UNIQUENESS IN THE CAUCHY PROBLEM FOR FIRST-ORDER LINEAR P.D.E.'s

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Introduction and Statement of Results

In [4] Strauss and Trèves proved the following result:

Theorem 1: Consider the following differential operator:

$$L = \partial_t - ib(x,t)\partial_x - c(x,t) \tag{0.1}$$

where the coefficients $b(x,t)$, $c(x,t)$ are C^∞ functions in an open neighborhood Ω of the origin in $\mathbb{R}^2$, b being real-valued. Suppose that $t \to b(0,t)$ vanishes of finite order at $t = 0$. Then, to every open neighborhood $U \subset \Omega$ of 0 there is another open neighborhood U' of 0 such that the following is true: If $u \in C^1(U)$ is such that $Lu = 0$ in U and $u = 0$ in $U^- = \{(x,t) \in U: t < 0\}$, then $u = 0$ in U'.

Remark 1: Theorem 1 is proved by obtaining a Carleman's inequality. Moreover, they observe that the result is valid under the following more general assumptions: $b(x,t)$ is real valued, $b(x,t)$ has distribution derivatives of order ≤ 2 with respect to x which belong to $L^\infty(\Omega)$, $c(x,t)$ has distribution derivatives of order ≤ 1 with respect to x which belong to $L^\infty(\Omega)$. In the present paper we shall prove:

Theorem 2: The same result as in Theorem 1 holds when distribution solutions are considered.

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1. Reduction to the case: $D_t + it^k \beta D_x + c$

Let us denote by k the order of the root $t = 0$ of the function $b(0,\cdot)$. Then by the preparation theorem and possibly on a smaller U,

$$b(x,t) = \beta(x,t)(t^k + a_{k-1}(x)t^{k-1} + \ldots + a_0(x)), \qquad (1.1)$$

where β is nowhere vanishing on U and $a_{k-1}(0) \ldots = a_0(0) = 0$. Let $\lambda_i(x)$, $i=1,\ldots,k$, denote the roots of the polynomial in (1.1). We define $n(x) = \max\{\text{order } \lambda_i(x) : i=1,\ldots,k\}$. Notice that $n(0) = k$.

Lemma 1.1: Denote by Z_k the zeros of $b(x,\cdot)$ of order k. Then if u is a distribution solution of $Pu = 0$, with support contained in $Z_k \cap U$, u is identically equal to zero.

Proof: It is clear that:

$$-a_{k-1}(x) = \lambda_1(x) + \ldots + \lambda_k(x).$$

If (x,t) is a zero of order k in U, we have:

$$\lambda_1(x) = \ldots = \lambda_k(x) = t.$$

It follows that $(x,t) = (x, -a_{k-1}(x)/k)$. In other words, $Z_k \cap U$ is contained in the image of γ, where γ is the path defined by $\gamma(x) = (x, -a_{k-1}(x)/k)$.

The conormal bundle of γ is given exactly by points of the form:

$$(x, -a_{k-1}(x)/k;\ a'_{k-1}(x)/k, \tau), \text{ with } \tau \in \mathbb{R};$$

on the other hand, a simple computation shows that the *wave front set* of u, $WF(u)$, is contained in the set $(x,t,\xi,0)$, $\xi \neq 0$,

$b(x,t) = 0$, and this is a contradiction to the wellknown fact that $WF(u)$ must contain the conormal bundle of the image of γ (see [1]).

Remark: If $n(0) = 0$, the operator (0.1) is elliptic at the origin and Theorem 1 applies.

Lemma 1.2: Assume that Theorem 2 is true when $n(x) = k$ is constant; then Theorem 2 follows in general.

Proof: We are going to prove Lemma 1.2 by induction on k. If $n(0)$ is equal to one, formula (1.1) implies that in a neighborhood of the origin $n(x) = 1$, hence uniqueness follows from our assumption. Suppose now that we have proved the lemma when $n(0)<k$ and let us assume that $n(0) = k$. If $n(x) = k$ in a neighborhood of the origin, we have nothing to prove. So let us assume that there exists $\bar{x}_0$ such that $n(\bar{x}_0) < k$; it follows then that u is zero in a neighborhood of the vertical segment $x = \bar{x}_0$. Moreover we assume that $b(x,\pm\eta)$ is different from zero for η sufficiently small and $|x| < \delta$, where δ and η are fixed. We may also assume, by shrinking Ω if necessary, than in $]-\delta,\delta[\times]-\eta,\eta[= \Omega$ the factorization (1.1) is valid. We consider (x_0,t') in Ω and suppose that $b(x_0,t') \neq 0$; let (x_0,t_0) be a zero of $b(x_0,\cdot)$ of order k, then (x_0,t_0) is the only zero of b in the vertical segment $x = x_0$. Let us assume that $\lambda_1(x)(\lambda_k(x))$ is the smallest (largest) of the real roots of the polynomial (1.1). If t' is less than $\lambda_1(x_0)$ we have that u is zero in a neighborhood of (x_0,t'). In fact, u is zero in the set (x,t) with $t < \lambda_1(x)$ because this is a connected open set where P is elliptic. (We remind that u is zero for $t < 0$). If $t' > 0$, we know that (x_0,t') belongs to the connected open region defined by (x,t) with $t > \lambda_k(x)$. Since this region intersects the open neighborhood of the vertical segment $x = \bar{x}_0$, where $n(\bar{x}_0) < k$, u vanishes in the region defined by $t > \lambda_k(x)$ and, consequently, u is zero at (x_0,t').

If on the vertical segment $x = x_0$, $n(x_0) < k$, we obtain, by our inductive hypothesis, that u is zero in a neighborhood, not only of (x_0, t'), but also of the whole vertical segment $x = x_0$.

It follows then that the support of u is contained in Z_k, and hence, by Lemma 1.1, u is zero in $]-\delta,\delta[\times]-\eta,\eta[$. Q.E.D.

As a consequence of Lemma 1.2 it is sufficient to consider operators (0.1) of the form:

$$P = \partial_t - i\beta(x,t)(t-\lambda(x))^k \partial_x - c(x,t).$$

By an obvious change of coordinates and wellknown results of continuation of zeros of solutions of first-order elliptic equations, the proof of Theorem 2 can be reduced further to the case where

$$P = \partial_t - it^k\beta(x,t)\partial_x - c(x,t), \quad \beta \neq 0.$$

2. Reduction to the case where k is even

In this section we consider the operator:

$$P = \partial_t + it^k\beta(x,t)\partial_x + c(x,t), \quad \beta \neq 0, \tag{2.1}$$

and we will show how to reduce the proof of Theorem 2 to the case where k is even.

We remind that P is hypoelliptic for even k (See [5]) in which case, Theorem 2 coincides with Theorem 1. From now on we shall be concerned with odd k.

It is wellknown that the operator (2.1) is partially hypoelliptic with respect to the variablet t. Therefore the solutions of $Pu = 0$ are C^∞ functions in $t, |t| < \eta$, valued in the space of distributions in x, $|x| < \delta$.

Let $[\alpha]$ be the greatest integer less than or equal to α.

We will consider the $C^{[\alpha]}$ function $\psi(t) = t^\alpha$, $t \geq 0$, $\psi(t) = -|t|^\alpha$, $t < 0$, and let $v(x,t) = u(x,\psi(t))$; it follows that v is a function of class $C^{[\alpha]}$ in t valued in the space of distributions in x. It is easy to show that v satisfies an equation of the following form:

$$\bar{P}v = 0, \quad v = 0 \quad \text{for} \quad t < 0, \tag{2.2}$$

where $\bar{P} = \partial_t + it^{2\mu}\beta(x, t^{2\mu+1/1+k})\partial_x + \bar{c}(x,t)$. Here μ denotes a non-negative integer that will be chosen later. In fact we have for $t > 0$:

$$\partial_t v = \alpha t^{\alpha-1}\partial_t u(x,t^\alpha),$$
$$\partial_x v = \partial_x u(x,t^\alpha). \tag{2.3}$$

We obtain, after multiplying the second line of (2.3) by $it^{\alpha k}\beta(x,t^\alpha)\alpha t^{\alpha-1}$:

$$\partial_t v + i\alpha t^{\alpha(1+k)-1}\beta(x,t^\alpha)\partial_x v + \alpha t^{\alpha-1}c(x,t^\alpha)v \equiv 0. \tag{2.4}$$

When $t = 0$, we use the fact that v is a $C^{[\alpha]}$ function of t valued in the space of distributions in x which vanishes for $t < 0$. Setting $2\mu = \alpha(1+k)-1$ we can rewrite (2.4) as:

$$[\partial_t + it^{2\mu}\bar{\beta}(x,t)\partial_x + \bar{c}]v = 0 \tag{2.5}$$

with $\bar{\beta}(x,t) = \alpha\beta(x,t^\alpha)$, $\bar{c}(x,t) = \alpha t^{\alpha-1}c(x,t^\alpha)$. Equation (2.5) has the advantage that if we choose μ as a positive integer, we obtain a zero of even order; the disadvantage is that $\bar{\beta}, \bar{c}$ are not C^∞ functions any longer. More precisely $\bar{\beta}$ is of class C^M with $M = [\alpha]$ and $\bar{c}$ is of class C^{M-1}.

3. Existence of a parametrix for k odd

The proof of the existence of a parametrix will follow [5]. The main modifications are a consequence of the fact that in our case the coefficients belongs to C^M and not to C^∞ and that, moreover, the domain of the operator $\bar{P}$ is restricted to distributions of order less than or equal to the order of differentiability of the coefficients.

We remind that the principal part $\bar{P}_0$ of (2.5) is given by:

$$\bar{P}_0 = \partial_t + it^{2\mu}\bar{\beta}(x,t)\partial_x \tag{3.1}$$

In order to simplify the notation we will write simply $\bar{P}_0 = P$ and $\bar{\beta} = \beta$.

We define the operator E by

$$(Ef)(x',t') = \int e(x',t',x,t)f(x,t)dtdx \tag{3.2}$$

whose kernel is given by

$$e(x',t',x,t) = \left\{2\pi\left[\int_t^{t'} b(x,z)dz + i(x'-x)\right]\right\}^{-1} \tag{3.3}$$

where $b(x,z) = z^{2\mu}\beta(x,z)$.

It follows that e is a C^M function outside the diagonal of $-]\delta,\delta[\times]-\eta,\eta[$.

We will write T, dT instead of (x,t) and $dxdt$.

Proposition 3.1: Let ω be an appropriate neighborhood of the origin; then for every ρ, $0 \leq \rho < (2\mu+1)^{-1}$, there exists a constant M_ρ such that;

$$\sup_{T\in\omega}\int_\omega |e(T',T)|^{1+\rho}\,dT' \leq M_\rho \tag{3.4}$$

$$\sup_{T'\in\omega}\int_\omega |e(T',T)|^{1+\rho}\,dT \leq M_\rho \tag{3.4'}$$

Furthermore for every ρ, $0 < \rho < 1$, there exists a constant N_ρ such that for T, T', T'' in ω, we have:

$$|e(T',T)-e(T'',T)| \leq N_\rho |T'-T|^\rho(|e(T',T)|^{1+\rho} + |e(T'',T)|^{1+\rho}).$$

Proof: We will give the modifications required for adapting Susuki's arguments to our case. We set:

$$\gamma = \min\{t,t+s\} \quad \text{and} \quad \zeta = \max\{t,t+s\}.$$

We obtain

$$|B| \equiv \left|\int_t^{t+s} b(x,z)dz\right| = \int_\gamma^\zeta b(x,z)dz \geq c(\zeta^{1+k} - \gamma^{1+k}).$$

from which follows that

$$|B| \geq c|(t+s)^{1+k} - t^{1+k}|.$$

We observe that $(t+s)^{1+k} - t^{1+k}$ is a distinguished polynomial in s of degree $1+k$ and from this, the proof goes like in [5].

Proposition 3.2: For each ρ, $0 < \rho < (1+2\mu)^{-1}$, there exists a constant C such that:

$$|||Ef|||_\rho \leq C||f||_0, \quad f \in C_0^\infty(\omega).$$

We have used here the notation:

$$|||u|||_s^2 = \int_\omega |u|^2 dT + \iint_{\omega\times\omega} |u(T)-u(T')|^2|T-T'|^{-2-2s}dTdT'.$$

Observe that this norm is equivalent to the usual Sobolev norm $\|\ \|_s$.

Proof: See [5].

Proposition 3.3: The operator E is a left parametrix of P in the following sense: there exists a constant C such that:

$$|||EP\phi - \phi|||_\rho \leq C||\phi||_0, \quad \phi \in C_0^\infty(\omega). \tag{3.5}$$

Proof: See [5].

From (3.5), we get an a priori estimate for P

$$||\phi||_\rho \le C(||P\phi||_0 + ||\phi||_0), \quad \phi \in C_0^\infty(\omega). \tag{3.6}$$

Moreover (3.6) is still valid if we add to P a bounded operator in $L^2(\omega)$.

Remark 3.4: From (3.6) we obtain, if P were a classical pseudo-differential operator, the inequality

$$||\phi||_{\rho+s} \le C(||P\phi||_s + ||\phi||_s), \quad \text{for all } s \in \mathbb{R}. \tag{3.7}$$

This fact follows from the continuity of $[\beta,\Lambda]\Lambda^{1-s}$ in $H^0(\Omega)$, $\Lambda = (1-\Delta)^{1/2}$ and $\beta \in C^\infty(\mathbb{R}^2)$. If $\beta \in C_0^M(\mathbb{R}^2)$, we can write $[\beta,\Lambda^s]\Lambda^{1-s}$ as a sum of two operators of the form:

$$K(x,D)\phi = \int e^{ix\eta} c(\eta) k(x,\eta)\hat{\phi}(\eta)d\eta,$$

here $c(\eta) = (1+|\eta|^2)^{\frac{1-s}{2}}$, $b(\eta) = D_{\eta_j}\{(1+|\eta|^2)^{s/2}$, $a(x,y) \in C_0^{M-1}(\mathbb{R}^2\times\mathbb{R}^2)$ and $k(x,\eta) = \iint e^{-iy\xi}\, a(x,x+y)b(\xi+\eta)dyd\xi$. It is possible to prove that $K(x,\eta) = c(\eta)k(x,\eta)$ satisfies:

$$|D_x^\alpha D_\eta^\beta K(x,\eta)| \le C(1+|\eta|)^{-|\beta|}, \quad \text{if } |\alpha|, |\beta| \le 2.$$

The above inequality implies the continuity of $K(x,D)$ in H^0 (see [2]) and, consequently, the continuity of $[\beta,\Lambda^s]\Lambda^{1-s}$, if $-M+8<s<M-5$; the restriction on s is forced by the following fact (here $h(x,y) = a(x,x+y)$):

$$D_x^\alpha D_\eta^\beta k(x,\eta) = \iint e^{-iy\xi}\Delta_y^j D_x^\alpha h(x,y) D_\xi^\beta b(\xi+\eta)|\xi|^{-2j}\, dyd\xi$$

and, in order to obtain $|D_x^\alpha D_\eta^\beta k(x,\eta)| = O(|\eta|^{s-1-|\beta|})$, it is sufficient that $|\xi|^{|s-1-|\beta||} = O(|\xi|^{-2-\varepsilon})$, that is, $|s-|\beta|-1|-2j<-2$ and $2j+2<M-1$; both of these inequalities imply that $-M+8<s<M-5$.

End of the proof of Theorem 2:

We can assume that the distribution u has compact support and therefore belongs to some H_s, s large negative. By choosing μ a sufficiently large positive integer, we may suppose, since $\alpha = \frac{2\mu+1}{k+1}$, that $M = [\alpha]$ is big enough so that $-M+8<s<M-5$. Consequently, by repeatedly using inequality (3.7) we gain the necessary regularity for u in order to apply Theorem 1. This finishes the Proof of Theorem 2.

References

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