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On the uniqueness of the maximizing measure for rational maps

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I. Introduction

If f is a continuous transformation of a compact metric space X, the Dinaburg-Goodman-Goodwyn theorem states that

$$h_{top}(f) = sup\{h_\mu(f) \mid \mu \in \mathcal{M}(f)\}$$

where $h_{top}(f)$ is the topological entropy of f, $\mathcal{M}(f)$ is the space of f-invariant probabilities on the Borel σ-algebra of X and $h_\mu(f)$ denotes the μ-entropy of f. A measure $\mu \in \mathcal{M}(f)$ such that $h_\mu(f) = h_{top}(f)$ is called a *maximizing measure*. It is not difficult to exhibit examples of continuous maps without maximizing measures (see [2] p. 148 for instance). More delicate is the construction of C^r diffeomorphisms of compact manifolds $(1 \leq r < \infty)$ not having maximizing measures (Misiurevicz [6]) and it is still unknown whether there exist similar examples in the C^∞ class. Even more particular are the continuous transformations having a *unique* maximizing measure. Several interesting classes of transformations with this property are known, like transitive finite type subshifts, basic sets of Axiom A diffeomorphism and automorphisms of compact groups. The purpose of this paper is to prove that this property holds for analytic endomorphisms of the Riemann sphere.

Theorem. *Analytic endomorphisms of the Riemann Sphere* $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *with degree* ≥ 2 *have a unique maximizing measure.*

Analytic endomorphisms of the Riemann sphere are given by rational functions $f(z) = P(z)/Q(z)$, where the polynomials P and Q have no common roots. Its degree (that coincides with its topological degree) is defined as the maximum of the degrees of P and Q. According to the results of Gromov [4] and Misiurevicz-Przyticki [7], its topological entropy is given by the logarithm of the degree.

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Given an analytic endomorphism $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ with degree $d\geq 2$, an f-invariant measure μ_f with maximum entropy has been constructed by Freire, Lopes and Mañé [5] (and, according with the theorem above, is the unique one with this property). Let us briefly explain the construction of μ_f. Given $a\in\overline{\mathbb{C}}$ denote by $z_1^{(n)}(a),\ldots,z_{d^n}^{(n)}(a)$ the roots of the equation $f^n(z)-a=0$. Define a probability $\mu^{(n)}(a)$ by

$$\mu^{(n)}(a)=\frac{1}{d^n}\sum_{i=1}^{d^n}\delta_{z_i^{(n)}(a)}.$$

In [5], it is proved that for every $a\in\overline{\mathbb{C}}$ (with two possible exceptions easy to describe), the sequence of probabilities $\{\mu^{(n)}(a)\mid n\geq 1\}$ converges in the weak topology to an f-invariant measure μ_f, independent of a. Moreover it is proved that, with respect to μ_f, f is a K-system and that μ_f is the *unique f-invariant probability satisfying*

$$\mu_f(f(A))=d\mu_f(A) \tag{1}$$

for every Borel set where f/A is injective. This characterization will play a key role in the proof of the Theorem above, whose proof we now proceed partially outline.

Suppose that μ is an f-invariant probability. Assume that f is absolutely continuous with respect to μ, i.e. that $\mu(f(A))=0$ if $\mu(A)=0$. Then it is not difficult to find a μ-integrable function $J_\mu:\overline{\mathbb{C}}\to[0,+\infty)$ such that:

$$\mu(f(A))=\int_A J_\mu d\mu \tag{2}$$

for every Borel set $A\subset\overline{\mathbb{C}}$ where f/A is injective. Using a convenient partition of $\overline{\mathbb{C}}$, it is easy to prove that:

$$\int_{\overline{\mathbb{C}}} J_\mu d\mu=d. \tag{3}$$

Much more delicate is the proof of the formula for its entropy:

$$\int_{\overline{\mathbb{C}}}\log J_\mu d\mu=h_\mu(f). \tag{4}$$

Then, if μ is a maximizing measure:

$$\log d=h_\mu(f)=\int_{\overline{\mathbb{C}}}\log J_\mu d\mu\leq\log\int_{\overline{\mathbb{C}}}J_\mu d\mu=\log d. \tag{5}$$

Therefore:

$$\log\int_{\overline{\mathbb{C}}}J_\mu d\mu=\int_{\overline{\mathbb{C}}}\log J_\mu d\mu=\log d. \tag{6}$$

But this implies that $J_\mu=d$ μ-a.e.. Hence, from (2) and the absolute continuity of f, it follows that μ satisfies property (1) and we conclude that $\mu=\mu_f$. However, to prove the theorem, we have to introduce some changes in this method in order to make it work also when f is not absolutely continuous with respect to μ. Without the absolute continuity hypotheses, (2) cannot be expected to hold. What can be actually proved is that there exists a Borel set $A_\mu\subset\overline{\mathbb{C}}$, with $\mu(A_\mu^c)=0$ and such that (2) holds if f/A is injective and $A\subset A_\mu$. Equality (3) must then be corrected. It still holds replacing $=$ by $\leq$ and that is enough to get (6) using (5). Then we can reach the conclusion $J_\mu=d$ μ-a.e. With some minor technical work, this property still implies that μ has property (1), but is slightly less immediate than in the absolutely continuous case. Let us remark that (4) will be in fact proved only when μ is ergodic and $h_\mu(f)>0$. This is not an obstruction in our scheme because when a transformation has more than one maximizing measure, it has more than one maximizing ergodic measure ([2], Prop. 13.3).

The core (and most difficult part) of the proof is the entropy formula. As the rest of the proof of the Theorem, it becomes much simpler to prove in the absolutely continuous case.

There is a criteria, due to Bowen [1] and based on the method used by Parry to prove the uniqueness of the maximizing measure for finite type subshifts, from which this property can be deduced for finite type subshifts, basic sets of Axiom A diffeomorphisms and group automorphisms. We weren't able to deduce our theorem from this criteria. It is not clear at all whether the measure μ_f satisfies the very strong uniform estimates that are required by that criteria.

II. **Proof of the Theorem**

Let $f:\overline{\mathbb{C}}\circlearrowleft$ be an analytic endomorphism of degree $d\geq 2$. To formalize the sketch of the proof given in the introduction, the first step will be to prove existence of J_μ and A_μ.

Lemma II.1. *For every $\mu\in\mathcal{M}(f)$ without atoms, there exist a μ-integrable function $J_\mu:\overline{\mathbb{C}}\to[1,+\infty)$ and a Borel set A_μ with $\mu(A_\mu)=1$ such that*

$$\mu(f(A))=\int_A J_\mu d\mu \tag{*}$$

for every Borel set $A\subset A_\mu$ such that f/A is injective.

Proof. Using that μ has no atoms, it is easy to construct a family of disjoint topological disks (i.e., sets homeomorphic to a disk) not containing critical values of f and satisfying $\mu\left(\left(\bigcup_{i=1}^{m} U_i\right)^c\right)=0$. Let $g_j^{(i)}: U_i \to \overline{\mathbb{C}}, j=1,\dots,d$, $i=1,\dots,m$, be the branches of f^{-1}/U_i. Then if $A \subset U_i$ is a Borel set, we have

$$\mu(g_j^{(i)}(A)) \le \mu(A) \tag{1}$$

for every j because

$$\mu(g_j^{(i)}(A)) \le \mu(f^{-1}(A)) = \mu(A).$$

In particular, every $g_j^{(i)}$ transforms measure zero sets in measure zero sets. Then, by Radon-Nykodim theorem, there exist μ-integrable functions $H_j^{(i)}: U_i \to [0,+\infty)$ such that

$$\mu(g_j^{(i)}(A)) = \int_A H_j^{(i)} d\mu \tag{2}$$

for every $1 \le i \le m$, $1 \le j \le d$ and every Borel set $A \subset U_i$. Define

$$A_\mu = f^{-1}(\bigcup_i U_i) \cup \{g_j^{(i)}(x) \mid 1 \le i \le m,\ 1 \le j \le d,$$
$$x \in U_i,\ H_j^{(i)}(x) \ne 0\}.$$

From (2) it follows easily that $\mu(A_\mu^c)=0$ or $\mu(A_\mu)=1$. Moreover, (1) and (2) imply:

$$\int_A H_j^{(i)} d\mu \le \mu(A)$$

for every Borel set $A \subset U_i$. Hence, $H_j^{(i)}(x) \le 1$ for a.e. $x \in U_i$. Without loss of generality, we can then assume that

$$H_j^{(i)}(x) \le 1 \tag{3}$$

for all $1 \le i \le m$, $1 \le j \le d$, $x \in U_i$. Define $J_\mu: \mathbb{C} \to [1,+\infty)$ as

$$J_\mu(x) = (H_j^{(i)}(f(x)))^{-1} \tag{4}$$

if $x \in A_\mu \cap g_j^{(i)}(U_i)$ and

$$J_\mu(x) = 1$$

if $x \notin A_\mu$. The definition of A_μ grants that $H_j^{(i)}(f(x)) \ne 0$ in (4). Moreover, (3) implies that $J_\mu(x) \ge 1$. To verify (*) is sufficient to consider the case when A is contained in a set $g_j^{(i)}(U_i) \cap A_\mu$. To obtain from this the general case observe that if $A \subset A_\mu$ is a Borel set and f/A is injective, then the sets

$$A_j^{(i)} = g_j^{(i)}(f(A) \cap U_i)$$

satisfy

$$\bigcup_{i,j} A_j^{(i)} = A \cap f^{-1}(\bigcup_i U_i) \tag{5}$$

$$\bigcup_{i,j} f(A_j^{(i)}) = f(A) \cap (\bigcup_i U_i) \tag{6}$$

and

$$f(A_j^{(i)}) \cap f(A_\ell^{(k)}) = \varnothing \tag{7}$$

if $(i,j) \ne (k,\ell)$. The first two equalities are trivial. To prove the third suppose that the intersection is $\ne \varnothing$. Then $i=k$ because $f(A_j^{(i)}) \subset U_i$ and $f(A_\ell^{(k)}) \cap U_k$. Take y in the intersection. We can write $y=f(x_1)=f(x_2)$ with $x_1 \in A_j^{(i)}$, $x_2 \in A_\ell^{(i)}$. Since f/A is injective, it follows that $x_1=x_2$. Hence, $A_j^{(i)} \cap A_\ell^{(i)} \ne \varnothing$. This means $g_j^{(i)}(U_i) \cap g_\ell^{(i)}(U_i) \ne \varnothing$ that implies $j=\ell$ completing the proof of (7). From (5), (6), (7), it follows that:

$$\mu(f(A)) = \sum_{i,j} \mu(f(A_j^{(i)})) = \sum_{i,j} \int_{A_j^{(i)}} J_\mu d\mu = \int_{\bigcup_{i,j} A_j^{(i)}} J_\mu d\mu = \int_A J_\mu d\mu.$$

This completes the proof of the reduction step.

Suppose now that $A \subset g_j^{(i)}(U_i) \cap A_\mu$. Set

$$S_n = \{x \in f(A) \mid H_j^{(i)}(x) \ge 1/n\}.$$

The property $A \subset A_\mu$ implies that

$$\bigcup_{n \ge 1} S_n = f(A).$$

Moreover, the definition of $H_j^{(i)}$ implies:

$$\mu(S_n) = \int_{S_n} d\mu = \int_{S_n} H_j^{(i)}(x)(H_j^{(i)}(x))^{-1} d\mu(x) =$$

$$= \int_{S_n} H_j^{(i)}(J_\mu \circ g_j^{(i)}) d\mu = \int_{g_j^{(i)}(S_n)} J_\mu d\mu.$$

Hence:

$$\mu(f(A)) = \lim_{n\to+\infty} \mu(S_n) = \lim_{n\to+\infty} \int_{g_j^{(i)}(S_n)} J_\mu d\mu.$$

By the monotone convergence theorem, it follows that J_μ is integrable on

$$\bigcup_{n\ge1} g_j^{(i)}(S_n) = g_j^{(i)}(\bigcup_{n\ge1} S_n) = g_j^{(i)}(f(A)) = A$$

and

$$\int_A J_\mu d\mu = \lim_{n\to+\infty} \int_{g_j^{(i)}(S_n)} J_\mu d\mu = \mu(f(A)).$$

$A_{\mu_2} = \overline{\mathbb{C}}$. Applying the Lemma to μ_1, we obtain J_{μ_1} and A_{μ_1}. Then it is easy to see that $J_\mu = \lambda J_{\mu_1} + (1-\lambda)J_{\mu_2}$ and $A_\mu = A_{\mu_1}$ satisfy the required properties.

Corollary. $\int_{\overline{\mathbb{C}}} J_\mu d\mu \le d.$

Proof.

$$\int_{\overline{\mathbb{C}}} J_\mu d\mu = \sum_{i,j} \int_{g_j^{(i)}(U_i)} J_\mu d\mu = \sum_{i,j} \int_{g_j^{(i)}(U_i)\cap A_\mu} J_\mu d\mu =$$

$$= \sum_{i,j} \mu(f(g_j^{(i)}(U_i) \cap A_\mu)) \le \sum_{i,j} \mu(f(g_j^{(i)}(U_i))) =$$

$$= \sum_i \left(\sum_j \mu(f(g_j^{(i)}(U_i)))\right) = d \sum_i \mu(U_i).$$

Lemma II.2. *If $\mu \in \mathcal{M}(f)$ is ergodic and $h_\mu(f) > 0$, then:*

$$h_\mu(f) = \int_{\overline{\mathbb{C}}} \log J_\mu d\mu.$$

Proof. Endow $\overline{\mathbb{C}}$ with the standard Riemann structure. Then, if $\|(f^n)'(z)\|$ denotes the norm with respect to this structure of the linear maps $(f^n)'(z) : T_z\overline{\mathbb{C}} \to T_{f^n(z)}\overline{\mathbb{C}}$, we have

$$\|(f^n)'(z)\| = \prod_{j=0}^{n-1} \|f'(f^j(z))\|.$$

Denote by $d(\cdot\,,\cdot)$ the metric on $\overline{\mathbb{C}}$ associated to this Riemann structure. We shall need several lemmas to complete the proof of II.2. The first one concerns the integrability of $\log\|f'\|$ and the Lyapounov exponents of f.

Lemma II.3. *For every ergodic $\mu \in \mathcal{M}(f)$ with $h_\mu(f) > 0$, the function $z \to \log\|f'(z)\|$ is μ-integrable and*

(*) $$\int_{\overline{\mathbb{C}}} \log\|f'(z)\|\, d\mu(z) \ge \frac{1}{2} h_\mu(f).$$

Moreover,

(**) $$\lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \int_{\overline{\mathbb{C}}} \log\|f'(z)\|\, d\mu(z)$$

for μ – a.e. $x \in \overline{\mathbb{C}}$.

Proof. The function $\log\|f'\|$ is obviously measurable and upper bounded. Then, since μ is ergodic, either $\log\|f'\|$ is not μ-integrable and then

(1) $$\lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\|f'(f^j(x))\| = -\infty$$

for μ – a.e $x \in \overline{\mathbb{C}}$, or $\log\|f'\|$ is μ-integrable and:

(2) $$\lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \int_{\overline{\mathbb{C}}} \log\|f'\|\, d\mu$$

for μ-a.e. $x \in \overline{\mathbb{C}}$. When (1) holds, the Lyapounov exponent of f is $-\infty$. By Ruelle inequality [9], this implies $h_\mu(f) = 0$, contradicting the hypothesis. Then $\log\|f'\|$ is μ-integrable and (2) holds. Clearly, (2) and (**) state the same property. Moveover, (2) implies that the Lyapounov exponent (with multiplicity 2) of f is

$$\chi = \int_{\overline{\mathbb{C}}} \log\|f'\|\, d_\mu.$$

Using again Ruelle's inequality, we get $h_\mu(f) \le \max\{2\chi, 0\}$. From $h_\mu(f) > 0$, we obtain $\chi > 0$ and then

$$h_\mu(f) \le 2\chi = 2 \int_{\overline{\mathbb{C}}} \log\|f'\|\, d_\mu.$$

Corollary. *If z_0 is a critical point of f, the function $z \to \log d(z, z_0)$ is μ-integrable for every ergodic $\mu \in \mathcal{M}(f)$ with $h_\mu(f) > 0$.*

Proof. The μ-integrability of $z \to \log d(z, z_0)$ follows from the fact that it is bounded below by a constant times $\log\|f'(z)\|$.

Lemma II.3 and its Corollary will be used to prove the following result that is the key step in the proof of II.2.

Lemma II.4. *If $\mu \in \mathcal{M}(f)$ is ergodic and $h_\mu(f) > 0$, there exists a partition $\mathcal{P}$ of $\overline{\mathbb{C}}$ and a sequence of partitions $\mathcal{R}_0 \le \mathcal{R}_1 \le \ldots$ satisfying the following properties:*

and

$$\int_A J_\mu d\mu = \lim_{n\to+\infty} \int_{g_j^{(i)}(S_n)} J_\mu d\mu = \mu(f(A)).$$

$A_{\mu_2} = \overline{\mathbb{C}}$. Applying the Lemma to μ_1, we obtain J_{μ_1} and A_{μ_1}. Then it is easy to see that $J_\mu = \lambda J_{\mu_1} + (1-\lambda)J_{\mu_2}$ and $A_\mu = A_{\mu_1}$ satisfy the required properties.

Corollary. $\int_{\overline{\mathbb{C}}} J_\mu d\mu \le d.$

Proof.

$$\int_{\overline{\mathbb{C}}} J_\mu d\mu = \sum_{i,j} \int_{g_j^{(i)}(U_i)} J_\mu d\mu = \sum_{i,j} \int_{g_j^{(i)}(U_i)\cap A_\mu} J_\mu d\mu =$$

$$= \sum_{i,j} \mu(f(g_j^{(i)}(U_i)\cap A_\mu)) \le \sum_{i,j} \mu(f(g_j^{(i)}(U_i))) =$$

$$= \sum_i (\sum_j \mu(f(g_j^{(i)}(U_i)))) = d \sum_i \mu(U_i).$$

Lemma II.2. *If* $\mu \in \mathcal{M}(f)$ *is ergodic and* $h_\mu(f) > 0$, *then*:

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Proof. Endow $\overline{\mathbb{C}}$ with the standard Riemann structure. Then, if $\|(f^n)'(z)\|$ denotes the norm with respect to this structure of the linear maps $(f^n)'(z) : T_z\overline{\mathbb{C}} \to T_{f^n(z)}\overline{\mathbb{C}}$, we have

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Denote by $d(\cdot\,,\cdot)$ the metric on $\overline{\mathbb{C}}$ associated to this Riemann structure. We shall need several lemmas to complete the proof of II.2. The first one concerns the integrability of $\log\|f'\|$ and the Lyapounov exponents of f.

Lemma II.3. *For every ergodic* $\mu \in \mathcal{M}(f)$ *with* $h_\mu(f) > 0$, *the function* $z \to \log\|f'(z)\|$ *is* μ*-integrable and*

$$(*) \qquad \int_{\overline{\mathbb{C}}} \log\|f'(z)\|\, d\mu(z) \ge \frac{1}{2} h_\mu(f).$$

Moreover,

$$(**) \qquad \lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \int_{\overline{\mathbb{C}}} \log\|f'(z)\|\, d\mu(z)$$

for μ*-a.e.* $x \in \overline{\mathbb{C}}$.

Proof. The function $\log\|f'\|$ is obviously measurable and upper bounded. Then, since μ is ergodic, either $\log\|f'\|$ is not μ-integrable and then

$$(1) \qquad \lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\|f'(f^j(x))\| = -\infty$$

for μ-a.e $x \in \overline{\mathbb{C}}$, or $\log\|f'\|$ is μ-integrable and:

$$(2) \qquad \lim_{n\to+\infty} \frac{1}{n} \log\|(f^n)'(x)\| = \int_{\overline{\mathbb{C}}} \log\|f'\|\, d\mu$$

for μ-a.e. $x \in \overline{\mathbb{C}}$. When (1) holds, the Lyapounov exponent of f is $-\infty$. By Ruelle inequality [9], this implies $h_\mu(f) = 0$, contradicting the hypothesis. Then $\log\|f'\|$ is μ-integrable and (2) holds. Clearly, (2) and (**) state the same property. Moveover, (2) implies that the Lyapounov exponent (with multiplicity 2) of f is

$$\chi = \int_{\overline{\mathbb{C}}} \log\|f'\|\, d_\mu.$$

Using again Ruelle's inequality, we get $h_\mu(f) \le \max\{2\chi, 0\}$. From $h_\mu(f) > 0$, we obtain $\chi > 0$ and then

$$h_\mu(f) \le 2\chi = 2\int_{\overline{\mathbb{C}}} \log\|f'\|\, d_\mu.$$

Corollary. *If* z_0 *is a critical point of* f, *the function* $z \to \log d(z, z_0)$ *is* μ*-integrable for every ergodic* $\mu \in \mathcal{M}(f)$ *with* $h_\mu(f) > 0$.

Proof. The μ-integrability of $z \to \log d(z, z_0)$ follows from the fact that it is bounded below by a constant times $\log\|f'(z)\|$.

Lemma II.3 and its Corollary will be used to prove the following result that is the key step in the proof of II.2.

Lemma II.4. *If* $\mu \in \mathcal{M}(f)$ *is ergodic and* $h_\mu(f) > 0$, *there exists a partition* $\mathcal{P}$ *of* $\overline{\mathbb{C}}$ *and a sequence of partitions* $\mathcal{R}_0 \le \mathcal{R}_1 \le \ldots$ *satisfying the following properties*:

a) $\bigvee_{n=0}^{\infty} f^{-n}(\mathscr{P})$ is the Borel σ-algebra of $\overline{\mathbb{C}}$.

b) If $\mathscr{P}_n = \bigvee_{j=0}^{n} f^{-j}(\mathscr{P})$ and we denote by $\mathscr{P}_n(x)$, $\mathscr{R}_n(x)$ the atoms of $\mathscr{P}_n$ and $\mathscr{R}_n$ containing x, then, for every $n \geq 0$ and μ-a.e. x, f maps $\mathscr{R}_n(x)$ bijectively onto almost all $\mathscr{P}_n(f(x))$ and $\mathscr{P}_n(x) = \mathscr{R}_{n-1}(x) \cap \mathscr{P}(x)$.

c) $\bigvee_{n=0}^{\infty} \mathscr{R}_n$ is the Borel σ-algebra.

d) The μ-entropy of $\mathscr{P}$ and $\mathscr{R}_0$ is finite.

Before proving this lemma, we shall show how it is used to complete the proof of II.2. By properties (a) and (d) of II.4, in order to prove II.2 it is sufficient to show that the sequence of functions:

$$\phi_n(x) = -\frac{1}{n} \log \mu(\mathscr{P}_n(x))$$

converge in $\mathscr{L}^1$ to the μ-integral of $\log J_\mu$. Set:

$$F_n(x) = \log \frac{\mu(\mathscr{P}_{n-1}(f(x)))}{\mu(\mathscr{P}_n(x))}.$$

Then:

$$\phi_n(x) = -\frac{1}{n} \sum_{j=0}^{n-1} \log \frac{\mu(\mathscr{P}_{n-j}(f^j(x)))}{\mu(\mathscr{P}_{n-j-1}(f^{j+1}(x)))} - \frac{1}{n} \log \mu(\mathscr{P}(f^n(x))) =$$

$$= \frac{1}{n} \sum_{j=0}^{n-1} F_{n-j}(f^j(x)) - \frac{1}{n} \log \mu(\mathscr{P}(f^n(x))).$$

But since the μ-entropy of $\mathscr{P}$ is finite, the function $x \to |\log \mu(\mathscr{P}(x))|$ is μ-integrable and its integral is the μ-entropy of $\mathscr{P}$. The f-invariance of μ implies that the same is true for the functions $x \to |\log \mu(\mathscr{P}(f^n(x)))|$. Then the last term converges to zero in $\mathscr{L}^1$. Our problem is now reduced to show that the sequence of functions:

$$\frac{1}{n} \sum_{j=0}^{n-1} F_{n-j} \circ f^j$$

belong to $\mathscr{L}^1$ and converges in $\mathscr{L}^1$ to the integral of $\log J_\mu$. In order to insure this, it is enough to prove that the functions F_n are in $\mathscr{L}^1$ and converge in $\mathscr{L}^1$ to $\log J_\mu$, because if this is true, we write

$$\frac{1}{n} \sum_{j=0}^{n-1} F_{n-j} \circ f^j = \frac{1}{n} \sum_{j=0}^{n-1} (F_{n-j} - \log J_\mu) \circ f^j + \frac{1}{n} \sum_{j=0}^{n-1} (\log J_\mu)\, f^j$$

and by the ergodicity of μ the last term converges in $\mathscr{L}^1$ to the integral of $\log J_\mu$ and the first term satisfies:

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} (F_{n-j} - \log J_\mu)\, f^j \right\|_1 \leq \frac{1}{n} \sum_{j=0}^{n-1} \| F_{n-j} - \log J_\mu \|_1 =$$

$$= \frac{1}{n} \sum_{j=1}^{n} \| F_j - \log J_\mu \|_1$$

that converges to zero if $F_n \to \log J$ in $\mathscr{L}^1$. To prove this convergence write:

$$F_n(x) = \log \frac{\mu(\mathscr{P}_{n-1}(f(x)))}{\mu(\mathscr{P}_n(x))} = \log \frac{\mu(\mathscr{P}_{n-1}(f(x)))}{\mu(\mathscr{R}_{n-1}(x) \cap \mathscr{P}(x))} =$$

$$= -\log \frac{\mu(\mathscr{R}_{n-1}(x) \cap \mathscr{P}(x))}{\mu(\mathscr{R}_{n-1}(x))} + \log \frac{\mu(\mathscr{P}_{n-1}(f(x)))}{\mu(\mathscr{R}_{n-1}(x))} =$$

$$(1) \quad = -\log \frac{\mu(\mathscr{R}_{n-1}(x) \cap \mathscr{P}(x))}{\mu(\mathscr{R}_{n-1}(x))} - \log \frac{\mu(f(A_\mu \cap \mathscr{R}_{n-1}(x)))}{\mu(\mathscr{P}_{n-1}(f(x)))} +$$

$$+ \log \frac{\mu(f(A_\mu \cap \mathscr{R}_{n-1}(x)))}{\mu(\mathscr{R}_{n-1}(x))}.$$

Observe that:

$$\int_{\overline{\mathbb{C}}} \left| \log \frac{\mu(\mathscr{R}_{n-1}(x) \cap \mathscr{P}(x))}{\mu(\mathscr{R}_{n-1}(x))} \right| d\mu(x) = -\int_{\overline{\mathbb{C}}} \log \frac{\mu(\mathscr{R}_{n-1}(x) \cap \mathscr{P}(x))}{\mu(\mathscr{R}_{n-1}(x))}\, d\mu(x)$$

$$= H_\mu(\mathscr{P}/\mathscr{R}_{n-1}).$$

But by II.4 (c) we know that $H(\mathscr{P}/\mathscr{R}_{n-1}) \to 0$ if $n \to +\infty$. This shows that the first term in (1) converges to zero in $\mathscr{L}^1$. The same technique can be used to show the convergence to zero in $\mathscr{L}^1$ of the second term. However, this case is slightly more delicate. Take an atom $P \in \mathscr{P}$ and $R \in \mathscr{R}_0$ such that f maps R bijectively onto P. Consider the partition $\mathscr{M}$ of P with atoms $f(A_\mu \cap R)$ and $f(A_\mu^c \cap R)$. Consider also the partitions $\mathscr{M}_n$ of P whose atoms are those atoms of $\mathscr{P}_n$ contained in P. If $\mathscr{B}$ denotes the Borel σ-algebra of P, it follows from II.4 (a) that $\bigvee_0^{\infty} \mathscr{M}_n = \mathscr{B}$ and obviously $\mathscr{M}_0 \leq \mathscr{M}_1 \leq \ldots$. Define a probability μ_0 on $\mathscr{B}$ by $\mu_0(S) = \mu(S)/\mu(P)$. It is easy to check that:

$$\int_P \left| \log \frac{\mu(\mathscr{M}_n(x) \cap \mathscr{M}(x))}{\mu(\mathscr{M}_n(x))} \right| d\mu(x) = \mu(P) H_{\mu_0}(\mathscr{M}/\mathscr{M}_n).$$

The map $f/R : R \to P$ satisfies $\mu((f/R)^{-1}(S)) \leq \mu(S)$ for every Borel set S contained in P. Hence:

$$\int_R \left| \log \frac{\mu(\mathcal{M}_n(f(x)) \cap \mathcal{M}(f(x)))}{\mu(\mathcal{M}_n(f(x)))} \right| d\mu(x) \leq \int_P \left| \log \frac{\mu(\mathcal{M}_n(x) \cap \mathcal{M}(x))}{\mu(\mathcal{M}_n(x))} \right| d\mu(x)$$

$$= \mu(P) H_{\mu_0}(\mathcal{M}/\mathcal{M}_n).$$

Now observe that if $x \in R \cap A_\mu$, we have:

$$\mathcal{M}_n(f(x)) \cap f(A_\mu \cap R) = \mathcal{P}_n(f(x)) \cap f(A_\mu \cap R) =$$
$$= f(\mathcal{R}_n(x)) \cap f(A_\mu \cap R) = f(A_\mu \cap R \cap \mathcal{R}_n(x)) =$$
$$= f(A_\mu \cap \mathcal{R}_n(x)).$$

Then, using that $\mu(A_\mu \cap R) = \mu(R)$, we obtain

$$\int_R \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) = \int_R \left| \log \frac{\mu(\mathcal{M}_n(f(x)) \cap \mathcal{M}(f(x)))}{\mu(\mathcal{M}_n(f(x)))} \right| d\mu(x) \leq$$

$$\leq \mu(P) H_{\mu_0}(\mathcal{M}/\mathcal{M}_n).$$

Since $f^{-1}(P)$ contains d atoms of $\mathcal{R}$, we conclude that:

$$\int_{f^{-1}(P)} \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) \leq d\mu(P) H_{\mu_0}(\mathcal{M}/\mathcal{M}_n). \tag{2}$$

Given $\varepsilon > 0$ take a set S that is a finite union of atoms of $\mathcal{P}$ (say k atoms) and such that $\mu(S^c) \leq \varepsilon/(2d \log 2)$. Since $H_{\mu_0}(\mathcal{M}/\mathcal{M}_n) \to 0$ for every atom P, it follows from (2) that we can choose N such that if $n \geq N$ then

$$\int_{f^{-1}(P)} \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) \leq \frac{\varepsilon}{k}$$

for every atom $P \in \mathcal{P}$ contained in S. Then

$$\int_{f^{-1}(S)} \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) \leq \varepsilon \tag{3}$$

if $n \geq N$. Moreover, since the partition $\mathcal{M}$ contains only two atoms, (2) implies that

$$\int_{f^{-1}(P)} \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) \leq d\mu(P) \log 2.$$

Then

$$\int_{f^{-1}(S^c)} \left| \log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{P}_n(f(x)))} \right| d\mu(x) \leq d \log 2 \ \mu(S^c) \leq \frac{\varepsilon}{2}. \tag{4}$$

From (3) and (4), it follows that the second term in (1) converges to zero in $\mathcal{L}^1$ as we wished to show. To complete the proof of II.2, it remains to show that the third term in (1) converges to $\log J_\mu$ in $\mathcal{L}^1$. But

$$\mu(\mathcal{R}_n(x)) = \mu(\mathcal{R}_n(x) \cap A_\mu) \leq \mu(f^{-1}(f(A_\mu \cap \mathcal{R}_n(x)))) =$$
$$= \mu(f(A_\mu \cap \mathcal{R}_n(x))).$$

Then

$$\frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{R}_n(x))} \geq 1. \tag{5}$$

Moreover,

$$\frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{R}_n(x))} = \frac{1}{\mu(\mathcal{R}_n(x))} \int_{A_\mu \cap \mathcal{R}_n(x)} J_\mu d\mu = \frac{1}{\mu(\mathcal{R}_n(x))} \int_{\mathcal{R}_n(x)} J_\mu d\mu. \tag{6}$$

Then (6), together with the fact that $\bigvee_0^\infty \mathcal{R}_n$ is the Borel σ-algebra, implies that

$$\frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{R}_n(x))} \to J_\mu(x) \tag{7}$$

in $\mathcal{L}^1$ (also for a.e. x, but this will not be used). But $J_\mu(x) \geq 1$ for a.e. x and the same is true for the functions $x \to \mu(f(A_\mu \cap \mathcal{R}_n(x)))/\mu(\mathcal{R}_n(x))$ by (5). Hence, (7) implies that

$$\log \frac{\mu(f(A_\mu \cap \mathcal{R}_n(x)))}{\mu(\mathcal{R}_n(x))} \to \log J_\mu(x)$$

in $\mathcal{L}^1$.

Now we shall prove Lemma II.4. We shall need the following property:

Lemma II.5. *For all $0 < k < 1$ there exists a continuous function $\rho : \overline{\mathbb{C}} \to [0, +\infty)$ satisfying the following properties:*

a) There exist constants $G > 0$, $\alpha > 0$ such that:

$$\rho(x) \geq G \prod_{i=1}^{m} d(x, x_i)^\alpha,$$

where $x_1, \ldots, x_m$ are critical points of f.

b) *If x is not a critical point of f and $d(y_1, x) < \rho(x)$, $d(y_2, x) < \rho(x)$, then*

$$d(f(y_1), f(y_2)) \geq k \, \| f'(x) \| \, d(y_1, y_2).$$

Proof. If x is not a critical point, let $\rho_1(x)$ be the maximum positive number such that f is injective when restricted to the disk $\{y \mid d(y, x) < \rho(x)\}$. Set $\rho_1(x) = 0$ when x is a critical point. It is easy to see that the function $\rho_1 : \overline{\mathbb{C}} \to \mathbb{R}$ is continuous, bounded away from zero on compact sets not containing critical points, and, using that, nearby a critical point, f acts as $z \to z^{\ell}$ with $2d - 1 \geq \ell \geq 2$, that in a neighborhood of a critical point, x_i is bounded below by $G_i d(x, x_i)^{\alpha_i}$, where G_i and α_i are positive. Therefore, there exist $A > 0$, $\alpha > 0$ such that:

$$\rho_1(x) \geq A \prod_{i=1}^{m} d(x, x_i)^{\alpha} \tag{1}$$

for all $x \in \overline{\mathbb{C}}$. We claim that there exists a continuous function $\rho_0 : \overline{\mathbb{C}} \to [0, +\infty)$ satisfying (a), upper bounded by ρ_1 and such that

$$\| f'(y) \| \geq k \, \| f'(x) \| \tag{2}$$

if $d(y, x) \leq \rho_0(x)$. To construct ρ_0, we take disks $D_1, \ldots, D_m$ centered at the critical points $x_1, \ldots, x_m$ such that f'/D_i can be written as

$$f'(z) = (z - x_i)^{d_i} g_i(z),$$

where $g_i : D_i \to \mathbb{C}$ has no zeroes. Then there exists $r > 0$ such that if $1 \leq i \leq m$, $x \in D_i$, $y \in D_i$ and $d(x, y) < r$ then:

$$|g_i(y)| > \sqrt{k} \, |g_i(x)|. \tag{3}$$

Now take the continuous functions $\rho^{(i)} : D_i \to [0, +\infty)$ defined by:

$$\rho^{(i)}(x) = \min \{r, (1 - k^{1/2d_i}) \, |x - x_i|\}. \tag{4}$$

Let $\widehat{D}_1, \ldots, \widehat{D}_m$ be disks centered at $x_1, \ldots, x_m$ whose closures are contained in $D_1, \ldots, D_m$ respectively. Set $r_0 = \min_i d(\widehat{D}_i, D_i^c)$ and choose $\delta > 0$ such that if $x \in (\bigcup_i \widehat{D}_i)^c$ and $d(y, x) \leq \delta$ then $\| f'(y) \| \geq k \, \| f'(x) \|$. If $x \in \widehat{D}_i$ and $d(y, x) < \min \{r_0, \rho^{(i)}(x)\}$ it follows that $y \in D_i$ (because $d(y, x) < r_0$) and $d(y, x) < \rho^{(i)}(x) < r$. Hence (3) implies:

$$\frac{\| f'(y) \|}{\| f'(x) \|} = \left(\frac{|y - x_i|}{|x - x_i|} \right)^{d_i} \cdot \frac{|g_i(y)|}{|g_i(x)|} \geq \left(\frac{|y - x_i|}{|x - x_i|} \right)^{d_i} \sqrt{k}.$$

If we now use (4), we obtain:

$$\frac{\| f'(y) \|}{\| f'(x) \|} \geq \left(1 - \frac{|y - x|}{|x - x_i|} \right)^{d_i} \sqrt{k} \geq \left(1 - \frac{\rho^{(i)}(x)}{|x - x_i|} \right)^{d_i} \sqrt{k} \geq \tag{5}$$

$$\geq (1 - (1 - k^{1/2d_i}))^{d_i} \sqrt{k} = k.$$

Now take a continuous function $\rho_0 : \overline{\mathbb{C}} \to [0, +\infty)$ such that:

$$\rho_0(x) = \min \{\rho_1(x), \delta\} \text{ if } x \notin \bigcup_i D_i,$$

$$\rho_0(x) = \min \{\rho_1(x), r_0, \rho^{(i)}(x)\}$$

if $x \in U_i$, $1 \leq i \leq m$ where U_i is neighborhood of x_i with closure contained in $\widehat{D}_i$ and

$$0 < \rho_0(x) \leq \min \{\rho_1(x), r_0, \rho^{(i)}(x)\}$$

if $x \in \widehat{D}_i - U_i$, $1 \leq i \leq m$. By (5) and by the way we choose δ it follows that ρ_0 satisfies the required properties. Now define ρ by

$$\rho(x) = \frac{1}{16} \rho_0(x)$$

Clearly ρ satisfies (a) because ρ_0 does. To complete the proof of II.5, we have to show that it also satisfies (b). Suppose that x is not a critical point. Since $\rho_0(x) < \rho_1(x)$, we know that $f/\{y \mid d(y, x) < \rho_0(x)\}$ is injective. By Koebe's theorem [3] we have:

$$f(\{y \mid d(y, x) < \rho_0(x)\}) \supset \{y \mid d(y, f(x)) < \frac{1}{4} \rho_0(x) \, \| f'(x) \| \}. \tag{6}$$

Let g be the branch of $f^{-1}/\{y \mid d(y, f(x)) < \left(\frac{1}{4}\right) \rho_0(x) \, \| f'(x) \| \}$ that maps $f(x)$ in x. Applying Koebe's theorem to g, we obtain:

$$g(\{y \mid d(y, f(x)) < \frac{1}{4} \rho_0(x) \, \| f'(x) \| \}) \supset \{y \mid d(y, x) < \frac{1}{16} \rho_0(x)\} =$$

$$= \{y \mid d(y, x) < \rho(x)\}.$$

Then:

$$f(\{y \mid d(y, x) < \rho(x)\}) \subset \{y \mid d(y, f(x)) < \frac{1}{4} \rho_0(x) \, \| f'(x) \| \}. \tag{7}$$

Therefore, if $d(y_1, x) < \rho(x)$ and $d(y_2, x) < \rho(x)$, we obtain from (7) that

$$d(f(y_i),\ f(x)) < \frac{1}{4}\rho_0(x) \| f'(x) \|,\ i = 1,2.$$

Hence, all the segment joining $f(y_1)$ and $f(y_2)$ is contained in the disk $\{y \mid d(y, x) \le \left(\frac{1}{4}\right)\rho_0(x) \| f'(x) \|\}$. Then:

$$d(y_1, y_2) = d(g(f(y_1)),\ g(f(y_2))) \le$$

$$\le d(f(y_1),\ f(y_2)) \sup \{\| g'(z) \| \mid d(z, f(x)) < \frac{1}{4}\rho_0(x) \| f'(x) \| \}$$

But $d(z, f(x)) < \left(\frac{1}{4}\right)\rho_0(x) \| f'(x) \|$ implies by (6) that:

$$d(g(z), x) \le \rho_0(x).$$

Hence

$$\| g'(z) \| = \| (f'(g(z)))^{-1} \| \le k^{-1} \| f'(x) \|^{-1}$$

and then

$$d(y_1, y_2) \le k^{-1} \| f'(x) \|^{-1} d(f(y_1),\ f(y_2)),$$

as we wished to prove.

Continuing with the proof of II.4, we take a partition $\mathscr{P}$ with finite μ-entropy and satisfying:

(8)
$$\operatorname{diam} \mathscr{P}(x) < \rho(x).$$

Such a partition exists because the function $x \to \log \rho(x)$ is integrable by II. 3. Therefore we can apply Lemma 2.3 of [8]. For each $P \in \mathscr{P}$, we choose a point $a \in P$ and then (8) implies that P is contained in the disk $D(P) = \{x \mid d(x, a) < \rho(x)\}$. Take a topological disk $\hat{D}(P) \subset D(P)$ with $\mu(\hat{D}(P)) = \mu(D(P))$ not containing critical values of f. Then there exist branches $g_i^{(P)} : \hat{D}(P) \to \overline{\mathbb{C}}$ of $f^{-1}/\hat{D}(P)$, $i = 1, \ldots, d$. Let $\mathscr{R}_n$ be the partition with atoms of the form $g_i^{(P)}(A)$, where P is an atom of $\mathscr{P}$, A an atom of $\mathscr{P}_n$ contained in P and $1 \le i \le d$. From this definition, it is clear that $\mathscr{P}$ and $\mathscr{R}_0 \le \mathscr{R}_1 \le \ldots$ satisfy condition (b) of II.4. Condition (d) holds by the way we choose $\mathscr{P}$. To prove conditions (a), it is sufficient to show that:

(9)
$$\lim_{n \to +\infty} \operatorname{diam} \mathscr{P}_n(x) = 0$$

for μ-a.e x. Once this is proved, by the way $\mathscr{R}_n$ was constructed it follows that $\lim_{n \to +\infty} \operatorname{diam} \mathscr{R}_n(x) = 0$ for μ-a.e. x. In order to show (9), observe that if $y_i \in \mathscr{P}_n(x)$, $i = 1,2$, then $f^j(y_i) \in \mathscr{P}(f^j(x))$, $i = 1,2$, for all $0 \le j \le n$. Hence, since $\operatorname{diam} \mathscr{P}(f^j(x)) \le \rho(f^j(x))$, we have:

$$d(f^j(y_i),\ f^j(y_2)) \ge k \| f'(f^{j-1}(x)) \| d(f^{j-1}(y_1),\ f^{j-1}(y_2))$$

for all $1 \le j \le n$. Then:

$$d(f^n(y_1),\ f^n(y_2)) \ge k \| (f^n)'(x) \| d(y_1, y_2).$$

But $d(f^n(y_1),\ f^n(y_2)) \le \operatorname{diam} \mathscr{P}_n(f^n(x))$. The last term is bounded, say by $G > 0$, for every x and $n \ge 0$. Hence,

$$d(y_1, y_2) \le \frac{G}{k} \| (f^n)'(x) \|^{-1}$$

and then

(10)
$$\operatorname{diam} \mathscr{P}_n(x) \le \frac{G}{k} \| (f^n)'(x) \|^{-1}$$

By (*) and (**) of II.3, (10) implies (9) completing the proof of II.4.

Now we shall prove the theorem. As we mentioned in the introduction, the f-invariant measure μ_f constructed in [5] satisfies $h_{\mu_f}(f) = \log d$ and is characterized by the property

$$\mu_f(f(A)) = d_{\mu_f}(A),$$

for all Borel set A such that f/A is injective. If by contradiction we assume there exists another f-invariant measure $v \ne \mu_f$ with $h_v(f) = \log d$ then it is well known that there exists an f-invariant *ergodic* measure $\mu_f \ne \mu$ with $h_\mu(f) = \log d$ (this follows for instance from [2] Prop. 13.3). Associated to μ, we have the function J_μ and the set A_μ given by Lemma II.1. Then applying II.2, the Corollary of II.1 and the convexity of the logarithm we get

$$\log d = h_\mu(f) = \int_{\overline{\mathbb{C}}} \log J_\mu d_\mu \le \log \int_{\overline{\mathbb{C}}} J_\mu d_\mu \le \log d.$$

Hence:

$$\int_{\overline{\mathbb{C}}} \log J_\mu d_\mu = \log \int_{\overline{\mathbb{C}}} J_\mu d_\mu = \log d.$$

Then $J_\mu = d$ μ-a.e. Now let us show that $\mu(f(A_\mu^c)) = 0$. Take a family of disjoint connected and simply connected open sets $\{U_1, \ldots, U_\ell\}$ such that they do not contain critical values and

$$\mu((\bigcup_i U_i)^c) = 0. \tag{11}$$

Then for each $1 \le i \le k$, we can find open sets $D_1^{(i)}, \ldots, D_d^{(i)}$ such that $f/D_j^{(i)}$ is a homeomorphism onto U_i and $\bigcup_i D_j^{(i)} = f^{-1}(U_i)$. It follows from (11) that

$$\mu(\bigcup_{i,j} D_j^{(i)}) = 1. \tag{12}$$

Hence:

$$\mu(f(A_\mu \cap D_j^{(i)})) = \int_{A_\mu \cap D_j^{(i)}} J_\mu d\mu =$$

$$= d\mu(A_\mu \cap D_j^{(i)}) = d\mu(D_j^{(i)}).$$

Therefore,

$$\sum_j \mu(f(A_\mu \cap D_j^{(i)})) = d\mu \sum_j \mu(D_j^{(i)}) = d\mu(f^{-1}(U_i)) = d\mu(U_i). \tag{13}$$

Since:

$$\mu(f(A_\mu \cap D_j^{(i)})) \le \mu(f(D_j^{(i)})) = \mu(U_i) \tag{14}$$

for all i, j, we obtain from (13) and (14):

$$\mu(f(A_\mu \cap D_j^{(i)})) = \mu(U_i)$$

for all i, j. Then:

$$\begin{aligned} 0 = \mu(U_i) - \mu(f(A_\mu \cap D_j^{(i)})) &= \mu(f(D_j^{(i)})) - \mu(f(A_\mu \cap D_j^{(i)})) = \\ &= \mu(f(A_\mu^c \cap D_j^{(i)})). \end{aligned} \tag{15}$$

Since this holds for all i and j, it follows from (15) and (12) that $\mu(f(A_\mu^c)) = 0$. Now consider a Borel set A where f/A is injective. We have:

$$f(A) = f(A \cap A_\mu) \cup f(A \cap A_\mu^c)$$

and then:

$$\mu(f(A)) = \mu(f(A \cap A_\mu)) + \mu(f(A \cap A_\mu^c)) =$$

$$= \mu(f(A \cap A_\mu)) = \int_{A \cap A_\mu} J_\mu d\mu = \int_A J_\mu d\mu = d\mu(A).$$

Therefore $\mu = \mu_J$.

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