TORSION UNITS IN THE INTEGRAL GROUP RING OF S_4

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Abstract: Any torsion unit in $\mathbb{Z}S_4$ is rationally conjugate to a trivial unit.

Introduction

Let G be a finite group and $U_1(\mathbb{Z}G)$ the group of units of augmentation one of the integral group ring $\mathbb{Z}G$. A conjecture of Zassenhaus (ZC) states that every torsion unit of $U_1(\mathbb{Z}G)$ is rationally conjugate to an element of G. More precisely, if u is a torsion unit of $U_1(\mathbb{Z}G)$, then there exists an element g of G and a unit γ of the rational group algebra $\mathbb{Q}G$ such that $u = \gamma^{-1} g \gamma$.

This conjecture has been confirmed for nilpotent class two groups in [5], metacyclic groups $\langle a\rangle \rtimes \langle x\rangle$ with $(o(a), o(x)) = 1$ in [4] and some metabelian groups in [3] and [7]. In this work we shall prove that it is also true for the symmetric group on four letters S_4. We observe that S_4 is the only non metabelian group for which (ZC) is proved so far.

In §1 we give some results on S_4 that shall be needed in the sequel.

In §2 we prove that every unit of order 2, 3 or 4 of $U_1(\mathbb{Z}S_4)$ is rationally conjugate to an element of S_4 and, then, we prove that no other possibilities arise.

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§1. Some results on S_4

Let $u = \sum u(g)g \neq 1$ be a torsion unit of $U_1(\mathbb{Z}S_4)$ and, for each $g \in S_4$, let $\tilde{u}(g) = \sum_{h \sim g} u(h)$ be the sum of coefficients of u over the conjugacy class of g in S_4. We write, for simplicity, $\tilde{u}g$ instead of $\tilde{u}(g)$, for $g \in S_4$. Thus, for instance, we write $\tilde{u}(123)$ instead of $\tilde{u}((123))$.

By [7, p. 45, Corollary 1.2], $u(1) = 0$, and if we consider the character table of S_4, that is calculated, for instance, in [2, p. 277],

	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	-1	0	-1	1
χ_5	3	1	0	-1	-1

and extend χ_i, $1 \leq i \leq 5$, to $\mathbb{Z}S_4$, we have the following linear system in $\tilde{u}(g)$, $g \in S_4$:

$$\chi_1(u) = \tilde{u}(12) + \tilde{u}(123) + \tilde{u}(12)(34) + \tilde{u}(1234)$$

$$\chi_2(u) = -\tilde{u}(12) + \tilde{u}(123) + \tilde{u}(12)(34) - \tilde{u}(1234)$$

(1) $$\chi_3(u) = -\tilde{u}(123) + 2\tilde{u}(12)(34)$$

$$\chi_4(u) = -\tilde{u}(12) - \tilde{u}(12)(34) + \tilde{u}(1234)$$

$$\chi_5(u) = \tilde{u}(12) - \tilde{u}(12)(34) - \tilde{u}(1234)$$

Then, since $\chi_1(u)$ is the augmentation of u and $\chi_2(u)$ is a unit in $\mathbb{Z}$, we must have

(2) $$\chi_1(u) = 1 \quad \text{and} \quad \chi_2(u) = \pm 1.$$

Now, let H be the subgroup of S_4 fixing 4 and

$K = \{(1),(12)(34),(13)(24),(14)(23)\}$. Clearly, H is isomorphic to S_3 and K is a normal subgroup of S_4 such that $\frac{S_4}{K}$ is isomorphic to H.

Let $f: \mathbb{Z}S_4 \to \mathbb{Z}(\frac{S_4}{K})$ and $g: \mathbb{Z}(\frac{S_4}{K}) \to \mathbb{Z}H$ be the natural homomorphisms. Since (ZC) is true for H, there exists an element h of H and a unit γ of QH such that

$$(g \circ f)(u) = \gamma^{-1} h \gamma = (g \circ f)(\gamma^{-1} h \gamma);$$

hence

(3) $$o(h) \mid o(u) \quad \text{and} \quad u - \gamma^{-1} h \gamma \in \Delta_{\mathbb{Z}}(S_4, K),$$

where $\Delta_{\mathbb{Z}}(S_4, K)$ is the kernel of f.

Now, it is well known that $\Delta_{\mathbb{Z}}(S_4, K)$ is generated, as $\mathbb{Z}S_4$-submodule, by the set $\{k-1 \in \mathbb{Z}S_4 : k \in K\}$.

On the other hand, if ρ_i, $1 \leq i \leq 5$, is the irreducible representation of S_4 such that χ_i is the character of S_4 afforded by ρ_i, then K is contained in the kernel of ρ_i, for $i = 1,2,3$.

Therefore, if we extend ρ_i, $1 \leq i \leq 5$, to $\mathbb{Z}S_4$, then $\rho_i(\Delta_{\mathbb{Z}}(S_4, K)) = \{0\}$ and, hence, $\chi_i(\Delta_{\mathbb{Z}}(S_4, K)) = \{0\}$, for $i = 1,2,3$.

§2. (ZC) for S_4

We shall make use of the following two results that are proved in [3].

Theorem 1. Let G be a finite group and U a subgroup of torsion units of $U_1(\mathbb{Z}G)$. Then the following are equivalent:

(i) For each $u \in U$ there exists $g_0 \in G$, unique up to conjugacy, such that $\tilde{u}(g_0)$, the sum of coefficients of u over the conjugacy class of g_0 in G, is non zero;

(ii) For each $u \in U$ there exists $g_0 \in G$ such that u is rationally conjugate to g_0.

In (i) u is rationally conjugate to g_0 and in (ii) $\tilde{u}(g_0) = 1$.

Theorem 2. Let G be a finite group, u a torsion unit of $U_1(\mathbb{Z}G)$ and $g \in G$. If there exists a prime p such that $p|o(g)$ and $p \nmid o(u)$, then $\tilde{u}(g) = 0$.

We still need the following result that can be found in [7, p. 4, Lemmas 1.4, 1.5, and p. 110, Lemma 6.5].

Theorem 3. Let G be a group and $\Lambda = [\mathbb{Z}G, \mathbb{Z}G]$ the $\mathbb{Z}$-linear span of all Lie products $[\alpha,\beta] = \alpha\beta - \beta\alpha$, $\alpha,\beta \in \mathbb{Z}G$.

(i) $\Lambda = \{\alpha = \sum \alpha(g)g \in \mathbb{Z}G : \tilde{\alpha}(g) = 0$, for all $g \in G\}$;

(ii) $(\sum \alpha(g)g)^p \equiv \sum \alpha(g)^p g^p \ (\mathrm{mod}(\Lambda + p\mathbb{Z}G))$, for all prime p and $\alpha = \sum \alpha(g)g \in \mathbb{Z}G$;

(iii) $\Lambda^p \subseteq \Lambda + p\mathbb{Z}G$, for all prime p.

Now we are ready to prove the results announced in §1.

Theorem 4. Let $u = \sum u(g)g \neq 1$ be a torsion unit of $U_1(\mathbb{Z}S_4)$.

a) If $o(u) = 2$, then u is rationally conjugate to (12) or (12)(34);

b) If $o(u) = 3$, then u is rationally conjugate to (123);

c) If $o(u) = 4$, then u is rationally conjugate to (1234).

Proof. We will write, for clarity, $\tilde{u}(g).g$ instead of $\tilde{u}(g)g$, for $g \in S_4$.

(a) By Theorem 2, $\tilde{u}(123) = 0$ and, by Theorem 3(i),

$$u - \tilde{u}(12)\cdot(12) - \tilde{u}(123)\cdot(123) - \tilde{u}(12)(34)\cdot(12)(34) - \tilde{u}(1234)\cdot(1234) \equiv 0 \ (\mathrm{mod}\ \Lambda)$$

Then, by Theorem 3(ii) and (iii),

$$1 + \tilde{u}(12)^2 + \tilde{u}(12)(34)^2 + \tilde{u}(1234)^2\cdot(13)(24) \equiv 0 \ (\mathrm{mod}\ (\Lambda + 2\mathbb{Z}S_4)),$$

whence $2|\tilde{u}(1234)$.

Now, by (2), we can have

(a_1) $\chi_2(u) = 1$ or (a_2) $\chi_2(u) = -1$.

Besides, since $o(u) = 2$, $\rho_i(u)$, for $i = 4,5$, is similar to I_3, $-I_3$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$, where I_3 is the 3×3 identity matrix. Then $\chi_i(u)$, for $i = 4,5$, can only be 3, -3, 1 or -1.

Suppose we have (a_1). Then, by (1) and (2), it follows that

$$2 = \chi_1(u) + \chi_2(u) = 2\tilde{u}(12)(34)$$

$$1 = \chi_1(u) = \tilde{u}(12) + \tilde{u}(12)(34) + \tilde{u}(1234)$$

$$\chi_4(u) = -\tilde{u}(12) - \tilde{u}(12)(34) + \tilde{u}(1234),$$

whence

$$1 = \tilde{u}(12)(34)$$

$$0 = \tilde{u}(12) + \tilde{u}(1234)$$

$$1 + \chi_4(u) = 2\tilde{u}(1234)$$

Thus, we can construct the following table:

	$\chi_4(u)$	$\tilde{u}(12)$	$\tilde{u}(123)$	$\tilde{u}(12)(34)$	$\tilde{u}(1234)$
(a_{11})	3	-2	0	1	2
(a_{12})	-3	1	0	1	-1
(a_{13})	1	-1	0	1	1
(a_{14})	-1	0	0	1	0

From (1), $\chi_5(u) = \tilde{u}(12) - \tilde{u}(12)(34) - \tilde{u}(1234)$. Then, the case (a_{11}) brings the contradiction $\chi_5(u) = -5$.

The cases (a_{12}) and (a_{13}) also can not occur, since $2|\tilde{u}(1234)$.

Now, if we have (a_{14}) and take, in Theorem 1, U as the subgroup generated by u, we can conclude that u is rationally conjugate to (12)(34).

If we proceed at the same way in (a_2), we obtain

$$0 = \tilde{u}(12)(34)$$

$$1 = \tilde{u}(12) + \tilde{u}(1234)$$

$$1 + \chi_4(u) = 2\tilde{u}(1234).$$

Thus, in this case we have the following table:

	$\chi_4(u)$	$\tilde{u}(12)$	$\tilde{u}(123)$	$\tilde{u}(12)(34)$	$\tilde{u}(1234)$
(a_{21})	3	-1	0	0	2
(a_{22})	-3	2	0	0	-1
(a_{23})	1	0	0	0	1
(a_{24})	-1	1	0	0	0

In (a_{21}) we have $\rho_1(u) = 1$, $\rho_2(u) = -1$, $\rho_4(u) = I_3$ and $\rho_5(u) = -I_3$. Then, by [8, p. 49, Proposition 11], it follows that

$$u(g) = \frac{1}{24}\left(1-\chi_2(g^{-1}) + 2\chi_3(g^{-1}u) + 3\chi_4(g^{-1}) - 3\chi_5(g^{-1})\right), \text{ for all } g \in S_4.$$

Now, as we observe in §1, K is contained in the kernel of ρ_3; thus, since $(12)(1423) \in K$, $\rho_3(12) = \rho_3(1423)$.

Then,

$$u(12) = \frac{1}{24}\left(1 + 1 + 2\chi_3(g^{-1}u) - 3 - 3\right) = \frac{1}{12}\left(\chi_3(g^{-1}u)-2\right)$$

and

$$u(1324) = \frac{1}{24}\left(1 + 1 + 2\chi_3(g^{-1}u) + 3 + 3\right) = \frac{1}{12}\left(\chi_3(g^{-1}u)+4\right);$$

hence

$$\chi_3(g^{-1}u) - 2 \equiv 0 \pmod{12} \quad \text{and} \quad \chi_3(g^{-1}u) + 4 \equiv 0 \pmod{12},$$

that it is a contradiction.

Again since $2|\tilde{u}(1234)$, (a_{22}) and (a_{23}) must not happen and, from Theorem 1, (a_{24}) implies that u is rationally conjugate to (12).

(b) By Theorem 2, $\tilde{u}(g)$ and $\tilde{u}^{-1}(g)$ are non zero only if g is conjugate to (123). Then, by Theorem 1, we can conclude that u is rationally conjugate to (123).

(c) By (3), there exists $h \in H$ and a unit $\gamma \in \mathbb{Q}H$ such that $o(h)|o(u)$ and $u - \gamma^{-1}h\gamma \in \Delta_{\mathbb{Z}}(S_4, K)$. Then we can suppose

$$u-1 \in \Delta_{\mathbb{Z}}(S_4,K) \quad \text{or} \quad u-\gamma^{-1}(12)\gamma \in \Delta_{\mathbb{Z}}(S_4,K).$$

Now, K is a group of exponent 2. Then, it is not difficult to prove that $2\Delta_{\mathbb{Z}}(S_4,K) \subseteq \Delta_{\mathbb{Z}}(S_4,K)^2$, since $1 = (1 + (k-1))^2 = 1 + 2(k-1) + (k-1)^2$, for all $k \in K$. Besides, since K is abelian, it follows, from [1, p. 175, Lemma 3.3], that the group of units in $1 + \Delta_{\mathbb{Z}}(S_4,K)^2$ is torsion free.

Thus, if $u-1 \in \Delta_{\mathbb{Z}}(S_4,K)$, then $u^2 \in 1 + \Delta_{\mathbb{Z}}(S_4,K)^2$, that brings the contradiction $u^2 = 1$.

Therefore, we must have $u-\gamma^{-1}(12)\gamma \in \Delta_{\mathbb{Z}}(S_4,K)$.

On the other hand, we have seen in §1 that $\chi_2(\Delta_{\mathbb{Z}}(S_4,K)) = \{0\}$; hence we can conclude that $\chi_2(u) = -1$.

Now, by Theorem 2, $\tilde{u}(123) = 0$. Then, by (1) and (2),

$$0 = \chi_1(u) + \chi_2(u) = 2\tilde{u}(12)(34)$$

$$1 = \chi_1(u) = \tilde{u}(12) + \tilde{u}(12)(34) + \tilde{u}(1234)$$

$$\chi_4(u) = -\tilde{u}(12) - \tilde{u}(12)(34) + \tilde{u}(1234);$$

thus

$$0 = \tilde{u}(12)(34)$$

$$1 = \tilde{u}(12) + \tilde{u}(1234)$$

$$1 + \chi_4(u) = 2\tilde{u}(1234).$$

Besides, since $o(u) = 4$ and $\chi_4(u) \in \mathbb{Z}$, $\rho_4(u)$ is similar to I_3, $-I_3$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$ or $\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$. Then $\chi_4(u)$ can only be 3, -3, 1 or -1.

Therefore, we can construct the same table obtained in (a_2). Of course, here, we must consider the possibilities (a_{21}), (a_{22}), (a_{23}) and (a_{24}) for $o(u) = 4$.

In any case, by Theorem 3, we have

$$u^2 + \tilde{u}(12)^2 + \tilde{u}(1234)^2.(13)(24) \equiv 0(\mathrm{mod}\ (\Lambda + 2\,\mathbb{Z}S_4)).$$

On the other hand, u^2 is rationally conjugate to (12)(34), since $o(u^2) = 2$ and $\rho_2(u^2) = \rho_2(u)^2 = 1$.

Therefore, by Theorem 1, we can conclude that $2 | 1+\tilde{u}(1234)$, which implies that (a_{21}) and (a_{24}) must not occur.

In (a_{22}), we should have $\rho_4(u) = -I_3$. Then, $\rho_4(u^2) = I_3$, which brings the contradiction $3 = \chi_4(u^2) = \chi_4((12)(34))$.

Finally, if we have (a_{23}), by Theorem 1, we can conclude that u is rationally conjugate to (1234).

Theorem 5. (ZC) holds for S_4.

Proof. Let u be a torsion unit in $U_1(\mathbb{Z}S_4)$. By [7, p. 177, Theorem 2.1], $o(u)$ can only be 2, 3, 4, 6 or 12.

We will prove that if $u^6 = 1$, then $u^2 = 1$ or $u^3 = 1$. Clearly, from this and Theorem 4, it follows that (ZC) is true for S_4.

By (3), we can suppose that $u-1 \in \Delta_{\mathbb{Z}}(S_4, K)$ or there exists a unit $\gamma \in QH$ such that $u-\gamma^{-1}(12)\gamma \in \Delta_{\mathbb{Z}}(S_4, K)$ or $u-\gamma^{-1}(123)\gamma \in \Delta_{\mathbb{Z}}(S_4, K)$.

If $u-1 \in \Delta_{\mathbb{Z}}(S_4, K)$, then, as in the proof of Theorem 4(c), we can conclude that $u^2 = 1$.

If $u-\gamma^{-1}(12)\gamma \in \Delta_{\mathbb{Z}}(S_4, K)$, then $u^2-1 \in \Delta_{\mathbb{Z}}(S_4, K)$; thus, as before, $u^4 = 1$, which implies $u^2 = u^6(u^4)^{-1} = 1$.

Now, suppose that $u-\gamma^{-1}(123)\gamma \in \Delta_{\mathbb{Z}}(S_4, K)$ and let $v = (g \circ f)(u)$, where f and g are the homomorphisms considered in §1. Of course $v = \gamma^{-1}(123)\gamma$ and it is not difficult to see that $\tilde{v}(123) = \tilde{u}(123)$. Then, by Theorem 1, we can conclude that $1 = \tilde{v}(123) = \tilde{u}(123)$ and, hence, by Theorem 3,

$$u^3-\tilde{u}(12)^3.(12) -1 - \tilde{u}(12)(34)^3.(12)(34)-\tilde{u}(1234)^3.(1432) \equiv 0 (\mathrm{mod}(\Lambda+3\mathbb{Z}S_4$$

Therefore we must have $u^3 = 1$, since on the contrary, by [7, p. 45, Corollary 1.2], we should have the contradiction $1 \equiv 0(\mathrm{mod}\ 3)$.

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