underline{n}}) : (z_{\underline{j}}) \text{ is an } i_a\delta\text{-trajectory}$$

for some a with $0 < a < a_0$, for every j $z_{\underline{j}} \in DB$ and $z_{\underline{1}} \in (i_a\delta)(B)$ is a 1-dimensional manifold in $(\mathbb{R}^2)^n$. (Observe that the notation $(z_{\underline{1}}, \ldots, z_{\underline{n}})$ for an element of $\tilde{\gamma}_n$ contains the information about a. Namely $a = (z_{\underline{1}})_1$. For every $n > 1$ subtract from $\tilde{\gamma}_n$ a small neighborhood of the set of trajectories which have $a = 0$ and $z_1 = z_2 = B$ and denote the rest by γ_n.

We will show now that $\Gamma = \bigcup_{n > 0} \varepsilon_n(\gamma_n)$ is a 1-dimensional (topological manifold in $(\mathbb{R}^2)^\infty$ and we will analyze its endpoints.

Observe that the endpoints of the components of each $\tilde{\gamma}_n$ are exactly the $i_a\delta$-trajectories $(z_{\underline{1}}, \ldots, z_{\underline{n}})$ such that either

$$a \neq 0 \text{ and there exists } k : 0 < k \leq n, \text{ for which}$$

$$z_{\underline{k}} = D \text{ or } z_{\underline{k}} = B$$

$$\text{or} \quad a = 0$$

(for $a = 0$ we always assume to have an end point in a generic situation: no turn back points for $z_{\underline{k}}$ at $a = 0$ can happen).

No point $z \in \tilde{\gamma}_n$ with $a \neq 0$ is contained in $\varepsilon_k \tilde{\gamma}_k$ for $k \neq n$, or course. Exclusively for $k > n \geq 1$ and $a = 0$, $\varepsilon_n \tilde{\gamma}_n$ has common points with $\varepsilon_k \tilde{\gamma}_k$, namely the points of the form

$$\varepsilon_n(z_{\underline{1}}, \ldots, z_{\underline{n}}) = \varepsilon_k(\underbrace{B, \ldots, B}_{n-k \text{ times}}, z_{\underline{1}}, \ldots, z_{\underline{n}}) = (\ldots B, z_{\underline{1}} \ldots z_{\underline{n}})$$

If $\tilde{\gamma}$ is replaced by γ only the curves $\varepsilon_n \gamma_n$ and $\varepsilon_{n+1} \gamma_{n+1}$ meet like that. So Γ does not branch here, hence it is a manifold. The points $(\ldots B, B, z_{\underline{1}}, \ldots, z_{\underline{n}})$ for $n \geq 1$, $z_{\underline{1}} \neq B$ are not its end points.

Consider the component Γ' of Γ containing (starting at) the point $(\dots B,B,B) = \varepsilon_1(B)$. Γ' consists of the ε_n-images of components of the sets γ_n. Numerate these components by $\gamma_{n_1,i_1}, \gamma_{n_2,i_2}, \dots$ ($n_1 = 1$, γ_{n_1,i_1} starts at B). Later on it will be shown that in fact this sequence is finite.

By the construction all the endpoints of each γ_{n_j,i_j} except the end of the last γ corresponding to the end of Γ' have $a = 0$. Observe that for every $j > 1$ with $n_j > 1$, for every k with $1 \leq k < n_j$, there exists $t < j$ such that

$$(7) \qquad \pi_{1,\dots,k}(\gamma_{n_j,i_j}) \subset \gamma_{n_t,i_t}.$$

This can be easily seen by induction over j for every fixed difference $n_j - k$: Namely (7) for $j > 1$, $k \geq 1$ follows from (7) for $j-1$, $k+n_{j-1}-n_j$ (if $k+n_{j-1}-n_j \geq 1$). At the begin of each inductive sequence, for $k = 1$ and $n_{j-1}-n_j = -1$, we have $\pi_1(\gamma_{n_j,i_j}) \subset \gamma_{n_1,i_1}$.

Let us parametrize Γ' by $\phi\colon R^+ \to \Gamma'$ starting from the point $(\dots,B,B,B) = \varepsilon_1(B)$ (or by an interval $[0,t_0]$ if Γ' consists only of finite number of $\varepsilon_{n_j}(\gamma_{n_j,i_j})$ which will turn out to be the case in fact). If $\phi(t) = \varepsilon_n(\underline{z}_1,\dots,\underline{z}_n)$ we shall denote $n = n(t)$, $a = (\underline{z}_1)_1 = a(t)$ $n(t)$ is well defined except for $a = 0$. Denote $\varepsilon_{n_j}^{-1}\phi$, a parametrization of γ_{n_j,i_j}, also by ϕ.

Let $t = t_1$ be the parameter for which $\pi_{n(t)}\phi(t)$ has its right hand extremum for the first time (i.e. $\sup(\pi_{n(t)}\phi(t))_1$). Denote the point $\pi_{n(t_1)}(\phi(t_1))$ by E. Of course $E \neq B$. Observe also that $n(t_1) = 1$. Indeed, always $a(t) \geq (\pi_{n(t)}\phi(t))_1$, so $a(t_1) \geq E_1$. If $n(t_1) \geq 2$, in view of (7) we would have $\pi_1\phi(t_1) = \phi(t')$ for some $t' < t_1$. So of course $n(t') = 1$, $a(t_1) = a(t')$. So $a(t') \geq E_1$. But t_1 was the first parameter with this property, we have a contradiction.

(This argument proves also that the right hand extremum is really obtained for some point of Γ'. Indeed if it is approached by a sequence $(\pi_{n(t^{(m)})}(\phi(t^{(m)})))_1$ (with maybe $n(t^{(m)}) \to \infty$), then $\liminf a(t^{(m)}) \geq E_1$. But $a(t^{(m)}) = (\pi_1(\phi(t^{(m)})))_1 = (\pi_1(\phi(t^{(m)'})))_1$ for some $t^{(m)'}$ such that all $\phi(t^{(m)'})$ (for sufficiently large m) belong to the component of γ_1 containing E, so to the same γ_{n_j,i_j} $(n_j = 1)$. Take $t_1 = \lim t^{(m_s)'}$ for a subsequence m_s.)

We shall prove now that for every $t > t_1$

$$(8) \qquad \pi_{n(t)}\phi(t) \in DE \setminus E$$

Of course if $t > t_1$ and t is near t_1 then (8) holds. Suppose that (8) is false for some $t > t_1$. Then let $t = t_2$ be the first $t > t_1$ with $\pi_{n(t)}(\phi(t)) = E$. Then it must be $\pi_{n(t_2)-1}(\phi(t_2)) = B$. (Otherwise $a_{t_2} = (\pi_1(\phi(t_2)))_1 = (\phi(t'))_1 > E_1$ for some $t' < t_2$ in view of (7), a contradiction).

We have $n(t_2) \neq 1$, otherwise $t_2 = t_1$. Observe that $a(t_2) \neq 0$ (if $= 0$ then $(\pi_{n(t_2)}\phi(t))_1 \leq B_1 < E_1$). But then, in view of (7) $\pi_{1,\dots,n(t_2)-1}(\phi(t_2)) = \phi(t'') \in \Gamma'$ for some $t'' < t_2$ $a(t'') \neq 0$ and $\pi_{n(t'')}(\phi(t'')) = B$. So $\phi(t'')$ is an end point in Γ, a contradiction. (8) is proved.

Observe that the set $\{n_j\}$ is bounded from above. To see that denote $N = \max\{n(t) : t \leq t_1\}$; for every point $\varepsilon_n(\underline{z}_1,\dots,\underline{z}_n) \in \Gamma'$ with $a=0$, since $\underline{z}_k \in DE$ for $k > N$, we have

$$B - (z_{\underline{n}})_2 \geq (n-N)\operatorname{dist}(\Delta_0, DE).$$

Hence $n < N + \dfrac{\operatorname{dist}(\Delta_0, DE)}{B_2 - D_2}$.

Since for every n, for $a = 0$, there exists only a finite number of trajectories of B of length n, the number of $\varepsilon_{n_j}\gamma_{n_j, i_j}$ in Γ' is finite. So Γ' has a second end point (the first one is $(\dots, B, B))$ $\phi(t_3)$.

By (8) $\pi_{n(t_3)}(\phi(t_3))$ is far from B. The same holds for $\pi_k(\phi(t_3))$ for every $k \geq 2$, otherwise in view of (7) Γ' would end at $t' < t_3$. The conclusion is that $\pi_{n(t_3)}\phi(t_3) = D$, hence the curve $(a(t), (\pi_{n(t)}\phi(t))_2)$ for $\phi(t) \in \Gamma'$ joins the upper and lower sides of K. □

Proof of Theorem 5. Consider first the case when for every parameter $a < a_0$ the kneading sequence for h_s is smaller than the Feigenbaum-like (then because of the genericity we can assume this also for $a = a_0$). We shall prove that in the set $\{(a, z_2) : z$ is a periodic sink for h_a, $0 \leq a \leq a_0\}$ (h_a was defined in the assumption 2)) there exists a curve ζ joining the left and right sides of K:

It is known that every infinite kneading sequence, smaller than the Feigenbaum-like one, is periodic. The critical point c is attracted then either to a unique periodic sink orbit or to a "flip" bifurcation neutral periodic orbit x_a. So, since the periods of the x_a's are bounded, for a generic δ there exists a sequence $b_1 < \dots < b_k$ with $b_1 = 0$, $b_k = a_0$ such that on every interval $(b_i b_{i+1})$ the kneading sequence is constant and for every b_i $i = 2, \dots, k-1$ there happens a period doubling or the inverse to a period doubling bifurcation, see Fig. 15.

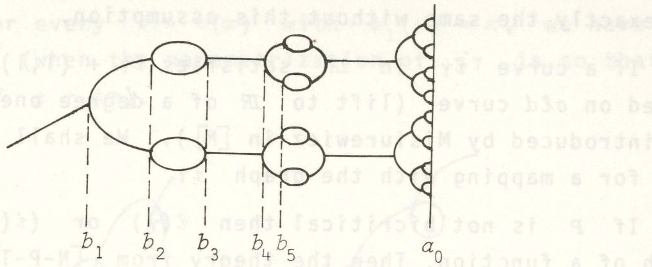


Fig. 15

Due to this description one can find a demanded curve very easily.

ζ intersects γ constructed in Lemma 2 at a point with $a = a_1$. Of course $h^n_{a_1}(B)$ converge to the sink h_{a_1}-orbit x_{a_1}, for generic δ.

Concluding, for $i_{a_1}\delta$ $B \in W^u(x_{a_1})$ for a source x_{a_1} and $(i_{a_1}\delta)^m(B) = x_{a_1}$ for some $m > 0$. We have the phenomenon (i) homoclinic tangency).

In the case when the parameter a_5 described in 3) exists we can obtain due to the assumption 3), by a small perturbation of h_{a_5} to h', a nonperiodic, eventually periodic kneading sequence. Then the forward h'-trajectory of C hits a source and, due to the nonexistence of homtervals, $C \in W^u(x)$ for h'. We obtain the situation of (i') (homoclinic tangency). □

5. Multivalued, degree one mappings on a circle

Assume that $W^s(P) \cap W^u(P)$ contains a closed curve γ going through P. So $i(\gamma)$ (more exactly $i(\gamma \setminus P)$) is a connected curve, imbedded in $\mathbb{R}^2$, such that there exists an integer m, for which $i(\gamma) + (m, m) = i(\gamma)$. To simplify the notation we shall assume that $m = 1$ (for example this is the case when $W^s(P) \cap W^u(P)$

contains only one closed curve going through P). All the proofs are exactly the same without this assumption.

If a curve $i\gamma$ in $\mathbb{R}$ satisfies $i\gamma + (1,1) = i\gamma$ it is called an *old* curve (lift to $\mathbb{R}$ of a degree one map, the term was introduced by Misiurewicz in [M]). We shall use the same term for a mapping with the graph $i\gamma$.

If P is not bicritical then $i(\gamma)$ or $(i(\gamma))^{-1}$ is a graph of a function. Then the theory from [N-P-T] proves that Conjecture a (homoclinic tangency), hence Conjecture A, are true.

In the next theorem we shall consider a larger class of $i\gamma$'s including some bicritical situations, namely the so-called *upper (lower) heavy curves*.

First for any curve η (or graph of a multivalued mapping η) parametrized by $\phi(t) = (\phi_1(t),\phi_2(t))$, $t \in \mathbb{R}$, define an *upper function* $F_u(\eta)$ by:

$$F_u(\eta)(x) = \sup\{\phi_2(t): \phi_1(t) \le x,\ t \in \mathbb{R}\}$$

and a *lower function*

$$F_\ell(\eta)(x) = \inf\{\phi_2(t): \phi_1(t) \ge x,\ t \in \mathbb{R}\}.$$

We call and old curve η *upper heavy* if the function $F_u(\eta)$ is continuous and *lower heavy* if $F_\ell(\eta)$ is continuous (See Fig. 16).

To produce a homoclinic tangency one needs to have at least one critical point. Here we shall assume about a continuous upper (lower) function that it has at least one "plateau". The formal condition is

$$(0) \qquad \text{graph } (F_u(i\gamma)) \not\subset i\gamma$$

Remark 1. It is easy to prove that (0) is true if an upper heavy $i\gamma$ has at least one strict maximum and either it has no self-intersections or it is upper heavy in the following stronger sense:

for every $x \in \mathbb{R}$ and $t(x)$ being the first parameter when $\phi_1(t(x)) = x$, for every $t > t(x)$ with $\phi_1(t) = x$, we have $\phi_2(t) \le \phi_2(t(x))$ (when the parametrization of $i\gamma$ is so that $\phi_{1(2)}(t) \to \pm\infty$ if $t \to \pm\infty$).

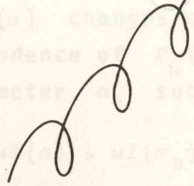


Fig. 16(a)

An upper heavy not lower heavy curve, satisfying (0)

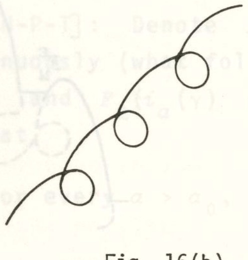


Fig. 16(b)

The condition (0) is not satisfied.

Theorem 6: If $i\gamma$ (or $(i\gamma)^{-1}$) is upper heavy (or lower heavy) and satisfies the condition (0), then there exists $\alpha \in \mathbb{R}$ with a homoclinic tangency for $i_\alpha G$ (Conjectures a and A).

Proof. Parametrize $i\gamma$ by $\phi(t) = (\phi_1(t),\phi_2(t))$, so that $\phi_i(t+1) = \phi_i(t) + 1$, and $\phi_i(t) \to \pm\infty$ for $t \to \pm\infty$, $i = 1,2$. Let $\tau_j + k$, $j=1,\dots,J$, $k \in \mathbb{Z}$ be the set of all local maxima of the function $\phi_2(t)$. For every j choose *small* numbers $\varepsilon_j^0, \varepsilon_j^1 > 0$ such that $\phi_2(\tau_j - \varepsilon_j^0) = \phi_2(\tau_j + \varepsilon_j^1)$. Replace in the curve $i\gamma$ the pieces $\phi|{<}\tau_j - \varepsilon_j^0, \tau_j + \varepsilon_j^1{>} + k$ by, the "plateau"'s $(\phi_1, \phi_2(t_0 - \varepsilon_j^0) + k)$. Denote the resulting curve with the flattened (truncated) maxima by $i\gamma^\varepsilon$ $(\varepsilon = (\varepsilon_1^0,\dots,\varepsilon_J^0))$. Then define a mapping $F_u^\varepsilon: \mathbb{R} \to \mathbb{R}$ by

$$F_u^\varepsilon(x) = \max(F_u(i\gamma^\varepsilon)(x),\ \sup\{\phi_2(t):\phi_1(t) = x\})$$

(If $i_\alpha\gamma$, with varying α, is considered we denote F_u^ε by $F_u^\varepsilon(i_\alpha\gamma)$) (Remind that we consider a generic γ, in particular all the v- and h-critical values are pairwise distinct. So, for all ε_j^0 sufficiently small $i\gamma^{\varepsilon}$ is upper heavy, hence F_u^{ε} is continuous, see Fig. 17)

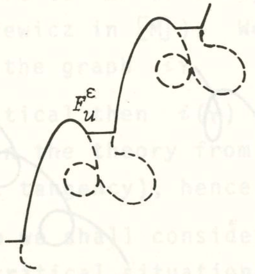


Fig. 17

For any old curve in $\mathbb{R}^2$ which is a graph of a continuous mapping h, or for h itself, denote by $I(h)$ its *rotation interval* (see [N-P-T] for the definition) and by $\ell I(h)$, $uI(h)$ its left and right ends. If $I(h)$ reduces to one number we use the term *rotation number* and the notation $\rho(h)$. Observe that if we smoothen the points of the nondifferentiability of F_u^{ε} replacing it by the function $\tilde{F}_u^{\varepsilon} \geq F_u^{\varepsilon}$ (see Fig. 18) then $I(\tilde{F}_u^{\varepsilon}) \subset I(F_u^{\varepsilon})$.

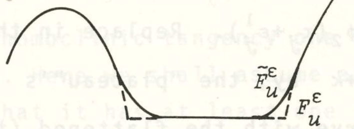


Fig. 18

(This follows from the facts that always $\ell I(h) = \rho(F_\ell(h))$ $uI(h) = \rho(F_u(h))$, if $h_1 \leq h_2$ are monotonic functions then $\rho(h_1) \leq \rho(h_2)$, and in our construction $F_u(F_u^{\varepsilon}) = F_u(\tilde{F}_u^{\varepsilon})$, $F_\ell(F_u^{\varepsilon}) \leq F_\ell(\tilde{F}_u^{\varepsilon})$.)

The function $\tilde{F}_u^{\varepsilon}$ satisfies the assumptions of the Block-Franke Theorem [B-F] (it has at least one strict maximum, since in view of (0) F_m^{ε} has at least one "plateau"), which asserts that for no $a \in \mathbb{R}$, $I(\tilde{F}_u^{\varepsilon}(i_a\gamma))$ reduces to a single irrational number. The conclusion is that the same is true for $I(F_u^{\varepsilon}(i_a\gamma))$.

Now one can proceed as in [N-P-T]: Denote $I(F_u^{\varepsilon}(i_a\gamma))$ by $I(a)$. Since $I(a)$ changes continuously (what follows from the continuous dependence of $F_u(i_a\gamma)$ and $F_\ell(i_a(\gamma)$ on a), one can find a parameter a_0 such that

$$(1) \qquad uI(a) > uI(a_0), \quad \text{for every } a > a_0,$$

and $I(a_0)$ consists of more than a single number. So there exists a rational number $\frac{r}{s}$ with

$$(2) \qquad \frac{r}{s} \in \operatorname{int} I(a_0).$$

For a generic γ every $i_a\gamma$-periodic point of period s is hyperbolic except a finite set A of the parameters a. Since $uI(a)$ depends continuously on a, there exists an infinite set of parameters arbitrarily close to a_0 $(>a_0)$ for which $uI(a)$ is not locally constant from the right hand side. So one can find a parameter $a_1 > a_0$, $a_1 \in A$, such that (1) holds for $a = a_1$ and $\frac{r}{s} \in I(a_1)$. So, there exists a point $x \in \mathbb{R}$, which is periodic under $F_u(i_{a_1}\gamma)$, of period s and rotation number $\frac{r}{s}$ and such that $\phi(t) = \text{graph}((F_u^{\varepsilon}(i_{a_1}\gamma))^s)$ $(\phi(t) = (\phi_1(t), \phi_2(t))$ is a parametrization of the graph curve, $\phi_i(t) \to \pm\infty$ when $t \to \pm\infty)$ crosses the diagonal from below to above at $\phi(t_0) = x$ (i.e. $\phi_2(t) \leq \phi_1(t)$ for $t < t_0$, $\phi_2(t) \geq \phi_1(t)$ for $t > t_0$, $t \approx t_0$). In particular x is not in any "plateau" (including the ends). So x stays periodic under $i_{a_1}\gamma$ and since $a_1 \notin A$, it must be a source.

It follows that there exists $a_2 > a_1$ $(a_2 \approx a_1)$ such that for every a, $a_1 \leq a \leq a_2$, there exists a point $x_a \in \mathbb{R}$ which is a periodic source for $F_u(i_a\gamma)$, of period s and rotation number $\frac{r}{s}$, x_a depends continuously on a and all x_a's are close to $x = x_{a_1}$.

Since $uI(a)$ changes when a goes from a_1 to a_2, and in view of (2) there exists an integer $k \geq 1$ such that

$$(3) \qquad d(a) = \sup_{x_a - 1 \leq y \leq x_a} (F_u^\varepsilon(i_a\gamma))^{ks}(y) - (F_u^\varepsilon(i_a\gamma))^{ks}(x_a) > 0$$

for every a: $a_1 \leq a \leq a_2$, and

$$(4) \qquad d(a_2) - d(a_1) > 1$$

Of course d depends on a continuously.

Due to (3) the function $F_u((F_u^\varepsilon(i_a\gamma))^{ks})$ is locally constant at x_a, if $a_1 \leq a \leq a_2$. But

$$F_u((F_u^\varepsilon(i_a\gamma))^{ks}) = (F_u(F_u^\varepsilon(i_a\gamma)))^{ks} = (F_u(i_a\gamma))^{ks} = F_u((i_a\gamma)^{ks})$$

(we used the general fact that for any composition $F_u(\eta_1 \circ \eta_2) = F_u(\eta_1) \circ F_u(\eta_2)$. Its proof is straightforward). So $F_u((i_a\gamma)^{ks})$ is locally constant at x_a. So by the definition of F_u for every a: $a_1 \leq a \leq a_2$ there exists a point $c_a \in \mathbb{R}$, $c_a < x_a$ and its $i_a\gamma$-trajectory $c_a = c_{a,0}, c_{a,1}, \ldots, c_{a,ks}$ (with the translation integers 0) such that the pair $(c_a, c_{a,ks})$ is h-critical (local maximum) for $(i_a\gamma)^{ks}$ and $c_{a,ks} = (F_u((i_a\gamma)^{ks}))(x_a)$. ($c_a$ need not depend continuously on a).

Thus, in view of (4) and continuity of $d(a)$ there exists $a_3 \in (a_1, a_2)$ such that for some positive integer m $c_{a_3,ks} = x_{a_3} + m$. One of the pairs $(c_{a_3,i}, c_{a_3,i+1}$ is of course h-critical. To end the Proof of the existence of tangency (i) one needs only to find a periodic source x' such that $W^u(x') \ni c_{a_3}$

Observe that (similarly as in [N-P-T]) there exists a point $y \in \mathbb{R}$ periodic for $F_u(i_{a_3}\gamma)$ of period s and rotation number $\frac{r}{s}$, whose orbit is a preserving orientation source from the right hand side such that the unstable manifold for $F_u(i_{a_3}\gamma)$ $W^u(y)$ is the whole $\mathbb{R}$. Otherwise we would have $uI(a_3) = \frac{r}{s}$. See Fig. 19.

($W^u(y) \subset \mathbb{R}$ is understood as the projection preimage of the unstable manifold in the circle. In the standard sense the claim is that the unstable manifold for the map $(F_u(i_{a_3}\gamma))^s - r$ is equal to the set $\{z \in \mathbb{R} : z \geq y\}$.)

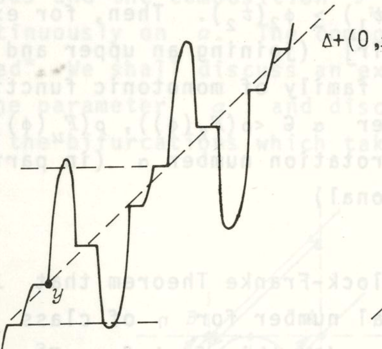


Fig. 19 $(F_u^\varepsilon(i_{a_3}\gamma))^s$

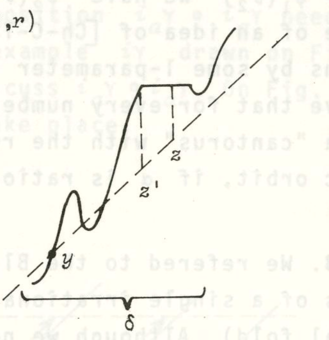


Fig. 20 $(F_u^\varepsilon(i_{a_3}\gamma))^{ks}$

Let $\delta \subset \mathbb{R}$ be an open interval containing y such that $\delta \subset W^u(y)$ the unstable manifold for the multifunction $i_{a_3}\gamma$. There exist integers $k > 0$, and a point $z \in \delta$ such that $(F_u^\varepsilon(i_{a_3}\gamma))^{ks}(z) = c_{a_3} + m$. So, there exists $z' \in (y, z> \subset \delta$ such that $(F_u^\varepsilon(i_{a_3}\gamma))^{ks}$ is not locally constant at z' and has the same value $c_{a_3} + m$ (Fig. 20). Then of course $(F_u^\varepsilon(i_{a_3}\gamma))^{ks}(z') \in (i_{a_3}\gamma)^{ks}(z')$.

Thus $W^u(y)$ contains $c_{a_3} + m$. One can take $x' = y$. The proof of Conjecture c(i) for upper heavy curves is finished.

To have $x = x'$, namely homoclinic tangency (Conjecture a), one should have chosen first x' for a with $W^u(x') = \mathbb{R}$ (as above for a_3) and then consider $F_u^{2\varepsilon}$ instead of F_u^ε. For $F_u^{2\varepsilon}(i_{a_1}\gamma)$ x' is a both side source (not in any closed "plateau" interval) so it can serve as x_{a_1} at the begin of the search for x_{a_3}, c_{a_3},... as before. (Observe that in view of $uI(a_1) > \frac{r}{s}$

the property $W^u(x_{a_1}) = \mathbb{R}$, with the map $F_u^{2\varepsilon}$ under consideration, is preserved for a close to a_1). $\square$

Remark 2. By analogy to M we can call an old curve $\phi(t) = (\phi_1(t), \phi_2(t))$ *heavy* if for every $t_1 \leq t_2$ such that $\phi_1(t_1) = \phi_1(t_2)$ we have $\phi_2(t_1) \geq \phi_2(t_2)$. Then, for example with use of an idea of [Ch-C-Tr] (joining an upper and lower functions by some 1-parameter family of monotonic functions) one can prove that for every number $\alpha \in \langle \rho(F_\ell(\phi)), \rho(F_u(\phi)) \rangle$ there exists a "cantorus" with the rotation number α (in particular a periodic orbit, if α is rational)

Remark 3. We refered to the Block-Franke Theorem that $I(\eta)$ never consists of a single irrational number for η of class C^2 (with a typical fold). Although we needed this fact for F_u^ε which is only continuous, we could smoothen the bad places by $\tilde{F}_u^\varepsilon$.

In general the smoothness assumption is substantial. One could consider any monotonic, degree one map on S^1, with irrational rotation number, having a "plateau" and then make a fold mapped to this "plateau" (Fig. 21)

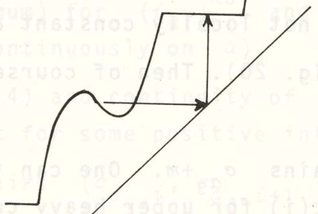


Fig. 21

In connection with that, Misiurewicz raised the following question: Let f be any, degree one, continuous endomorphism of a circle, which is not monotonic. Can the rotation interval of $f \circ \rho_a$ reduce to a single number for every $a \in \mathbb{R}$? (ρ_a denotes the rotation by a).

(Compare this Remark with Theorem 8 in the later part of this Section 5.)

If one does not assume that $i_a\gamma$ is upper heavy then one source of troubles is that the upper function $F_u(i_a\gamma)$ need not be continuous and the composition $F_u(i_a\gamma)\ F_u(i_a\gamma)$ need not depend continuously on a. The composition $i_a\gamma \circ i_a\gamma$ need not be connected*. We shall discuss an example $i\gamma$ drawn on Fig. 22. Increase the parameter a and discuss $i_a\gamma \circ i_a\gamma$. On Fig. 23 we draw up the bifurcations which take place.

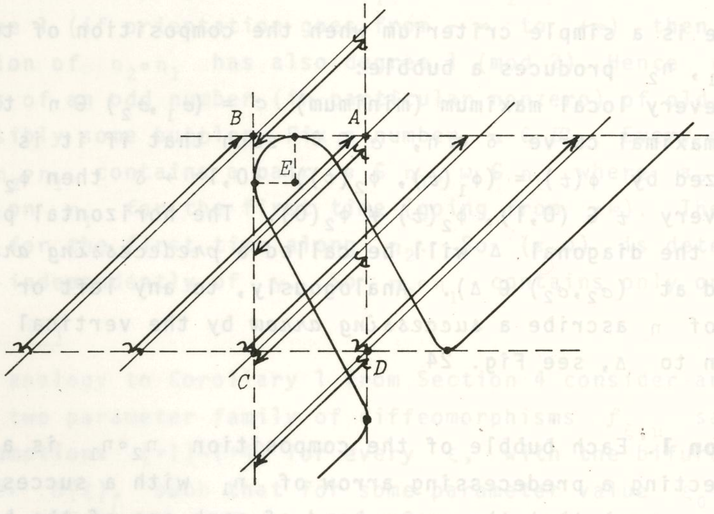


Fig. 22

If the diagonal Δ_a is above B then $i_a\gamma \circ i_a\gamma$ is an old curve, stage (i). When Δ_a passes B, a small bubble above an old curve $\eta(a)$ arises from nothing (stage (ii)). It grows (begins to selfintersect when Δ_a passes E) and when Δ_a passes A it joins the old curve $\eta(a)$ (stage (iii)). When Δ_a passes C a new large bubble separates from $\eta(a)$ from below (stage (iv)). It shrinks and dies when Δ_a passes D (see (v))

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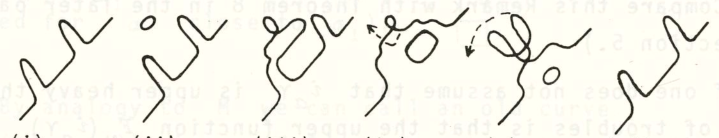
* Independently phenomena discussed on this and two next pages were discovered and studied in a series of preprints by J.-P. Dufour from Univ. Montpellier II (France).



(i) (ii) (iii) (iv) (v) (vi)

Fig. 23

Exercise (not accomplished by the author). Verify this bifurcation scheme with a help of a computer.

There is a simple criterium when the composition of two old curves η_1, η_2 produces a bubble:

For every local maximum (minimum) $c = (c_1, c_2) \in \eta$ there exists a maximal curve $\delta \subset \eta$, $c \in \delta$ such that if it is parametrized by $\phi(t) = (\phi_1(t), \phi_2(t)): \langle 0,1\rangle \to \delta$ then $\phi_2(0)=\phi_2(1)$ and for every $t \in (0,1)$, $\phi_2(t) \neq \phi_2(0)$. The horizontal projection of δ to the diagonal Δ will be called a *predecessing arrow* (the arrow-head at $(c_2, c_2) \in \Delta$). Analogously, to any left or right extremum of η ascribe a *successing arrow* by the vertical projection to Δ, see Fig. 24

Proposition 1. Each bubble of the composition $\eta_2 \circ \eta_1$ is a result of intersecting a predecessing arrow of η_1 with a successing arrow of η_2 such that the arrow-head of each one of the both arrows is inside the other one and the arrow-tail is not (Fig. 24)

Proof is straightforward.

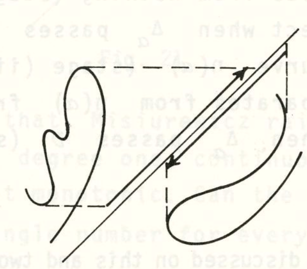


Fig. 24

The arrows → on Figure 22 are used for the critical points (B_1, E_2) and (E_1, B_2) and the arrows —⇠ for the other two.

Observe that due to the above criterium and periodicity the size of all the bubbles is uniformly bounded.

We shall finish this discussion with the following.

Proposition 2. $\eta_2 \circ \eta_1$ consists of a single old curve and possibly some bubbles.

Proof. Since the projections of η_1 and η_2 to the x-th axis are of degree 1 (if orientation goes from $-\infty$ to $+\infty$) then the projection of $\eta_2 \circ \eta_1$ has also degree 1 (mod 2). Hence $\eta_2 \circ \eta_1$ consists of an odd number (in particular nonzero) of old curves and possibly some bubbles. Fix a number $\alpha \in \mathbb{R}$. Every old curve γ of $\eta_2 \circ \eta_1$ contains a pair $z \in \eta_1$, $w \in \eta_2$ where $z_2 = \alpha$ is reached on η_1 for the first time (going from $-\infty$). Then w_1 is reached for the first time along η_2. So (z, w) is determined by α, independently of γ. So $\eta_2 \circ \eta_1$ contains only one old curve. $\square$

By analogy to Corollary 1 from Section 4 consider any smooth generic two parameter family of diffeomorphisms $f_{\xi,\mu}$ satisfying the assumptions (*1)-(*4) for every ξ, with the bifurcation parameter $b(\xi)$, such that for some parameter value ξ_0: $W^u(P(\xi_0)) \pitchfork W^s(P(\xi_0))$, $W^u(P(\xi_0)) \cap W^s(P(\xi_0))$ contains a curve $\gamma(\xi_0)$ going through $P(\xi_0)$ with $i(\gamma(\xi_0))$ being a graph of a function $\eta(x)$ with a point x_0 where η is nondifferentiable (i.e. $\frac{d\eta}{dx}(x_0) = +\infty$, or $-\infty$). For all $\xi \approx \xi_0$ we have curves $\gamma_\xi \subset W^u P(\xi) \cap W^s P(\xi)$ continuously depending on ξ. So $i\gamma(\xi)$ continuously depends on ξ, in C^∞-topology. Assume that for $\xi < \xi_0$ $i\gamma(\xi)$ is a graph of a smooth function and after ξ passes ξ_0 it gets a graph of a multi-function.

Theorem 7. If a family $f_{\xi,\mu}$ satisfies all the above assumptions then for every $i\gamma(\xi)$ such that $|\xi-\xi_0|$ is sufficiently small (including $\xi > \xi_0$) there exists $a \in \mathbb{R}$ such that for i_aG with $G = G(\xi)$ given by $f_{\xi,b(\xi)}$ a homoclinic tangency happens (see Conjectures a and A).

Proof. Due to the genericity assumption for $f_{\xi_0,\mu}$ it is enough to prove this Proposition for $i\gamma(\xi_0)$ (the property (i) is stable). For $i\gamma(\xi_0)$ one can proceed as in [N-P-T]. The unique thing which should be checked is the correctness of the *Block-Franke Theorem* in our case.

Theorem 8. Let $\eta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a degree one (1-valued) mapping with finite, nonzero number of strict local maxima, such that graph η is a C^2-curve in $\mathbb{R}^2/\mathbb{Z}^2$ and for every x there exists n such that $|\eta(x) - \eta(y)|^n \leq |x-y|$ for every y. Assume also that $\frac{d\eta}{dx} = 0$ can happen only at a strict maximum or minimum point. Then there exists a periodic point for η.

Proof. (a simplification of the Block-Franke Proof). Let $\bar\eta$ be a lift of η to $\mathbb{R}$. Consider the map $\bar\eta^{\varepsilon}$ with the strict maxima truncated (see the definition at the begin of Proof of Theorem 6). Consider the upper map $F_u(\bar\eta^{\varepsilon})$. Denote the maximal open intervals where $F_u(\bar\eta^{\varepsilon})$ is constant by $A_1+k,\dots,A_t+k$, $k \in \mathbb{Z}$. Let $K = \mathbb{R} \setminus (\bigcup_j \bigcup_{n\geq 0} (F_u(\bar\eta^{\varepsilon}))^{-n}(A_j+\mathbb{Z}))$. K is of course $F_u(\bar\eta^{\varepsilon})$ invariant and $F_u(\bar\eta^{\varepsilon})|_K = \bar\eta|_K$

Let K' be a subset of K such that $K'+1 = K'$ whose projection $\pi(K')$ to the circle is a *minimal set* (for the mapping $F_u(\eta^{\varepsilon}) = \pi \circ F_u(\bar\eta^{\varepsilon}) \circ \pi^{-1}$, $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ $\pi(x) = x(\text{mod } 1)$)

1. If $\rho(F_u(\bar\eta^{\varepsilon}))$ is rational then $\pi K'$ is an $F_u(\eta^{\varepsilon})$-periodic orbit, so it is also a periodic orbit for η

2. If $\rho(F_u(\bar\eta^{\varepsilon}))$ is irrational and the left end point x of some A_j belongs to K', then there exist integers $k > 0$ and m such that $x' = (F_u(\bar\eta^{\varepsilon}))^{-m}(x)$ is to the left of $x+k$ close to $x+k$. Then there exists x'' to the right of $x+k$ such that $\bar\eta(x') = \bar\eta(x'')$. Thus

$$\bar\eta^{m}(x') > x' - k, \quad \bar\eta^{m}(x'') < x'' - k$$

This implies the existence of a point y $(x' < y < x'')$ with $\bar\eta^{m}(y) = y-k$. So πy is η-periodic.

3. Suppose that $\rho(F_u(\bar\eta^{\varepsilon}))$ is irrational and $K' \cap (\bigcup_j \text{cl} A_j+\mathbb{Z}) = \emptyset$. Let $B_1,\dots,B_t$ be some open intervals in $\mathbb{R}\setminus K'$ such that $B_j \supset \text{cl} A_j$. So the function $\bar\eta$ is strictly increasing on $\mathbb{R} \setminus (\bigcup_j B_j+\mathbb{Z})$. So one can extend η to a mapping Θ on S^1 which is C^2 on $\bigcup_j \text{cl}\pi B_j$ and such that $\frac{d\Theta}{dx}(x) > 0$ for every $x \in \text{cl}\pi B_j$ (one should take care about to have $\frac{d\eta}{dx} \neq \infty$ at the ends of every B_j).

The situation contradicts the Denjoy Theorem applied to the inverse function Θ^{-1}, the version with non-flat singularities elaborated by Yoccoz [Y] (In view of the Yoccoz Theorem $\pi K'$ which is a minimal, invariant set for Θ^{-1} must be the whole circle.) $\square$

By analogy to Conjecture e (Section 4) the following is reasonable:

Conjecture f. For every smooth old curve $\eta \subset \mathbb{R}^2$, every finite number of points $x_1,\dots,x_k \in \mathbb{R}$ and for every point $z \in \mathbb{R}$ there exists a C^{∞}-arbitrarily small perturbation of η to a smooth old curve $\tilde\eta$, which does not change the first jets at the points $x_1,\dots,x_k$, such that there exists an η-trajectory $z = z_0, z_1, \dots$ converging to an orbit of a periodic sink (periodic after projecting to S^1).

Theorem 9. If Conjecture f is true then for every $i\gamma$ such that neither $i\gamma$ nor $(i\gamma)^{-1}$ is a graph of a function (a bi-critical case), for some arbitrarily small perturbation of the underlying family of diffeomorphisms f_μ there exists $a \in I\!R$ such that for i_aG we have a tangency (iii) (see Conjecture c(iii)) hence failure of mild stability for the family f_μ itself.

(cf. Remark 6 §.4. Its analog is valid here.)

Proof. There exist an h-critical point c_1 and a v-critical point c_2 for i. Let a be a parameter such that (c_1,c_2) is an h-critical pair in $i_a\gamma$, see Fig. 25.

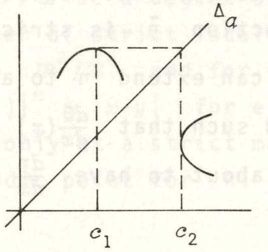


Fig. 25

Let (c_2,c_3) be our v-critical pair. By Conjecture f there exists $\tilde\eta$ close to $i_a\gamma$ with the first jets at (c_1,c_2) and (c_2,c_3) unchanged and such that an $\tilde\eta$-trajectory $c_3,c_4,\ldots$ converges to a sink and an $(\tilde\eta)^{-1}$-trajectory $c_1,c_0,c_{-1},\ldots$ converges to a source. It is easy now to make a subsequent perturbation so that the trajectory $\ldots c_1,c_2,c_3\ \ldots$ contains only (our) two critical pairs. □

Question. One can divide an old curve η into arcs between every two consecutive critical pairs of η (h- or v-critical). Ascribe symbols to these arcs and then for every critical value consider all its itineraries (kneading sequences). To what extend these kneading sequences help to undestand the whole dynamics?

Consider now the case when $W^s(P) \cap W^u(P)$ contains at least two closed curves going through P. Even then I am not able to prove neither Conjecture A, nor B, nor C. I can prove only:

Theorem 10. If $W^s(P) \cap W^u(P)$ contains two closed curves γ,η going through P, then we have the transversal homoclinic intersection phenomenon (as in Conjectures d and D).

Proof. If $i\gamma$ and $i\eta$ have a point of intersection then, by Remark 1 from Section 5, Conjecture d is true. So assume that $i\gamma \cap i\eta = \emptyset$ and that $i\gamma$ lies below $i\eta$. Let a_0 be a parameter such that for every $x \in I\!R$ $i_{a_0}\eta(x) \geq x$ (this means that for very $y \in i_{a_0}\eta(x)$, $y \geq x$), and there exists x_0 such that $i_{a_0}\eta(x\ ) \ni x_0$. By the genericity assumption x_0 is a left-side sink for $i_{a_0}\eta$ and for every $x \notin x_0 + \mathbb{Z}$, $i_{a_0}\eta(x) > x$. It is clear that either there exists a positive integer N such that

$$(5) \qquad (i_{a_0}(\eta))^N(i_{a_0}(\gamma))(x_0) > x_0$$

or for some $m > 0$ $(i_{a_0}(\eta))^m(i_{a_0}(\gamma))(x_0)$ contains a point y close to x_0+n for some integer $n \leq 0$ and at the left side of it. In the latter case y is in the basin of attraction of x_0 (in S^1). For a slightly changed a_0, x_0 is a sink, so Conjecture d holds.

Consider now the first case.

Let a_1 be a parameter such that there exists a point x_1 such that $i_{a_1}\eta(x_1) \ni x_1$ and $i_{a_1}\eta(x) \leq x$ for every $x \in I\!R$, see Fig. 26

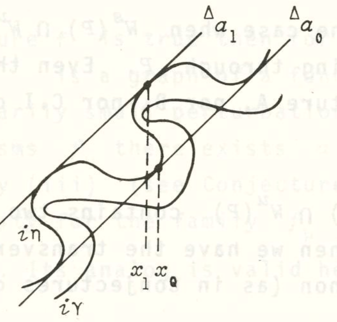


Fig. 26

Clearly

$$(i_{a_1}\eta)^N(i_{a_1}\gamma)(x_1) < x_1 \tag{6}$$

(opposite to (5)).

To end the Proof it is enough to find continuous curves x_τ, y_τ, a_τ, $\tau \in \langle 0,1\rangle$ such that $y_\tau \in (i_{a_\tau}\eta)^N(i_{a_\tau}\gamma)(x_\tau)$ and $x_\tau \in i_{a_\tau}\eta(x_\tau)$ because this, (5) and (6) imply the existence of $\tau' \in \langle 0,1\rangle$ such that $(i_{a_{\tau'}}\eta)^N(i_{a_{\tau'}}\gamma)(x_{\tau'}) \ni x_{\tau'}$, hence the existence of a homoclinic trajectory.

For every $a \in \mathbb{R}$ consider the set

$$h_a = \{(y_0,\ldots,y_{N+2}) \in \mathbb{R}^{N+3} : y_2 \in (i_a\gamma(y_1)$$

and

$$y_i \in (i_a\eta)(y_{i-1}) \quad \text{for} \quad i = 1,3,4 \ldots N+2\}$$

and its projection π to the curve $i_a\eta$ (the coordinates (y_0,y_1) If we parametrize $i\eta$ by $\phi(\tau)$, $\tau \in \mathbb{R}$ we can consider π as a projection to $\mathbb{R}$. It is clearly of degree 1 (mod 2) (Except for some isolated values of a, h_a is a 1-dimensional manifold). If one considers also a as a variable then the set $h = \bigcup_{a\in\mathbb{R}} h_a \times \{a\} \subset \mathbb{R}^N+4$ is a smooth surface (in a generic situation) and projects to the plane $\mathbb{R}$ of the variables (τ,a) again with degree 1 (mod 2).

Consider in $\mathbb{R}^2$ a curve $h'(\tau) = (\tau, a(\tau))$ such that $\phi_a(\tau)$ is a fixed point for $i_a\eta$ (i.e. $\phi_a(\tau)$ is in the diagonal). Then $\pi^{-1}(h')$ projects to h' with degree 1 (mod 2) (It is easy to show that for generic η,γ h' is in such a position that $\pi^{-1}(h')$ is a manifold).

Hence the set $\pi^{-1}(h')$ contains an old curve δ (old in the sense that after neglecting the coordinate a $\delta+(1,\ldots,1) = \delta$). In particular δ covers the points (x_0,x_0,a_0) and (x_1,x_1,a_1). $\square$

Let us end this section with a general discussion of a cycle saddle-node bifurcation, started in the Introduction. From the Newhouse-Palis-Takens theory it follows that every homoclinic tangency for a hyperbolic periodic orbit is accompanied by secondary homoclinic tangency phenomena for hyperbolic periodic orbits for neighbouring parameters. I conjecture that an analogous situation happens for any cycle saddle-node P. Namely that this bifurcation is accompanied by secondary saddle-node bifurcations, $u(s)$-critica provided P being $u(s)$-critical. Going in this direction I can prove the following:

Proposition 3. For every generic 1-parameter family f_μ of diffeomorphisms of M with f_b having a cycle saddle-node P for a parameter b, such that $W^u(P) \cap W^s(P)$ contains a curve with $i(\gamma)$ being an old upper-heavy curve satisfying the condition (0), there exists a sequence of parameters $b_n \to b$ such that f_{b_n} have periodic u-critical cycle saddle-nodes P_n. (We assume neither that the limit sets of some f, in particular f_b, are finite, nor that the cycle is a 1-cycle).

Sketch of Proof. There exists a parameter a_4 for which $\ell I(F_\ell i_{a_4}\gamma) < uI(F_u i_{a_4}\gamma) = uI(a_4) = \frac{r}{s}$ a rational number and $uI(a) < uI(a_4)$ for every $a < a_4$ (see the notation in Proof of Theorem 6). So there exists a saddle-node x for $i_{a_4}\gamma$, with rotation number $\frac{r}{s}$. Similarly as in Proof of Theorem 6 we see that $W^u(x) = \mathbb{R}$. In particular $W^u(x)$ contains a critical point (maximum) contained in $W^s(x)$. Now one finds P_n by the §3 (Reduction Theorem) technique.

6. DA-endomorphisms and multivalued expanding mappings on branched manifolds

This section is only roughly connected with the rest of the paper. We give examples of multivalued dynamics appearing naturally in much simpler situations than for bi-critical 1-cycle saddle-node bifurcations.

Make a construction like DA (derived Anosov) diffeomorphisms see [W₁], but for an algebraic, hyperbolic endomorphism f of the two-dimensional torus $\mathbb{T}^2$ rather than for a diffeomorphism. Namely, perturb f close to 0 along the leaves of the unstable foliation W^u so that 0 which is a saddle for f gets a sink for the new map g (Caution: One usually makes a source in the case of a diffeomorphism, but this way would be wrong in the case of an endomorphism). We call g a DA-endomorphism. Let U be a small open neighbourhood of 0 such that $g(U) \subset U$. Denote $R = \mathbb{T}^2 \setminus \bigcup_{n=0}^{\infty} g^{-n}(U)$. R is a repeller along which we have a contraction for g, (g is a nice example for an Axiom A endomorphism, unfortunately it is not Ω-stable. See [Prz][#] for a general theory)

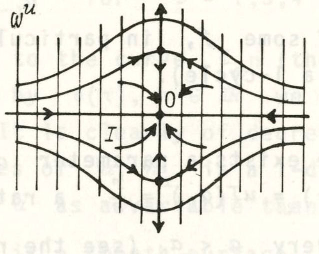


Fig. 27

So, for g^{-1} R is a *multivalued expanding attractor*. Now one can make the Williams branched manifold construction $[W_1]$ in the standard way: Let I be an open interval in the stable manifold of 0 for f (containing 0, with the ends in different leaves of W^u). Let K be the space of all the components of the leaves of W^u in $\mathbb{T}^2 \setminus I$ (topologically K is a pair of circles glued together along a common arc). g^{-1} maps components into components so there exists a multivalued factor-map h on K, expanding along K. Similarly as for DA-diffeomorphism g^{-1} on R is topologically conjugate to the left (multivalued) shift on the set of all backward trajectories of the system $\ldots \to K \xrightarrow{h} K$. g is conjugated to the left shift on the set of all forward trajectories of the system $K \xrightarrow{h^{-1}} K \xrightarrow{h^{-1}} K \to \ldots$. We will call these sets inverse and direct limits, although these names are not fully correct here.

If one considers the cartesian product of g on $\mathbb{T}^2$ with a Morse-Smale diffeomorphism ϕ on a 3-dimensional manifold N having a strong sink p, then after a perturbation of $g \times \phi$ to Φ on a neighbourhood of $\mathbb{T}^2 \times \{p\}$ along the fibers N one can obtain a basic set for the arising Axiom A endomorphism, on which Φ is 1-to-1 and is conjugated to the inverse limit of the system $\ldots \longrightarrow R \xrightarrow{g} R \xrightarrow{g} R$, i.e. inverse-direct limit (just the set of all trajectories) of $\ldots \to K \xrightarrow{h^{-1}} K \to \ldots$.

A similar construction can be done on a higher dimensional torus $\mathbb{T}^n$ $(n > 2)$, assumed $\dim W^u = 1$. Then the branched manifold K is $n-1$ dimensional.

A general question arises from the Williams theory for expanding attractors (see $[W_2]$ for example): What is true for multivalued expanding attractors (or contracting repellers)? Clearly the inverse - direct limit can be topologically realized on a basic set of some Axiom A endomorphism. Conversely, Williams theory asserts that every expanding attractor dynamics is an

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[#] A serious misprint happened in the statement of Theorem A in [Prz] and in the Abstract. "f is Ω C^r-stable" should stand there.

inverse limit of an expanding map on a branched manifold. However it is not clear what should be assumed about a basic set of Axiom A diffeomorphism so that the dynamics on it to be an inverse-direct limit of a multivalued expanding map on a branched manifold.

Another question concerns the structure of a codimension 1 multivalued expanding attractor. Must it be an attractor for the inverse map to a corresponding DA-endomorphism (under some reasonable assumptions)? (Kollmer-Plykin theory [K], [Pl] asserts that every expanding attractor for a diffeomorphism, of codimension 1, dimension at least 2, is an attractor for a DA-diffeomorphism).

A simplest possible example is just a circle $\{|z|=1\}$ and the mapping $z \to z^{n/m}$, $(n,m)=1$. The direct-inverse limit is a solenoid.

One can consider a 0-dimensional K, just a finite set of points. The multi-valued dynamics is just a directed graph with the set of vertices K. The direct-inverse limit can be realized as an invariant Cantor set for a horseshoe-like dynamics. As an example consider a DA-perturbation of $z \to z^2$ on the circle $\{|z|=1\}$ (the source 1 is perturbed to a sink) and then a subsequent perturbation in $S^1 \times I$ (analogous to the perturbation g in $T^2 \times N$). The reader can find an exact picture at the end of the paper [Prz].

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