THE STABILITY OF MINIMAL CONES OF CODIMENSION GREATER THAN ONE IN $\mathbb{R}^n$

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§0 - Introduction

J. Simons has proved in [3] that if M is a $(n-1)$-dimensional closed minimal submanifold in S^n, not totally geodesic, with $n \leq 6$, then the cone over M is not stable.

This result is fundamental for regularity of the solutions of the Plateau's problem and for the Bernstein's problem of codimension one, in $\mathbb{R}^{n+1}$ ([1], [3]).

For higher codimension, the first observation is that Simon's theorem cannot be extended if further hypotheses are not added. Indeed,

$$M = \{Z \in \mathbb{C}^n \mid Z \neq 0,\ P(Z) = 0\} \subset \mathbb{R}^{2n},$$

where $P(z) = \sum_{j=1}^{n} z_j^2$, is a 2-codimensional stable minimal cone. It is easy to verify that the immersion $M^{2n-2} \hookrightarrow \mathbb{R}^{2n}$ does not have flat normal bundle.

The existence of such manifolds as well as technical motivations led us to consider cones of higher codimension over submanifolds of S^n that have flat normal bundle.

The following theorem generalizes Simons' result:

Theorem 1. *Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal cone immersed in $\mathbb{R}^{n+m}$ such that $M \cap S^{n+m-1}$ is a closed manifold of S^{n+m-1} (the unitary ball in $\mathbb{R}^{n+m}$) and suppose that there is a global orthonormal family of parallel fields in the normal bundle of M. Then, if M is not totally geodesic in $\mathbb{R}^{n+m}$ and $n \leq 6$, M is not stable.*

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The following shows that this result is sharp.

Theorem 2. *Let* $M = \tilde{M} \times (0, +\infty) \hookrightarrow \mathbb{R}^{n+m}$; $(y,t) \longmapsto ty$ *where* $\tilde{M} = S^{n_1}(r_1) \times \dots \times S^{n_s}(r_s)$,

$$\sum_{i=1}^{s} n_i = n - 1, \qquad \sum_{i=1}^{s} r_i^2 = 1 \text{ and } m = s-1.$$

Then M *is a minimal cone in* $\mathbb{R}^{n+m}$ *having flat normal bundle, and* M *is stable if and only if* $n > 6$.

§1 - Basic Facts

1.1 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion and let $f \in C^1(U)$, where U is an open set of $\mathbb{R}^{n+m}$ such that $U \cap M \neq \emptyset$.

Let δf be the gradient in M of the restriction of f to M, that is

$$\delta f = \operatorname{grad}_M (f|_M).$$

(it is easy to see that δf depends only on the values of f over M) Then, if $\{\mu_i\}_{1 \le i \le n}$ is an orthonormal basis of T_pM, Df is the gradient of f in $\mathbb{R}^{n+m}$, that is,

$$Df = \left\{ \frac{\partial f}{\partial \mu_1}, \dots, \frac{\partial f}{\partial \mu_{n+m}} \right\};$$

and if $\{V^{(k)}\}_{1 \le k \le m}$ is an orthonormal frame of normal vector fields defined in $U \cap M$, then

(1.1.1) $$\delta f = Df - \sum_{k=1}^{m} \langle Df, V^{(k)} \rangle V^{(k)}.$$

Let $\{e_i\}_{1 \le i \le n+m}$ be an orthonormal basis in $\mathbb{R}^{n+m}$. Thus, if $V_i^{(k)} = \langle V^{(k)}, e_i \rangle$; $D_i f = \langle Df, e_i \rangle$; $\delta_i f = \langle \delta f, e_i \rangle$ we obtain

(1.1.2) $$\delta_i f = D_i f - \sum_{k=1}^{m} \langle Df, V^{(k)} \rangle V_i^{(k)}; \quad i = 1, 2, \dots, n+m.$$

1.2 - If H is the mean curvature field of the immersion, and

$$e_{n+k} = V^{(k)}(p); \quad k = 1, \dots, m,$$

then at p,

$$\delta_i = D_i \text{ if } i \le n, \quad \delta_{n+k} = 0$$

and

$$\langle H, V^{(k)} \rangle_p = -\sum_{i=1}^{n} \langle e_i, \bar{\nabla}_{e_i} V^{(k)} \rangle_p = -\sum_{i=1}^{n} D_i V_i^{(k)} = -\sum_{i=1}^{n+m} \delta_i V_i^{(k)}(p),$$

where $\bar{\nabla}$ is the riemannian connection of $\mathbb{R}^{n+m}$.

Since $\sum_{i=1}^{n+m} \delta_i V_i^{(k)}$ does not depend on the particular coordinate system, we obtain

(1.2.1) $$H = -\sum_{k=1}^{m} \left\{ \sum_{i=1}^{n+m} \delta_i V_i^{(k)} \right\} V^{(k)},$$

at any point of M and with respect to any coordinate system of $\mathbb{R}^{n+m}$.

Similarly, if $B(x,y) = (\bar{\nabla}_x Y)^{\perp}$ where $x, y \in T_pM$ and Y is an arbitrary extension of y that is tangent to M in a neighborhood of p, and

$$\langle A^v(x), y \rangle = \langle B(x,y), v \rangle; \quad \forall\, x,y \in T_pM, \quad \forall\, v \in (T_pM)^{\perp}, \quad \forall\, p \in M,$$

(1.2.2) we have, $$\langle A^{V^{(k)}}, A^{V^{(s)}} \rangle = \sum_{i,j=1}^{n+m} (\delta_i V_j^{(k)})(\delta_j V_i^{(s)}).$$

In what follows, we set

$$C_{ks} = \langle A^{V^{(k)}}, A^{V^{(s)}} \rangle$$

(1.2.3) $$C_k^2 = C_{kk}$$

$$C^2 = \sum_{k=1}^{m} C_k^2 = \|A\|^2$$

1.3 - Let X be a C^∞ vector field of M and $\mathrm{div}_M X$ be its divergence in M, that is, if $p \in M$,

$$\mathrm{div}_M X(p) = \mathrm{trace}\,(\omega \longmapsto \nabla_\omega X),$$

where $\omega \in T_p M$ and ∇ is the riemannian connection of M. Then

$$(1.3.1)\qquad \mathrm{div}_M X = \sum_{i=1}^{n+m} \delta_i X_i$$

at any point of M and in relation to any coordinate system of $\mathbb{R}^{n+m}$. Therefore, the laplacian Δf of $f \in C^2(M)$ is given by

$$(1.3.2)\qquad \Delta f = \sum_{i=1}^{n+m} \delta_i \delta_i f.$$

1.4 - If $\phi \in C^1(M)$ has compact support and $\Psi \in C^2(M)$, we obtain

$$(1.4.1)\qquad a)\ \int_M \delta\phi\, dM = -\int_M H\phi\, dM$$

$$(1.4.2)\qquad b)\ \int_M \phi\Delta\Psi\, dM = \int_M \Psi\Delta\phi\, dM = -\int_M \langle\delta\phi, \delta\Psi\rangle dM.$$

1.5 - Let $M^n \hookrightarrow N^{n+m}$ be an isometric immersion and let $\nabla^\perp$ be the induced connection in the normal bundle of M by the connection of N. Thus the normal curvature tensor of the normal bundle is defined by

$$(1.5.1)\qquad R^\perp(X,Y) = \nabla^\perp_x \nabla^\perp_y - \nabla^\perp_y \nabla^\perp_x - \nabla^\perp_{[X,Y]}$$

where X, Y are tangent fields of M.

Suppose that $R^\perp \equiv 0$; then we say that the immersion has flat normal bundle. The following well known results will be useful:

a) $R^\perp \equiv 0$ if and only if there is a local orthonormal family $\{V^{(k)}\}_{1\le k\le m}$ of parallel sections in the normal bundle, that is,

$$(1.5.2)\qquad \nabla^\perp_x V^{(k)} = 0;\quad k = 1,2,\ldots,m,\quad \forall\, x \in T_pM,\ \forall\, p \in M.$$

b) If N has constant curvature, then $R^\perp \equiv 0$ if and only if the tensors A^{e_α}, where $\{e_\alpha\}_{1\le\alpha\le m}$ is any orthonormal frame of normal fields, are simultaneously diagonalizable.

1.6 - Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion having flat normal bundle and let $\{V^{(k)}\}_{1\le k\le m}$ be an orthonormal family of parallel sections of the normal bundle of M. Let $p \in M$, and let be a coordinate system for $\mathbb{R}^{n+m}$ such that

$$V^{(k)}(p) = e_{n+k};\quad k = 1,2,\ldots,m.$$

Thus, from

$$\langle\bar\nabla_{e_i} V^{(k)}, e_j\rangle_p = \langle\bar\nabla_{e_j} V^{(k)}, e_i\rangle_p \qquad i,j=1,2,\ldots,n \quad k=1,2,\ldots,m$$

and $\delta_{n+s} \equiv 0$ at p,

and $0 = \langle\bar\nabla_{e_i} V^{(k)}, V^{(s)}\rangle_p = \delta_i V^{(k)}_{n+s}(p)$, if

$$i = 1,2,\ldots,n;\quad k,s = 1,2,\ldots,m$$

it follows:

$$\delta_i V^{(k)}_j(p) = \delta_j V^{(k)}_i(p);\qquad i,j=1,2,\ldots,n+m;\quad k=1,2,\ldots,m$$

with respect to $\{e_i\}_{1\le i\le n+m}$.

Considering another system $\{\bar e_i\}_{1\le i\le n+m}$ of coordinates, where

$$e_j = \sum_{i=1}^{n+m} \alpha^i_j \bar e_i$$

if $\bar V^{(k)}_i = \langle V^{(k)}, \bar e_1\rangle$ and $\bar\delta_i = \langle\delta, e_i\rangle$, we obtain

$$(\bar{\delta}_i \bar{V}_j^{(k)} - \bar{\delta}_j \bar{V}_i^{(k)})(p) = \sum_{h,\ell=1}^{n+m} \alpha_h^j \alpha_\ell^i (\delta_\ell V_h^{(k)} - \delta_h V_\ell^{(k)})(p) = 0.$$

Therefore

$$\bar{\delta}_i \bar{V}_j^{(k)}(p) = \bar{\delta}_j \bar{V}_i^{(k)}(p).$$

Thus, for any system of coordinates of $\mathbb{R}^{n+m}$, at each point of M,

(1.6.1) $$\delta_i V_j^{(k)} = \delta_j V_i^{(k)}\,; \qquad i,j = 1,2,\ldots,n+m, \quad k = 1,2,\ldots,m.$$

1.7. Proposition. *Let* $\{V^{(k)}\}_{1\le k\le m}$ *be as in* 1.6, *let* $\{e_i\}_{1\le i\le n+m}$ *be any coordinate system of* $\mathbb{R}^{n+m}$ *and let* $\delta_i = \langle\delta, e_i\rangle$. *Then*

(1.7.1) $$[\delta_i,\delta_j] = \delta_i\delta_j - \delta_j\delta_i = \sum_{k=1}^{m}\sum_{h=1}^{n+m} (V_i^{(k)}\delta_i V_h^{(k)} - V_j^{(k)}\delta_i V_h^{(k)})\delta_h$$

Proof. Having in mind that $R \equiv 0$ together with (1.1.2), (1.6.1) and the fact that

(1.7.2) $$\sum_{h=1}^{n+m} V_h^{(r)}\delta_j V_h^{(k)} = 0; \qquad j = 1,2,\ldots,n+m, \quad k,r = 1,2,\ldots,m,$$

the proof is a straightforward computation.

To prove (1.7.2) we choose a coordinate system $\{e_i\}_{1\le i\le n+m}$ of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$ for all k.

Then,

$$\sum_{h=1}^{n+m} V_h^{(r)}\delta_j V_h^{(k)}(p) = \langle\bar{\nabla}_{e_j} V^{(k)}, V^{(r)}\rangle_p = 0$$

because $R^{\perp} \equiv 0$. Now we argue as in the end of 1.6 for any system of coordinates.

1.8. Proposition. *With the hypothesis of* 1.6 *and supposing that the immersion is minimal,*

(1.8.1) $$\Delta V^{(k)} = -C_k^2 V^{(k)} - \sum_{\substack{k,r=1\\ k\ne r}}^{m} C_{kr} V^{(r)},$$

where $\Delta V^{(k)} = (\Delta V_1^{(k)},\ldots,\Delta V_{n+m}^{(k)})$ *and the components of* $V^{(k)}$ *are taken with respect to an arbitrary coordinate system of* $\mathbb{R}^{n+m}$.

Proof. The flatness of the normal bundle implies

$$\sum_{h=1}^{n+m} V_h^{(r)}\delta_j V_h^{(k)} = 0; \qquad j=1,2,\ldots,n+m, \quad k,r=1,2,\ldots,m.$$

Thus

$$\Delta V_j^{(k)} = \sum_{i=1}^{n+m} \delta_i\delta_i V_j^{(k)} = \sum_{i=1}^{n+m} \delta_i\delta_j V_i^{(k)} =$$

$$= \sum_{i=1}^{n+m} \delta_j\delta_i V_i^{(k)} + \sum_{r=1}^{m}\sum_{i,h=1}^{n+m} (V_i^{(r)}\delta_j V_h^{(r)} - V_j^{(r)}\delta_i V_h^{(r)})\delta_h V_i^{(k)}.$$

(1.8.2) $$\Delta V_j^{(k)} = -\sum_{r=1}^{m} V_j^{(r)}\Big[\sum_{i,h=1}^{n+m} (\delta_i V_h^{(r)})(\delta_h V_i^{(k)})\Big]$$

Therefore (1.8.1) follows from (1.2.2) and (1.2.3).

§2 - The Laplacian of the norm of the second fundamental form

2.1. Theorem. Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal isometric immersion having flat normal bundle. Let $\{V^{(k)}\}_{1\le k\le m}$ be a local orthonormal family of parallel sections of the normal bundle defined in an open subset U of M. If $\{e_i\}_{1\le i\le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$; $k=1,\dots,m$, $p \in M$, we have

$$(2.1.1) \qquad \left(\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{k,r=1\\ k\neq r}}^{m} C_{kr}\right)(p) = \sum_{hij=1}^{n} (\delta_i\delta_j V_h^{(k)})^2(p)$$

for all $k=1,2,\dots,m$.

Proof. Since $R^{\perp} = 0$, $C_k^2 = \sum_{ij=1}^{n+m} (\delta_i V_j^{(k)})^2$.

Therefore,

$$\frac{1}{2}\Delta C_k^2 = \frac{1}{2}\sum_{i=1}^{n+m} \delta_i\delta_i\left[\sum_{hj=1}^{n+m} (\delta_h V_j^{(k)})^2\right] =$$

$$= \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})^2 + \sum_{hij=1}^{n+m} (\delta_h V_j^{(k)})(\delta_i\delta_i\delta_h V_j^{(k)}).$$

Using $[\delta_i,\delta_h]$ twice we have:

$$\frac{1}{2}\Delta C_k^2 = \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})^2 + \sum_{hij=1}^{n+m} (\delta_h V_j^{(k)})\delta_h(\Delta V_j^{(k)}) +$$

$$\sum_{\ell=1}^{m}\sum_{hijs} V_i^{(\ell)}(\delta_h V_j^{(k)})(\delta_h V_s^{(\ell)})(\delta_s\delta_i V_j^{(k)}) -$$

$$\sum_{\ell=1}^{m}\sum_{hijs} (\delta_h V_j^{(k)})(\delta_i V_h^{(\ell)})(\delta_i V_s^{(\ell)})(\delta_s V_j^{(k)}).$$

From (1.8.1) we obtain:

$$\delta_h(\Delta V_j^{(k)}) = -(\delta_h C_k^2)V_j^{(k)} - (\delta_h V_j^{(k)})C_k^2 - \sum_{\substack{r=1\\ r\neq k}}^{m} \delta_h(C_{kr})V_j^{(r)} - \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}\delta_h V_j^{(r)}.$$

Therefore (1.2.3) and (1.7.2) imply:

$$(\delta_h V_j^{(k)})\delta_h(\Delta V_j^{(k)}) = -C_k^4 - \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}^2.$$

Thus

$$\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}^2 =$$

$$= \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})^2 - 2\sum_{\ell=1}^{m}\sum_{hijs=1}^{n+m} (\delta_h V_j^{(k)})(\delta_h V_s^{(\ell)})(\delta_s V_i^{(\ell)})(\delta_i V_j^{(k)}).$$

Using $[\delta_h,\delta_{n+r}]$ we have

$$\delta_h\delta_{n+r}V_j^{(k)}(p) = -\sum_{s=1}^{n+m} (\delta_h V_s^{(r)})(\delta_s V_j^{(k)})(p).$$

So at p:

$$\left(\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}^2\right)(p) =$$

$$= \sum_{hij=1}^{n+m} (\delta_i\delta_h V_j^{(k)})(p) - 2\sum_{\ell=1}^{m}\sum_{sj=1}^{n+m} (\delta_s\delta_{n+\ell}V_j^{(k)})^2(p).$$

Using again $[\delta_i,\delta_{n+\ell}](p)$ and

$$\delta_s V_{n+j}^{(k)}(p) = \langle\nabla^{\perp}_{e_s} V^{(k)}, e_{n+j}\rangle_p = 0$$

we obtain

$$\left(\frac{1}{2}\Delta C_k^2 + C_k^4 + \sum_{\substack{r=1\\ r\neq k}}^{m} C_{kr}^2\right)(p) = \sum_{hij=1}^{n} (\delta_i\delta_h V_j^{(k)})^2(p) +$$

$$+ \sum_{\ell=1}^{m}\sum_{ih=1}^{n} \delta_i\delta_h V_{n+\ell}^{(k)\,2}(p) + \sum_{\ell=1}^{m}\sum_{ij=1}^{n} (\delta_i\delta_{n+\ell}V_j^{(k)})^2(p) -$$

$$- 2 \sum_{\ell=1}^{m} \sum_{ij=1}^{n} (\delta_i \delta_{n+\ell} V_j^{(k)})^2 (p) =$$

$$= \sum_{hij=1}^{n} (\delta_i \delta_h V_j^{(k)})^2 (p).$$

2.2. Observation. Let $C^2 = \sum_{k=1}^{m} C_k^2$,

$$\Delta C^2 = \sum_{k=1}^{m} \Delta C_k^2 \quad \text{and} \quad C_{kr} = \langle A^{V^{(k)}}, A^{V^{(r)}} \rangle .$$

Thus,

$$\text{(2.2.1)} \qquad \sum_{k=1}^{m} C_k^4 + \sum_{\substack{kr=1 \\ r\neq k}}^{m} C_{kr}^2 \leq \sum_k C_k^4 + \sum_{k\neq r} C_k^2 C_r^2 = C^4 .$$

Therefore, with the same hipothesis of Theorem 2.1., (2.2.1) implies:

$$\text{(2.2.2)} \qquad (\tfrac{1}{2} \Delta C^2 + C^4)(p) \geq \sum_{k=1}^{m} \sum_{hij=1}^{n+m} (\delta_i \delta_h V_j^{(k)})^2 (p)$$

2.3. Theorem. *Let $M^n \hookrightarrow \mathbb{R}^{n+m}$ be a minimal cone isometrically immersed and having flat normal bundle. Let $\{V^{(k)}\}_{1\leq k\leq m}$ be a local orthonormal family of sections of the normal bundle defined in an open subset U of M. Then, if $\|x\|$ is the distance from $x \in M$ to the vertex of M, we have:*

$$\text{(2.3.1)} \qquad \frac{1}{2} \Delta C^2 + C^4 \geq |\delta c|^2 + \frac{2c^2}{\|x\|^2},$$

$$\text{(2.3.2)} \qquad \frac{1}{2} \Delta C_k^2 + C_k^4 + \sum_{\substack{r=1 \\ r\neq k}}^{m} C_{kr}^2 \geq |\delta C_k|^2 + \frac{2C_k^2}{\|x\|^2}; \qquad k=1,2,\ldots,m.$$

Proof. We may consider the vertex of the cone as the origin of $\mathbb{R}^{n+m}$,

$$U = \left\{ ty/y \in \tilde{U},\ \ t \in (0,+\infty) \right\}$$

where $\tilde{U}$ is an open subset of S^{n+m-1} (the euclidean unitary sphere), and that $V^{(k)}$ is constant through the rays of M.

Given $p \in M$, let $\{e_i\}_{1\leq i\leq n+m}$ be a coordinate system of $\mathbb{R}^{n+m}$ such that $V^{(k)}(p) = e_{n+k}$ for all k, and $p = \|p\| e_n$. Thus, for all $t \in (0,+\infty)$ and all k, we have:

$$D_n V^{(k)}(tp) = 0, \quad V_n^{(k)}(p) = 0,$$

$$\delta_i V_n^{(k)}(tp) = \delta_n V_i^{(k)}(tp) = (D_n V_i^{(k)} - \sum_{\ell=1}^{m} \langle V^{(\ell)}, DV_i^{(k)} \rangle V_n^{(\ell)})(tp) = 0.$$

Therefore,

$$\text{(2.3.3)} \qquad \delta_n(\delta_n V_i^{(k)})(tp) = \delta_n(\delta_i V_n^{(k)})(tp) = 0.$$

From (1.6.1) and (1.7.1) we obtain

$$\text{(2.3.4)} \qquad \delta_i \delta_j V_n^{(k)} = \delta_i \delta_n V_j^{(k)},$$

$$\text{(2.3.5)} \qquad \delta_i \delta_n V_j^{(k)}(p) = \delta_n \delta_i V_j^{(k)}(p).$$

So,

$$\Big[\sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - |\delta c|^2\Big](p) =$$

$$= \sum_{k=1}^{m} \sum_{ihr=1}^{n} (\delta_i \delta_h V_r^{(k)})^2 - \frac{1}{C^2} \sum_{k\ell=1}^{m} \sum_{ihrsj=1}^{n} (\delta_h V_r^{(k)})(\delta_i \delta_h V_r^{(k)})(\delta_s V_j^{(\ell)})(\delta_i \delta_s V_j^{(\ell)})$$

$$= \frac{1}{2C^2} \sum_{k\ell=1}^{m} \sum_{ihrsj=1}^{n} \Big[(\delta_h V_r^{(k)})^2 (\delta_i \delta_s V_j^{(\ell)})^2 + (\delta_s V_j^{(\ell)})^2 (\delta_i \delta_h V_r^{(k)})^2 -$$

$$- 2(\delta_h V_r^{(k)})(\delta_i \delta_h V_r^{(k)})(\delta_s V_j^{(\ell)})(\delta_i \delta_s V_j^{(\ell)}) \Big] =$$

$$= \frac{1}{2C^2}\sum_{k\ell=1}^{m}\sum_{ihrsj=1}^{n}\left[(\delta_h V_r^{(k)})(\delta_i\delta_s V_j^{(\ell)})-(\delta_s V_j^{(\ell)})(\delta_i\delta_h V_r^{(k)})\right]^2 =$$

$$= \frac{1}{2C^2}\sum_{k\ell=1}^{m}\left\{\sum_{i=1}^{n}\sum_{hjsr=1}^{n-1}\left[(\delta_h V_r^{(k)})(\delta_i\delta_s V_j^{(\ell)})-(\delta_s V_j^{(\ell)})(\delta_i\delta_h V_r^{(k)})\right]^2 + \right.$$

$$\left. + 4C_k^2\sum_{is=1}^{n}(\delta_n\delta_i V_s^{(\ell)})^2\right\}.$$

Therefore,

$$\text{(2.3.6)}\qquad \left[\sum_{k=1}^{m}\sum_{ihr=1}^{n}(\delta_i\delta_h V_r^{(k)})^2 - |\delta c|^2\right](p) \geq$$

$$\geq 2\sum_{k=1}^{m}\sum_{ih=1}^{n}(\delta_n\delta_i V_h^{(k)})^2(p).$$

Now, given any $\phi \in C^1(M)$, we have

$$\delta_n\phi(p) = \langle D\phi, e_n\rangle_p = \frac{1}{\|p\|}\langle D\phi, p\rangle = \frac{1}{\|p\|}\sum_{j=1}^{n+m} p_j D_j\phi(p).$$

Thus,

$$\text{(2.3.7)}\qquad \delta_n(\delta_i V_h^{(k)})(p) = \frac{1}{\|p\|}\sum_{j=1}^{n+m} p_j D_j(\delta_i V_h^{(k)})(p).$$

Since $V_h^{(k)}(x) = V_h^{(k)}(tx);\ \forall x \in M,\ \forall t \in (0,+\infty)$

$$\delta_i V_h^{(k)}(tx) = \frac{1}{t}\,\delta_i V_h^{(k)}(x)$$

which gives, by differentiation with respect to t,

$$\langle D(\delta_i V_h^{(k)})(tx), x\rangle = -\frac{1}{t^2}(\delta_i V_h^{(k)})(x).$$

In particular for $t = 1$,

$$\sum_{j=1}^{n+m} x_j D_j(\delta_i V_h^{(k)}) = \langle D(\delta_i V_h^{(k)})(x), x\rangle = -(\delta_i V_h^{(k)})(x), \qquad x \in M.$$

This together with (2.3.7) gives:

$$\sum_{ih=1}^{n}(\delta_n\delta_i V_h^{(k)})^2(p) = \frac{1}{\|p\|^2}\sum_{ih=1}^{n}(\delta_i V_h^{(k)})^2(p) = \frac{C_k^2(p)}{\|p\|^2}.$$

Thus, (2.3.6) implies

$$\text{(2.3.8)}\qquad \left[\sum_{k=1}^{m}\sum_{ihr=1}^{n}(\delta_i\delta_n V_r^{(k)})^2 - |\delta c|^2\right](p) \geq \frac{2C^2(p)}{\|p\|^2}.$$

From (2.2.2) and (2.3.8) we get (2.3.1). By a similar computation, we get for each k,

$$\left[\sum_{ihr=1}^{n}(\delta_i\delta_h V_r^{(k)})^2 - |\delta C_k|^2\right](p) \geq \frac{2C_k^2(p)}{\|p\|^2}$$

which together (2.2.1) gives (2.3.2).

§3 - Minimal Cones

3.1. The formula of the second variation of the area

Let $f:M^n \hookrightarrow \mathbb{R}^{n+m}$ be an isometric immersion of a compact orientable manifold M and let $F:M\times(-\varepsilon,\varepsilon) \to \mathbb{R}^{n+m}$ be a C^∞ variation of f such that

$$F_t(\partial M) = f(\partial M), \quad \text{for all } t.$$

Let $A(t)$ be the n-dimensional area of $F_t(M)$, $E(p) = \frac{\partial}{\partial t}F(p,0)$ the variational field, dM the volume form of M, $\tilde{A} = {}^tA\circ A$ and

$$\nabla^2 E^\perp = \sum_{i=1}^{n}\nabla^\perp_{\mu_i}\nabla^\perp_{\mu_i}E^\perp - \sum_{i=1}^{n}\nabla^\perp_{\nabla_{\mu_i}\mu_i}E^\perp,$$

where $\{\mu_i\}_{1\leq i\leq n}$ is an orthonormal tangent frame. Thus, if f is a minimal immersion having normal bundle and F is a variation such that $E^{\perp} = \sum_{k=1}^{m} h_k V^{(k)}$, where $\{V^{(k)}\}_{1 \le k \le m}$ is an orthonormal family of local parallel sections of the normal bundle of M, defined in an open set $U \subset M$, and each $h_k \in C_0^1(U)$, we have

$$(3.1.1) \qquad A''(0) = \int_M \langle -\nabla^2 E^{\perp} - \tilde{A}(E^{\perp}), E^{\perp} \rangle \, dM$$

(see [2])

3.2. Proposition. *With the hypothesis of 3.1,*

$$(3.2.1) \qquad A''(0) \int_M \left[\sum_{k=1}^{m} (|\delta h_k|^2 - h_k^2 C_k^2) - \sum_{\substack{kr=1 \\ k \ne r}}^{m} h_k h_r C_{kr} \right] dM.$$

Proof.

$$(3.2.2) \qquad \int_M \langle -\nabla^2 E^{\perp}, E^{\perp} \rangle dM = -\int_M \langle \nabla^{\perp} E^{\perp}, \nabla^{\perp} E^{\perp} \rangle dM,$$

where $\langle \nabla^{\perp} E^{\perp}, \nabla^{\perp} E^{\perp} \rangle = \sum_{i=1}^{n} \langle \nabla^{\perp}_{\mu_i} E^{\perp}, \nabla^{\perp}_{\mu_i} E^{\perp} \rangle$ for any orthonormal tangent frame $\{\mu_i\}_{1 \le i \le n}$. Moreover $R^{\perp} \equiv 0$ implies that

$$\langle \nabla^{\perp} E^{\perp}, \nabla^{\perp} E^{\perp} \rangle = \sum_{i=1}^{n} \sum_{k=1}^{m} (\mu_i [h_k])^2.$$

Thus, if $\{e_i\}_{1 \le i \le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ such that, for $p \in U$, $V^{(k)}(p) = e_{n+k}$; $k = 1, 2, \ldots, m$, we have

$$\langle \nabla^{\perp} E^{\perp}, \nabla^{\perp} E^{\perp} \rangle_p = \sum_{i=1}^{n} \sum_{k=1}^{m} (D_i h_k)^2 (p) =$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{m} (\delta_i h_k)^2 (p) = \sum_{k=1}^{m} \sum_{i=1}^{n+m} (\delta_i h_k)^2 (p) = \sum_{k=1}^{m} |\delta h_k|^2 (p).$$

Therefore, in M, we have:

$$(3.2.3) \qquad \langle \nabla^{\perp} E^{\perp}, \nabla^{\perp} E^{\perp} \rangle = \sum_{k=1}^{m} |\delta h_k|^2.$$

Since

$$-\langle \tilde{A}(E^{\perp}), E^{\perp} \rangle = -\langle A^{E^{\perp}}, A^{E^{\perp}} \rangle$$

$$(3.2.4) \qquad -\langle \tilde{A}(E^{\perp}), E^{\perp} \rangle = -\sum_{k=1}^{m} h_k^2 C_k^2 - \sum_{\substack{kr=1 \\ k \ne r}}^{m} h_k h_r C_{kr},$$

we have (3.2.1) from (3.1.1), (3.2.2), (3.2.3) and (3.2.4)

3.3. Proof of Theorem 1

Let $\{V^{(k)}\}_{1 \le k \le m}$ be an orthonormal family of parallel fields in the normal bundle of M. Thus, $V^{(k)}(y) = V^{(k)}(ty)$ for all t and $y \in M \cap S^{n+m-1}$, $k = 1, 2, \ldots, m$.

Supposing that M is stable and that E is the variational field of a variation of M such that

$$E^{\perp} = \sum_{k=1}^{m} h_k V^{(k)}; \; h_k \in C_0^1(M),$$

we have, from (3.2.1):

$$\int_M \left(\sum_{k=1}^{m} |\delta h_k|^2 - \sum_{k=1}^{m} h_k C_k^2 - \sum_{\substack{kr=1 \\ k \ne r}}^{m} h_k h_r C_{kr} \right) DM \ge 0.$$

In particular, if $h_k = \phi C_k$, $\phi \in C_0^1(M)$,

$$\delta h_k = C_k^2 |\delta \phi|^2 + \phi^2 |\delta C_k|^2 + \frac{1}{2} \langle \delta \phi^2, \delta C_k^2 \rangle, \quad \text{and}$$

$$(3.3.1) \qquad \int_M \phi^2 \left(\sum_{k=1}^{m} C_k^4 + \sum_{\substack{kr=1 \\ k \ne r}}^{m} C_k C_r C_{kr} \right) dM \le \int_M \left(C^2 |\delta \phi|^2 + \phi^2 \sum_{k=1}^{m} |\delta C_k|^2 + \frac{1}{2} \langle \delta \phi^2, \delta C^2 \rangle \right) dM,$$

which, together with (1.4.1) and (2.3.2), implies:

$$(3.3.2)\qquad \int_M \phi^2 \sum_{\substack{k,r=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}\, dM \le \int_M C^2|\delta\phi|^2\, dM - \int_M \phi^2 \frac{2C^2}{\|x\|^2}\, dM$$

(we observe that for $m = 1$, the first side of (3.3.2) is zero).

Now let ϕ such that $\phi(x) = \phi(ty) = f(t)$ where $x = ty$, $t = \|x\|$, $y \in M \cap S^{n+m-1}$ and $f \in C_0^1(0,+\infty)$. Thus, given $p \in M$, if $\{e_i\}_{1\le i\le n+m}$ is a coordinate system of $\mathbb{R}^{n+m}$ with origin at the vertex of M, and such that $V^{(k)}(p) = e_{n+k}$ for all $k = 1,2,\dots,m$ and $\|p\| e_n = p$, we have

$$\delta_i\phi(p) = D_i\phi(p);\quad i = 1,2,\dots,n,\quad k = 1,2,\dots,m$$

and so, $\delta_i\phi(p) = 0$ if $i \neq n$, $\delta_n\phi(p) = f'(\|p\|)$, $|\delta\phi|^2(p) = (f'(\|p\|))^2$.

Thus,

$$(3.3.3)\qquad |\delta\phi|^2(x) = |\delta\phi|^2(ty) = (f'(t))^2, \quad \forall\, x \in M.$$

Since

$$(3.3.4)\qquad \begin{cases} C^2(ty) = \dfrac{1}{t^2}\, C^2(y) \\ \displaystyle\sum_{\substack{k,r=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}(ty) = \frac{1}{t^4} \sum_{\substack{k,r=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}(y) \\ dM(ty) = t^{n-1}\, d\tilde{M}(y) \wedge dt;\quad \tilde{M} = M \cap S^{n+m-1} \end{cases}$$

we have, from (3.3.2), (3.3.3) and (3.3.4),

$$(3.3.5)\qquad \int_{\tilde{M}} \left\{\int_0^{+\infty} f^2(t)\, \frac{t^{n-1}}{t^4}\, dt\right\}\left\{\sum_{\substack{k,r=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}(y)\right\} d\tilde{M} \le$$

$$\le \int_{\tilde{M}} \left\{\int_0^{+\infty} (f'(t))^2\, \frac{t^{n-1}}{t^2} - f^2(t)\, \frac{2t^{n-1}}{t^4}\, dt\right\} C^2(y)\, d\tilde{M}, \quad \forall\, f \in C_0^1(0,+\infty).$$

Since $\int_{\tilde{M}} d\tilde{M} > 0$ and we may suppose that

$$K = \int_{\tilde{M}} \sum_{\substack{k,r=1\\ k\neq r}}^{m} (C_kC_r - C_{kr})C_{kr}\, d\tilde{M} \ge 0$$

(see 3.4), we have, from (3.3.5),

$$(3.3.6)\qquad \int_0^{+\infty} \left[(f'(t))^2 t^{n-3} - f^2(t)\, 2t^{n-5}\right] dt \ge 0, \quad \forall\, f \in C_0^1(0,+\infty).$$

(3.3.6) is still true for $f(t) = t^{\alpha} g(t)^{\beta}$, $t \in (0,+\infty)$ where $\alpha > 0$, $\alpha+\beta < 0$ and $g(t) = \max\{t,1\}$, if

$$(3.3.7)\qquad \int_0^{+\infty} f(t)\, t^{n-5}\, dt < +\infty$$

(see 3.5).

But

$$\int_0^{+\infty} f^2(t)\, t^{n-1}\, dt = \int_0^1 t^{n-5+2\alpha}\, dt + \int_1^{+\infty} t^{n-1+2(\alpha+\beta)}\, dt < +\infty$$

only if

$$(3.3.8)\qquad \alpha > \frac{4+n}{2},\quad \alpha+\beta < \frac{4+n}{2}.$$

Therefore, for these values of α and $\alpha+\beta$, (3.3.6) implies

$$(3.3.9)\qquad (\alpha^2-2)\int_0^1 t^{2\alpha+n-5}\, dt + ((\alpha+\beta)^2 - 2)\int_1^{+\infty} t^{2(\alpha+\beta)+n-5}\, dt \ge 0.$$

But if $\left(\frac{4+n}{2}\right)^2 < 2$, there is $\alpha > 0$, $\frac{4+n}{2} < \alpha < \sqrt{2}$, and β such that $\alpha+\beta > 0$ and $-\sqrt{2} < \alpha+\beta < \frac{4+n}{2} < \sqrt{2}$, which satisfy (3.3.8). For these values of α and β we have

$$\alpha^2 - 2 < 0 \quad\text{and}\quad (\alpha+\beta)^2 - 2 < 0$$

and this is incompactible with (3.3.9).

Therefore for those values of n such that $(\frac{4+n}{2})^2 < 2$, M cannot be stable. That is, if $2 \leq n \leq 6$, M is not stable.

3.4. Lemma. *If* $m \neq 1$, $K \geq 0$.

Proof. Let $a_{kr} = C_k C_r - C_{kr}$ and

$$S_i = \sum_{\substack{kr=1\\k\neq r}}^{m} \int_{\tilde M} a_{kr} \langle A^{V^{(k)}}, A^{V^{(r)}} \rangle d\tilde M, \quad i \geq 2.$$

By Schwarz inequality we have $a_{kr} \geq 0$. Thus changing, eventually, $V^{(2)}$ by $-V^{(2)}$,

$$S_2 = \int_{\tilde M} a_{12} \langle A^{V^{(1)}}, A^{V^{(2)}} \rangle d\tilde M \geq 0.$$

Supposing that for a given $i < m$,

$$S_i = \sum_{\substack{kr=1\\k\neq r}}^{m} \int_{\tilde M} a_{kr} \langle A^{V^{(k)}}, A^{V^{(r)}} \rangle d\tilde M \geq 0,$$

$$S_{i+1} = S_i + \sum_{k=1}^{i} \int_{\tilde M} a_{k(i+1)} \langle A^{V^{(k)}}, A^{V^{(i+1)}} \rangle d\tilde M$$

and changing, eventually, $V^{(i+1)}$ by $-V^{(i+1)}$ we have $S_{i+1} \geq 0$.

The result follows by induction.

3.5. Lemma. (3.3.6) *is true for* $f(t) = t^{\alpha} g(t)^{\beta}$ *where* $t \in (0,+\infty)$, *and* $\alpha > 0$ *and* $\alpha+\beta < 0$ *satisfy* (3.3.8).

Proof. Let

$$f_j(t) = \begin{cases} t^{\alpha}, & \text{if } \frac{1}{j} \leq t \leq 1 \\ t^{\alpha+\beta}, & \text{if } 1 \leq t \leq j \\ 0, & \text{if } t \notin [\frac{1}{j}, j], \quad j=1,2,\dots \end{cases}$$

Given $\varepsilon > 0$, let $\rho_\varepsilon(t) = \varepsilon\rho(\frac{t}{\varepsilon})$ where $\rho \in C_0^\infty(\mathbb{R})$, $\rho(t) \geq 0$, $\rho(t) = 0$ if $|t| \geq 1$ and $\int_{\mathbb{R}} \rho(t)dt = 1$.

Making the convolution of f_j with ρ_ε we get

$$f_{j,\varepsilon}(t) = \int_{\mathbb{R}} \rho_\varepsilon(t-\mu) f_j(\mu) d\mu.$$

Moreover, $\frac{d}{dt} f_{j,\varepsilon}(t) = f'_{j,\varepsilon}(t) = \int_{\mathbb{R}} \rho_\varepsilon(t-\mu) f'_j(\mu) d\mu.$

Since $f_{j,\varepsilon}$ and $f'_{j,\varepsilon}$ converge to f_j and f'_j, respectively, in $L^1(\mathbb{R})$, there is a subsequence $\{f_{j,\varepsilon_k}\}$ of $\{f_j\}$ such that $\{f_{j,\varepsilon_k}\}$ and $\{f'_{j,\varepsilon_k}\}$ converge almost everywhere to f_j and f'_j, respectively.

From the boundedness everywhere of f_j and f'_j, we have, as a consequence of the dominated convergence theorem,

$$\lim_{\varepsilon_k\to 0} \int_0^{+\infty} (f_{j,\varepsilon_k}(t))^2 t^{n-5} dt = \int_0^{+\infty} (f_j(t))^2 t^{n-5} dt,$$

$$\lim_{\varepsilon_k\to 0} \int_0^{+\infty} (f'_{j,\varepsilon_k}(t))^2 t^{n-3} dt = \int_0^{+\infty} (f'_j(t))^2 t^{n-3} dt.$$

Therefore, since (3.3.6) is true for any f_{j,ε_k} we have that (3.3.6) is also true for f_j. Now the result follows from the monotonic convergence theorem.

3.6. Proof of Theorem 2.

Let $\mathbb{R}^{n+m} = \mathbb{R}^{n_1+1} \times \dots \times \mathbb{R}^{n_s+1}$. It is easy to show that $\tilde M = S^{n_1}(r_1) \times \dots \times S^{n_s}(r_s) \subset S^{n+m-1}$ is a minimal submanifold of S^{n+m-1} and so that M is a minimal cone in $\mathbb{R}^{n+m}$.

Moreover, if $x = (x_1,\dots,x_s) \in \tilde M$,

$$e_0^i = \frac{1}{r_i}(0,\dots,0,\ x_i,\ 0,\dots,0); \quad i = 1,2,\dots,s,$$

$$R_k = \sum_{j=k+1}^{s} r_j^2; \quad R_s = 0, \text{ and}$$

(3.6.1) $$\tilde{V}^{(k)} = \frac{1}{\sqrt{1-R_k}\ \sqrt{1-R_{k+1}}}\left(r_1 r_{k+1} e_0^1 + \dots + r_k r_{k+1} e_0^k - (1-R_k) e_0^{k+1}\right),$$

$$k = 1,2,\dots,m$$

is straightforward to show that

a) $\{\tilde{V}^{(k)}\}_{1 \le k \le m}$ is an orthonormal family of parallel sections in the normal bundle of $\tilde{M}$ in S^{n+m-1};

b) With respect to it, $\tilde{C}_{k\ell} \equiv 0$ if $k \neq \ell$, $\tilde{C}_k^2 \equiv n-1$ and $\tilde{C}^2 \equiv m(n-1)$.

Thus setting $V^{(k)}(ty) = \tilde{V}^{(k)}(y)$; $y \in \tilde{M}$, $t \in (0,+\infty)$, we get a family of parallel orthonormal sections in the normal bundle of M in $\mathbb{R}^{n+m}$, such that with respect to it we have,

$$C_k^2(ty) = \frac{1}{t^2} \cdot \tilde{C}_k^2(y) = \frac{n-1}{t^2}$$

(3.6.2) $$C^2(ty) = \frac{\tilde{C}^2(y)}{t^2} = \frac{m(n-1)}{t^2}$$

$$C_{k\ell}(ty) = \frac{1}{t^2}\tilde{C}_{k\ell}(y) = 0, \text{ if } k \neq \ell.$$

Now let E be the variational field of a variation of M in $\mathbb{R}^{n+m}$ such that $E^{\perp} = \sum_{k=1}^{m} h_k V^{(k)}$, where $h_k \in C_0^{\infty}(M)$.

From (3.2.1) we have:

$$A''(0) = \int_M \sum_{k=1}^{m} \left(|\delta h_k|^2 - \frac{C^2}{m} h_k^2\right) dM = \sum_{k=1}^{m} \int_M \left(-h_k \Delta h_k - \frac{C^2}{m} h_k^2\right) dM.$$

The remainder of the proof follows step by step the one made by J. Simons [3, theorems 6.1.1 and 6.1.2, pag 97].

References

[1] H. Federer; *The singular sets of area minimizing*, Bull. A.M.S. (1970).

[2] J.B. Lawson; *Lectures on minimal submanifolds* (IMPA).

[3] J. Simons; *Minimal varieties in riemannian manifolds*, Ann. of Math., 88 (1968).

[4] E. Giusti; *Minimal Surfaces and Functions of Bounded Variation*, Notes on Pure Mathematics - Australian National University - 1977.

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