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Transverse intersections and stability of foliations

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Introduction.

Suppose there is given a pair of transverse integrable distributions $T\mathcal{F}$ and $T\mathcal{G}$ on a smooth manifold M, associated respectively to the codimension k and codimension ℓ foliation denoted by $\mathcal{F}$ and $\mathcal{G}$.

There is an obvious map, arising by taking the transverse intersection

$$(T\mathcal{F}, T\mathcal{G}) \mapsto T\mathcal{F} \pitchfork T\mathcal{G}$$

which according to Frobenius theorem defines a codimension $k+\ell$ foliation denoted by $\mathcal{F} \cap \mathcal{G}$.

Our first result says that $\mathcal{F} \cap \mathcal{G}$ depends continuously on the foliations $\mathcal{F}$ and $\mathcal{G}$ in the so called C^r – fine topology.

Theorem 1.1. *Let $\{\mathcal{F} \mid \mathcal{F} \pitchfork \mathcal{G}\}$ be the space of all C^r-codimension-k foliations transverse to the fixed C^r-foliation $\mathcal{G}$ of codimension ℓ on M. Then, $\mathcal{F} \mapsto \mathcal{F} \cap \mathcal{G}$ defined on $\{\mathcal{F} \mid \mathcal{F} \pitchfork \mathcal{G}\}$ is a continuous map in the space of all C^r-codimension $k+\ell$ foliations on M, where in both spaces the C^r-fine topology is given.*

We have to use the fine C^r-topology, since it guarantees that two C^r-close foliations possess the C^r-close holonomy maps as it was shown in [1]. This turns out to be the crucial feature of the construction of topological equivalence using the so called perturbed holonomy maps. By means of the above theorem we can approach the stability of foliation by reducing its dimension.

Our main goal is to prove the following result on structural stability.

Theorem 2.1. *Let $\mathcal{F}$ be a structurally stable C^r-foliation $(r \geq 1)$ on the smooth closed manifold M. A C^r-fibre bundle $F \to E \xrightarrow{\pi} M$, with F a closed manifold satisfying the condition $H^1(F, R) = 0$, is given.*

Then, the induced foliation $\pi^*(\mathcal{F})$ of E is structurally stable.

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This theorem connects two types of structurally stable foliations. The first is the foliation by compact leaves, which appears in the theorem as the fibration of E. The fact that these foliations are stable, under the particular condition $H^1(F, R) = 0$, is known from the papers of Reeb [10], Thurston [11] and Langevin-Rosenberg [7]. They use the perturbed holonomy map in order to establish a topological equivalence between the given foliation and its C^r-close integrable perturbation. Essential is that the topological equivalence is here a C^r-diffeomorphism.

An example for $\mathscr{F}$ may be chosen, in the second typical family of stable foliations, taking foliations which arise in connection with dynamical systems with transversely hyperbolic structure. The best known example for this is the "stable" or "unstable foliation" associated with an Anosow flow. See [6] for a detailed discussion.

The second example is a general suspended foliation which turns out to be structurally stable under appropriate conditions discussed in [9]. Remark that in these cases $\pi^*\mathscr{F}$ might consist of uncompact leaves only.

In some sense, theorem 2.1. can be viewed as a generalisation of the main theorem from [7]. If $\mathscr{F}$ is a foliation by points then $\pi^*\mathscr{F}$ is just the fibre bundle $F \to E \to M$ itself.

A different stability result, which will be proved along the same lines, is the statement about the local stability of a compact leaf.

Theorem 3.1. *Let $\mathscr{F}$ be a foliation of a manifold M. Suppose $\mathscr{F}$ has a locally stable compact leaf L. Assume $F \to E \xrightarrow{\pi} M$ is a fibre bundle with a closed fibre satisfying the condition $H^1(F, R) = 0$. Then, the leaf $\pi^{-1}(L)$ of $\pi^*\mathscr{F}$ is locally stable.*

This theorem will imply the existence of locally stable leaves which are not covered by the general theorem of Hirsch, see [4].

In a particular example, we will construct a Seifert fibration of higher dimension (higher than three) with an exceptional fibre being a locally stable leaf.

The main feature of the proofs is the reduction of $\pi^*\mathscr{F}$ to the structurally stable fibration $F \to E \to M$ by taking transverse intersections of $\pi^*\mathscr{F}$ with a sequence of partial foliations of the form $\pi^*\mathscr{G}_i$ where $\mathscr{G}_i$ is defined only locally, transversely to $\mathscr{F}$ and of complementary dimension. This is justified by theorem 1.1.

1. On the space of all foliations.

First we define the fixed pseudogroup of local diffeomorphisms of R^n.

Definition 1.1. The set of all local C^r-diffeomorphisms ($r \geq 1$)

$$(x, y, z) \mapsto (f(x, y, z),\ g(y),\ h(z))$$

of $R^{n-k-\ell} \times R^k \times R^\ell$ ($k + \ell \leq n$), where f is locally a submersion and g and h are local diffeomorphisms, has the pseudogroup structure. This pseudogroup will be denoted by $P(k, \ell)$.

The maximal atlas of the charts $\phi_i : U_i \to R^n$, such that all coordinate transformations belong to $P(k, \ell)$, represents a pair of transverse foliations of codimensions k and ℓ. It is said to be a $P(k, \ell)-$atlas.

Let the leaves of $\mathscr{F}$ be defined as maximal connected unions of the sets $\phi_i^{-1}(R^{n-k-\ell} \times pt \times R^\ell)$ and those of $\mathscr{G}$ as maximal connected unions of the sets $\phi_i^{-1}(R^{n-k-\ell} \times R^k \times pt)$. Then, $\{\phi_i\}$ also is a distinguished atlas of charts for the foliation $\mathscr{F} \cap \mathscr{G}$ which is defined in the obvious way by the integrable bundle $T\mathscr{F} \cap T\mathscr{G}$. We say $\{\phi_i\}$ is a simultaneously distinguished atlas for $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{F} \cap \mathscr{G}$.

Conversely, given a pair of transverse foliations $\mathscr{F}$ and $\mathscr{G}$ of codimensions k and ℓ, we construct an associated $P(k, \ell)$-atlas as follows:

Consider a pair of charts $\phi_i : U_i \to R^{n-k} \times R^k$ and $\psi_j : V_j \to R^{n-\ell} \times R^\ell$ containing $p \in U_i \cap V_j$ that are distinguished for $\mathscr{F}$ and $\mathscr{G}$ respectively. The map $x \mapsto (p_2\, \phi_i(x),\ p_2\, \psi_j(x)) \in R^k \times R^\ell$ is a local submersion at $(p_2 \cdot \phi_i(p), p_2\, \psi_j(p))$.

Each fibre of this submersion is contained in a leaf of $\mathscr{F} \cap \mathscr{G}$ and we take a chart $\xi : W_{i,j} \to R^{n-k-\ell} \times R^{k+\ell}$, $p \in W_{i,j}$, which parallelizes the fibres of that submersion. In other words

$$(p_2 \cdot \phi_i,\ p_2 \cdot \psi_j) \cdot \xi^{-1} : R^{n-k-\ell} \times R^{k+\ell} \to R^{k+\ell}$$

is locally a projection onto the second component. Now

$$x \mapsto (p_1 \xi(x),\ p_2 \cdot \phi_i(x),\ p_2 \cdot \psi_j(x))$$

is a simultaneously distinguished chart for $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{F} \cap \mathscr{G}$ defined on the neighbourhood of p. In this way the whole $P(k, \ell)$-atlas is obtained.

Next, we recall the definition of the so-called fine C^r-topology on the space of all codimension $- k$ foliations on M.

Definition 1.2. Let $(I, \{\phi_i\}, \{K_i\})$ denote a system of distinguished charts $\phi_i : U_i \to R^{n-k} \times R^k$ for $\mathscr{F}$, indexed by the set I, such that the compact sets $K_i \subset U_i$ exist and the following is true:

1. $\phi_i(K_i)$ is a cube with sides parallel to the axes of R^n
2. $\{K_i\}$ is a locally finite covering of M
3. $\{\text{int}\, K_i\}$ is also a covering for M

This is called a neighbourhood scheme for $\mathscr{F}$ and in the case of a compact M it contains only a finite number of charts.

Definition 1.3. Let $(I, \{\phi_i\}, \{K_i\})$ be a neighbourhood scheme for $\mathscr{F}$ and $\{\varepsilon_i\}_I$ a set of positive numbers. $N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ is the set of all foliations admitting a neighbourhood scheme $(I, \{\phi'_i\}, \{K'_i\})$ such that

1. $K_i \subset K'_i \subset U_i$
2. $|D\alpha(\phi'_i \cdot \phi_i^{-1} - id)| < \varepsilon_i$ on $\phi_i(K_i)$ for all $|\alpha| \le r$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the multiindex $|\alpha| = \alpha_1 + \ldots + \alpha_n$, is the "sup-norm", and

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

The sets $N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ for all $\{\varepsilon_i\}_I$ define the neighbourhoods of $\mathscr{F}$ in the fine C^r-topology.

Among the properties of this topology established in [1] we here recall the following:

The map $f \mapsto f_*(\mathscr{F})$, where $f_*(\mathscr{F})$ denotes the induced foliation by a diffeomorphism f, is a continuous map of $\mathrm{Diff}^r(M)$ in the space of all C^r-foliations on M. The holonomy maps of two C^r-close foliations are C^r-close local diffeomorphisms.

Proof of theorem 1.1.

By reducing the pseudogroup $P(k)$ to $P(k, \ell)$ we can introduce a new topology in $\{\mathscr{F} \mid \mathscr{F} \pitchfork \mathscr{G}\}$ which also gives the neighbourhoods of the pair of foliations $(\mathscr{F}, \mathscr{G})$ in the adequate space.

Definition 1.4. Let $(I, \{\phi_i\}, \{K_i\})$ be a $C^r - P(k, \ell)$ neighbourhood scheme for $\mathscr{F}$ and $\mathscr{G}$, i.e. $\{\phi_i\}$ are simultaneously distinguished charts. Let $N_{(k,\ell)}(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ be the set of all neighbourhood schemes $(I, \{\phi'_i\}, \{K'_i\}) \in N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ such that $\phi'_i : U_i \to R^{n-k-\ell} \times R^k \times R^\ell$ for $i \in I$ belongs to a $C^r - P(k, \ell)$-atlas of charts on M.

We may identify the scheme from the above with the maximal $P(k, \ell)$-atlas to which it belongs, i.e. with the pair of transverse foliations.

The neighbourhoods $N_{(k,\ell)}(\ldots\ldots)$ define the $C^r - P(k, \ell)$ topology on the set of all pairs of transverse foliations $(\mathscr{F}, \mathscr{G})$ of codimensions as above.

Proposition 1.1. *Both topologies, $P(k, \ell)$-fine and the $P(k) -$ fine topology on the space of C^r-foliations $\{\mathscr{F} \mid \mathscr{F} \pitchfork \mathscr{G}\}$ are equivalent.*

Proof. The relation $N_{(k,\ell)}(J, \{\psi_j\}, \{L_j\}, \{\delta_j\}) \subset N(J, \{\psi_j\}, \{L_j\}, \{\delta_j\})$ holds trivially.

Now, take some fixed neghbourhood $N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$. Suppose the sets K_i are small enough, so that the simultaneously distinguished charts $\psi_i : V_i \to R^{n-k-\ell} \times R^k \times R^\ell$ for $\mathscr{F}$ and $\mathscr{G}$ are defined, $\psi_i(L_i)$ is a cube neighbourhood of 0 in R^n and $K_i \subset L_i$.

Let further $\psi'_i : V_i \to R^{n-k-\ell} \times R^k \times R^\ell$ be a $P(k, \ell)$-chart from the neighbourhood scheme $(I, \{\psi'_i\}, \{L_i\})$ of the foliations $\mathscr{F}'$ and $\mathscr{G}'$ such that

$$|D^\alpha(\psi'_i \psi_i^{-1} - id)| < \delta_i \quad \text{on} \quad \psi_i(L_i).$$

We construct the compact set K'_i with $K_i \subset K'_i \subset L_i$ and the distinguished chart $\phi'_i : K'_i \to R^n$ for $\mathscr{F}'$ such that

$$|D^\alpha(\phi'_i \phi_i^{-1} - id)| < \varepsilon_i \quad \text{on} \quad \phi_i(K_i).$$

Consider the mapping

$$\psi_i \phi_i^{-1} : \phi_i(K_i) = I^n \to \psi_i(L_i) \subset R^{n-k} \times R^k,$$

where $I^n = [1, 1]^n$.

Let us take the notations

$$J^n = (\psi_i \phi_i^{-1})(I^n)$$
$$J^n_\mu = (\psi_i \phi_i^{-1})([-1-\mu, 1+\mu]^n)$$

for some positive constant μ such that $[-1-\mu, 1+\mu]^n \subset \mathrm{Im}\, \phi_i$. It holds

$$J^n \subset J^n_\mu \quad \text{and} \quad \psi_i^{-1}(J^n_\mu) \supset \psi_i^{-1}(J^n) = K_i.$$

If $\delta_i < \inf \{|\psi_i u - \psi_i v| : u \in \partial(K_i),\ v \in \partial\psi_i^{-1}(J^n_\mu)\}$ and $|D^\alpha(\psi'_i \psi_i^{-1} - id)| < \delta_i$ on $\psi_i(L_i)$, then $K_i \subset (\psi'_i)^{-1} J^n_\mu$.

Choose $(\psi'_i)^{-1} J^n_\mu = K'_i$ and define the chart

$$\phi'_i = \phi_i \psi_i^{-1} \psi'_i : V'_i \to R^n.$$

We check

$$|\phi'_i(K'_i) = [-1-\mu, 1+\mu]^n \text{ and}$$

$$|D^\alpha(\phi'_i \phi_i^{-1} - id)| = |D^\alpha(\phi_i \psi_i^{-1} \psi'_i \phi_i^{-1} - id)| =$$
$$= |D^\alpha(\phi_i \psi_i^{-1}(\psi'_i \phi_i^{-1} - \psi_i \phi_i^{-1})| =$$
$$= |D^\alpha(\phi_i \psi_i^{-1}(\psi'_i \psi_i^{-1} - id)\psi_i \phi_i^{-1}|.$$

Thus, δ_i can be taken such that

$$|D^\alpha(\phi'_i \phi_i^{-1} - id)| < \varepsilon_i \quad \text{on} \quad \phi_i(K_i).$$

Therefore, if the sets K_i from $N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ are small enough, we can find the $P(k, \ell)$-scheme $(I, \{\psi_i\}, \{L_i\})$ for $\mathscr{F}$ and $\mathscr{G}$ and the numbers δ_i such that

$$N_{(k,\ell)}(I, \{\psi_i\}, \{L_i\}, \{\delta_i\}) \subset N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\}).$$

To complete this part of equivalence proof, we must exhibit the following plausible statement:

A neighbourhood $N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ of $\mathscr{F}$ can be refined by taking an arbitrary fine refinement of the sets K_i.

More precisely, for $\phi_1 : U_1 \to R^n$ a distinguished chart in $(I, \{\phi_i\}, \{K_i\})$, we define:

$$N(\phi_1, K_1, \varepsilon_1) = \{(I, \{\phi_i'\}, \{K_i'\}) : K_1 \subset K_1' \text{ and } |D^\alpha(\phi_1' \phi_1^{-1} - id)| < \varepsilon_1\}.$$

Lemma 1.1. *For every distinguished chart $\phi_1 : U_1 \to R^n$ from $(I, \{\phi_i\}, \{K_i\})$ and every ε_1, the set K_1 can be covered by a union of a finite number of compact sets $P_j, j \in J$ of an arbitrary small diameter (e.g. diam $P_j < \delta$ for all j), such that $\phi_1(P_j)$ are cubes in R^n and the numbers δ_j can be found such that*

$$N(J, \{\phi_1 \mid P_j\}, \{P_j\}, \{\delta_j\}) \subset N(\phi_1, K_1, \varepsilon_1).$$

Remark. All cubes appearing later will have sides parallel to the axes in R^n.

By $N(J, \{\phi_1 \mid P_j\}, \{P_j\}, \{P_j\}, \{\delta_j\})$ we denote the set

$$\{(S, \{\phi_i'\}, \{K_i\}) : \forall_j \exists_i |D^\alpha(\phi_i' \phi_1^{-1} - id)| < \delta_j \text{ on } \phi_1(P_j)\}$$

A similar statement has also been proved in [1].

Proof of the lemma.

Let $\phi_1(K_1) = I^n = [0, 1]^n$ be decomposed by hyperplanes $x_i = \dfrac{r}{q}$, $i = 1, \ldots, n, r = 0, 1, \ldots, q-1$, in the union of smaller cubes:

$$\left[\frac{i_1}{q}, \frac{i_1+1}{q}\right] \times \left[\frac{i_2}{q}, \frac{i_2+1}{q}\right] \times \ldots \times \left[\frac{i_n}{q}, \frac{i_n+1}{q}\right] \text{ for}$$

$$i_1, \ldots, i_n \in \{0, 1, \ldots, q-1\}.$$

A positive number $0 < \alpha < \dfrac{1}{2q}$ is chosen and the cubes

$$\left[\frac{i_1 - \alpha}{q}, \frac{i_1 + \alpha + 1}{q}\right] \times \ldots \times \left[\frac{i_n - \alpha}{q}, \frac{i_n + \alpha + 1}{q}\right] \text{ considered.}$$

Multiindices $(i_1, \ldots i_n)$ can be ordered lexicographically and then, the above cube will be denoted by $I_j^n(\alpha)$, where $j = 1, \ldots, q^n$. I_j^n wil stand for $I_j^n(0)$.

For α small, we have $I^n \subset \bigcup_{j=1}^{q^n} I_j^n(\alpha) \subset \phi_1(U_1)$.

We define the sets $P_j = \phi_1^{-1}(I_j^n(\alpha))$.

It is clear that, for q large, the condition on diameters will hold. Consider now the neighbourhood scheme $(J, \{\phi_1 \mid P_j\}, \{P_j\})$ $J = \{1, \ldots, q^n\}$ associated to $\mathscr{F}$ on the neighbourhood of K_1. If $(J, \{\psi_j\}, \{Q_j\}) \in N(J, \{\phi_1 \mid P_j\}, \{P_j\}, \{\delta_j\})$ then $P_j \subset Q_j$ and

$$|D^k(\psi_j \phi_1^{-1} - id)| < \delta_j \text{ on } \phi_1(P_j) = I_j^n(\alpha) \text{ for all } |k| \leq r.$$

We may assume $Q_j \subset U_1$ for all j, which is easily achieved, taking all Q_j slightly smaller. Consider the covering $\{Q_j\}_{j \in J}$ of K_1. We want to prove that, given δ_j small, the sets $\psi_j \cdot \phi_1^{-1}(I_j^n(\alpha))$ can be deformed by local diffeomorphisms $\tilde{\psi}_j \cdot \psi_j^{-1}$ from $P(k)$ in such a way that

$$\tilde{\psi}_j \phi_1^{-1} \mid I_j^n(\beta) \cap I_k^n(\beta) = \tilde{\psi}_k \phi_1^{-1} \mid I_j^n(\beta) \cap I_k^n(\beta)$$

for some $0 < \beta < \alpha$ and all j, k.

Then, the charts $\tilde{\psi}_j$ can be glued together giving rise to the chart $\psi_1' : U \to R^n$.

Let $\tilde{\psi}_1 \mid P_1 = \psi_1 \mid P_1$. The local diffeomorphism $\psi_1 \psi_2^{-1} \mid \psi_2(P_1 \cap P_2)$ is used to construct a deformation of $\psi_2 : P_2 \to R^n$ to $\tilde{\psi}_2 : P_2 \to R^n$ such that

$$\tilde{\psi}_2 \mid P_1' \cap P_2' = \tilde{\psi}_1 \mid P_1' \cap P_2', \text{ where}$$

$$P_j' = \phi_1^{-1}(I_j^n(\alpha')) \subset P_j \text{ for } 0 < \alpha' < \alpha.$$

Let $\psi_1 \psi_2^{-1}$ have the form

$$\psi_1 \psi_2^{-1}(x, y) = ((g(x, y), h(y)) \in R^{n-k} \times R^k.$$

Further, $\psi_1 \psi_2^{-1} - id = (\psi_1 \phi_1^{-1} - id)((\psi_2 \phi_1^{-1})^{-1} - id) +$

$$+ ((\psi_2 \phi_1^{-1})^{-1} - id) + (\psi_1 \phi_1^{-1} - id).$$

Setting $\psi_2 \phi_1^{-1} = \gamma$, we can evaluate

$$\left|\frac{\partial(\gamma^{-1})}{\partial x_j}\right| = \frac{\left|P\left(\dfrac{\partial \gamma^\ell}{\partial x_k}\right)\right|}{det\left(\dfrac{\partial \gamma^\ell}{\partial x_k}\right)} \leq \frac{\left|P\left(\dfrac{\partial \gamma^\ell}{\partial x_k}\right)\right|}{c}$$

using formula for the inverse matrix. Here P is the homogeneus polynomial of grad $= n-1$ and $0 < c < 1$.

Thus

$$\lim_{\delta_2 \to 0} \left| \frac{\partial(\gamma^{-1})^i}{\partial x_k} \right| = \delta_{i,k} \quad \text{and}$$

$$\lim_{\delta_2 \to 0} \left| \frac{\partial^n(\gamma^{-1})^i}{\partial x_{i_1} \ldots \partial x_{i_n}} \right| = 0 \text{ for all } i, i_1, \ldots, i_n, \text{ if } n > 1.$$

The estimate

$$|D^\alpha(\psi_2 \phi_1^{-1} - id)| < \delta_2 \quad \text{on} \quad \phi_1(P_2)$$

for all $|\alpha| \leq r$ thus implies

$$\sup_{0 \leq |\alpha| \leq r} |D^\alpha((\psi_2 \phi_1^{-1})^{-1} - id)| = s_1(\delta_2)$$

is bounded and $\lim_{\delta_2 \to 0} s_1(\delta_2) = 0$.

Therefore

$$|D^\alpha(\psi_1 \psi_2^{-1} - id) | \psi_2(P_1 \cap P_2)| \leq \delta_1 s_1(\delta_2) + \delta_1 + s_1(\delta_2).$$

The local diffeomorphism $h | p_2 \psi_2(P_1 \cap P_2)$ from the representation of $\psi_1 \psi_2^{-1}$ is thus C^1-near identity, consequently there exists an isotopy of embeddings

$$\{h_t\} : p_2 \psi_2(P_1 \cap P_2) \to R^k, \; h_0 = id, \; h_1 = h$$

and this isotopy extends to a diffeotopy $\{\tilde{h}_t\} : R^k \to R^k$ in the following way: $\{h_t\}$ defines the map

$$H : p_2 \psi_2(P_1 \cap P_2) \times I \to R^k \times I$$

$H(x, t) = (h_t(x), t)$. The tangent vectors to the curves $H(\{x\} \times I)$ define the vector field X of the form $(X_h, \partial/\partial t)$ with X_h the R^k-component. We may multiply X_h with a bump function $\rho : R^k \times I \to [0, 1]$ such that:

1. $\rho | H(p_2 \psi_2 (P'_1 \cap P'_2) \times I) = 1$
2. supp $\rho \subset$ Image (H)

The vector field $(\rho X_h, \partial/\partial t)$ is extended by $(0, \partial/\partial t)$ to the vector field on $R^k \times I$ and the flow of this new vector field is of the form $(x, t) \mapsto \mapsto (\tilde{h}_t(x), t)$. So, the diffeotopy $\{\tilde{h}_t\}$ results. Since $|D^\alpha \rho| < c_1$, we conclude:

$$|D^\alpha(\rho X_h)| < c_1 |D^\alpha X_h| \text{ and } |D^\alpha(\tilde{h}_1 - id)| < c_1(\delta_1 s_1(\delta_2) + \delta_1 + s_1(\delta_2))$$

Further, $\tilde{h}_1 | p_2 \psi_2(P'_1 \cap P'_2) = h | p_2 \psi_2(P'_1 \cap P'_2)$ and

$$|D^\alpha((p_1, \tilde{h}_1) - id)| < c_1(\delta_1 s_1(\delta_2) + \delta_1 + s_1(\delta_2)).$$

Similarly, (g, id) is close to id $| \psi_2(P_1 \cap P_2)$ and $(g, id) | \psi_2(P'_1 \cap P'_2)$ may be extended to a diffeomorphism $(\tilde{g}_1, id)$ of R^n such that

$$|D^\alpha((\tilde{g}_1, id) - id)| \leq c_2(\delta_1 s_1(\delta_2) + s_1(\delta_2) + \delta_1).$$

Hence, $(\tilde{g}_1, \tilde{h}_1)$ is the global diffeomorphism of R^n extending

$$\psi_1 \psi_2^{-1} | \psi_2(P'_1 \cap P'_2) \text{ and } \tilde{\psi}_2 = (\tilde{g}_1, \tilde{h}_1) \cdot \psi_2$$

is the required deformation of ψ_2.

One defines $\theta_2 : P'_1 \cup P'_2 \to R^n$ by $\theta_2 | P'_1 = \tilde{\psi}_1$, $\theta_2 | P'_2 = \tilde{\psi}_2$.

It holds:

$$|D^\alpha(\theta_2 \theta_1^{-1} - id)| \leq |D^\alpha((\tilde{g}_1, \tilde{h}_1) - id)(\psi_2 \phi_1^{-1} - id)| +$$
$$+ |D^\alpha((\tilde{g}_1, \tilde{h}_1) - id)| + |D^\alpha(\psi_2 \phi_1^{-1} - id)| \leq r_2(\delta_1, \delta_2) \text{ on } \phi_1(P'_2).$$

We proceed by induction:

In the inductive step from I^n_j to I^n_{j+1} the sets $P^j_k = \phi_1^{-1}(I^n_k(\alpha_j))$, for all k, are deminished into $P^{j+1}_k = \phi_1^{-1}(I^n_k(\alpha_{j+1}))$, $0 < \alpha_{j+1} < \alpha_j$.

Assume $\theta_j : P^{(j)}_1 \cup \ldots \cup P^{(j)}_j \to R^n$ is defined,

$$|D^\alpha(\theta_j \phi_1^{-1} - id)| < r_j(\delta_1, \ldots, \delta_j) \text{ on } \phi_1(P^{(j)}_1 \cup \ldots \cup P^{(j)}_j) \text{ and}$$

$|D^\alpha(\psi_{j+1} \phi_1^{-1} - id)| < \delta_{j+1}$ on $\phi_1(P^{(j)}_{j+1})$. We prove:

$$|D^\alpha(\theta_j \psi_{j+1}^{-1} - id)| < r_j s(\delta_{j+1}) + r_j + s(\delta_{j+1})$$

on $\psi_{j+1}(P^{(j)}_{j+1} \cap (P^{(j+1)}_1 \cup \ldots \cup P^{(j+1)}_j))$ analogously as in the case $j = 1$ from above.

$\theta_j \psi_{j+1}^{-1}$, defined on $\psi_{j+1}(P^{(j+1)}_{j+1} \cap (P^{(j+1)}_1 \cup \ldots \cup P^{(j+1)}_j))$, can be extended to a global diffeomorphism $(\tilde{g}_j, \tilde{h}_j)$ which is then used to define $\tilde{\psi}_{j+1} = (\tilde{g}_j, \tilde{h}_j) \circ \psi_{j+1}$. This chart can be glued together with θ_j, giving so θ_{j+1} such that $|D^\alpha(\theta_{j+1} \phi_1^{-1} - id)| < r_{j+1}(\delta_1, \ldots, \delta_{j+1})$ on $\phi_1(P^{(j+1)}_{j+1})$. One finally gets the chart θ_{q^n}, and the constants $\delta_1, \ldots, \delta_{q^n}$ can be found, such that $\delta_1, r_2; \ldots, r_{q^n} < \varepsilon_1$ hold. A cube I'^n is chosen, such that

$$I^n \subset I'^n \subset \theta_{q^n}(P^{(q^n)}_1 \cup \ldots \cup P^{(q^n)}_{q^n}) \text{ and}$$

$\partial I^n \cap \partial I'^n = \emptyset$. Set $K'_1 = \theta^1_{q^n}(I'^n)$. For $\min\{\delta_1, r_2, \ldots, r_{q^n}\{ < \min_{v,w}\{|\phi_1 v - - \phi_1 w| : v \in \partial K'_1, \; w \in \partial \phi_1^{-1}(I^n)\}$ it holds $\theta_{q^n} \phi_1^{-1}(I^n) \subset I'^n$ and therefore, $\phi_1^{-1} I^n = K_1 \subset \theta_{q^n}^{-1}(I'^n) = K'_1$ and $|D^\alpha(\theta_{q^n} - id)| < \varepsilon_1$ on $\phi_1(K_1)$. This gives the final estimate for $\delta_1, r_2, \ldots, r_{q^n}$ and completes the proof of the lemma. Doing so for all charts $\phi_i : U_i \to R^n$ from the neighbourhood scheme $(I, \{\phi_i\} \{K_i\})$, we obtain the $P(k, \ell)$-neighbourhood scheme $(J, \{\psi_j\}, \{L_j\})$ for $\mathscr{F}$ and the numbers $\{\delta_j\}$ such that

$$N_{(k,\ell)}(J, \{\psi_j\}, \{L_j\}, \{\delta_j\}) \subset N(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\}).$$

One analogously proves:

Lemma 1.1′. *For every neighbourhood $N_{(k,\ell)}(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ of the foliation pair $(\mathscr{F}, \mathscr{G})$, there exists a new $P(k, \ell)$-neighbourhood scheme $(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\})$ for $(\mathscr{F}, \mathscr{G})$ with sets $K_{i,j}$ of arbitrary small diameter and a system of numbers $\delta_{i,j}$ such that*

$$N_{(k,\ell)}(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\}) \subset N_{(k,\ell)}(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$$

In order to complete the proof of proposition 1.1., it remains to show the opposite relation for the equivalence of both topologies.

According to lemma 1′ the neighbourhood $N_{(k,\ell)}(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$ of $\mathscr{F}$ can be refined by $N_{(k,\ell)}(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\})$ with $K_{i,j}$, s of small diameter. Let $\mathscr{F}'$ be contained in $N((I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\})$ and represented by a neighbourhood scheme $(I \times J, \{\psi_{i,j}\}, \{K'_{i,j}\})$. Consequently

$$(*) \qquad |D^\alpha(\psi_{i,j}\phi_{i,j}^{-1} - id)| < \delta_{i,j} \quad \text{on} \quad \phi_{i,j}(K_{i,j}).$$

We may assume:

1. both sets $K'_{i,j} \supset K_{i,j}$ are small enough
2. $\psi_{i,j}\phi_{i,j}^{-1} : R^{n-k-\ell} \times R^k \times R^\ell \to R^{n-k-\ell} \times R^k \times R^\ell$ takes the form of

$$(**) \qquad (x, y, z) \to (f_1(x, y, z),\ f_2(x, y, z),\ f_3(x, y, z))$$

Next, the existence of the $(\mathscr{F}', \mathscr{G})$-scheme $(I \times J, \{\psi'_{i,j}\}, \{L_{i,j}\})$ will be established, such that

1. $K'_{i,j} \subset L_{i,j}$
2. $\psi'_{i,j}(L_{i,j})$ is a cube and
3. $|D^\alpha(\psi'_{i,j}\phi_{i,j}^{-1} - id)| < \delta_{i,j}$ on $\phi_{i,j}(K_{i,j})$.

First we construct $\psi'_{i,j}$:
The projections of $R^{n-k-\ell} \times R^k \times R^\ell$ on the respective factors are denoted by p_1, p_2 and p_3. Then set

$$\psi'_{i,j}(x) = (p_1\psi_{i,j}(x),\ p_2\phi_{i,j}(x),\ p_3\psi_{i,j}(x)).$$

Since $\phi_{i,j} | K_{i,j} = \phi_i | K_{i,j}$, both $\phi_{i,j}$ and $\psi_{i,j}$ can be taken to be defined on some open neighbourhood of $K'_{i,j}$ denoted by $U_{i,j}$.
Since $p_2\phi_{i,j}$ is close to $p_2\psi_{i,j}$ one has $\psi_{i,j}(K'_{i,j}) \supset \text{Image}\ \psi'_{i,j}$. There also exists a cube I'', for which $\psi_{i,j}(K'_{i,j}) \underset{\neq}{\subseteq} I'' \subset \text{Image}\ \psi'_{i,j}$ and we define $L_{i,j} = \psi'^{-1}_{i,j}(I'')$. Further, it holds

$$\begin{aligned} \psi'_{i,j}\phi_{i,j}^{-1}(x, y, z) &= \psi'_{i,j}\psi_{i,j}^{-1}\psi_{i,j}\phi_{i,j}(x, y, z) = \\ (***) \qquad &= \psi'_{i,j}\psi_{i,j}^{-1}(f_1(x, y, z),\ f_2(x, y, z),\ f_3(x, y, z)) = \\ &= (f_1(x, y, z),\ y,\ f_3(x, y, z)) \quad \text{on} \quad \phi_{i,j}(K_{i,j}) \end{aligned}$$

Thus, the assumptions $\mathscr{F}' \in N(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\})$ and $K_{i,j}$ are small, imply the existence of the $(\mathscr{F}', \mathscr{G})$-scheme $(I \times J, \{\psi'_{i,j}\}, \{L_{i,j}\})$. The formulas (*), (**), (***) yield $|D^\alpha(\psi'_{i,j}\phi_{i,j}^{-1} - id)| < \delta_{i,j}$ on $\phi_j(K_{i,j})$ for $|\alpha| \le r$. Hence, $\mathscr{F}'$ represented by the $P(k, \ell)$-scheme $(I \times J, \{\psi'_{i,j}\}, \{L_{i,j}\})$ is contained in $N_{(k,\ell)}(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\})$. This proves the remaining inclusion

$$N(I \times J, \{\phi_{i,j}\}, \{K_{i,j}\}, \{\delta_{i,j}\} \subset N_{(k,\ell)}(I, \{\phi_i\}, \{K_i\}, \{\varepsilon_i\})$$

and completes the proposition.

Finally, we turn over to the proof of theorem 1.1. Let $N(J, \{\phi_j\}, \{K_j\}, \{\varepsilon_j\})$ be a neighbourhood of $\mathscr{F} \cap \mathscr{G}$. We may assume, according to lemma 1.1, that all K_j are sufficiently small. Then, a $P(k, \ell)$-neighbourhood scheme $(J, \{\psi_j\}, \{L_j\})$ for the pair $(\mathscr{F}, \mathscr{G})$ exists, such that $K_j \subset L_j$ for every j. There also exist the numbers δ_j such that

$$N_{(k,\ell)}(J, \{\psi_j\}, \{L_j\}, \{\delta_j\}) \subset N(J, \{\phi_j\}, \{K_j\}, \{\varepsilon_j\}).$$

By proposition 1.1, $N_{(k,\ell)}(J, \{\psi_j\}, \{L_j\}, \{\delta_j\})$ is a neighbourhood of $\mathscr{F}$ in the fine C^r-topology and so the continuity follows.

2. On structural stability.

Definition 2.1. The C^r-foliation $\mathscr{F}$ on the manifold M is said to be structurally stable in the space of all foliations of fixed codimension, if for every ε-neighbourhood $\mathcal{O}_\varepsilon$ of id_M in Homeo (M), there exists a neighborhood $\mathscr{U}$ of $\mathscr{F}$ in the space of all foliations, such that for every $\mathscr{G} \in \mathscr{U}$ there is a homeomorphism $h : M \to M$, $h \in \mathcal{O}_\varepsilon$ with the property that $\mathscr{G} = h(\mathscr{F})$, i.e., h is a topological equivalence between $\mathscr{F}$ and $\mathscr{G}$.

Remark. The first requirement on ε enables one to have only topological equivalences h which are homotopic identity.

Proof of theorem 2.1.

Let $\mathscr{F}'$ be a foliation close to $\pi^*\mathscr{F}$ in the fine C^r-topology. We will use theorem 1.1. to reduce $\mathscr{F}'$ to the fibration $F \to E \to M$. Then, the deformation proof from [7], with some minor modifications, will be repeated.

It uses the perturbed holonomy maps and makes the foliation F-saturated. Finally, the foliation thus obtained, will be proved projectable under the fibre bundle projection π, and the stability of $\mathscr{F}$ will be used.

Let a C^r-triangulation of M be given with simplices so small that:

a) for every n-simplex Δ_i^n, $i = 1, \ldots, m$, there exists a neighbourhood U_i such that the fibration trivialises over U_i;

b) there also exists some $\mathscr{F}$-distinguished chart $\phi_i : U_i \to R^{n-k} \times R^k$ for every i.

We associate with every such a chart ϕ_i the foliation $\mathscr{G}_i$ on U_i with the leaves $\phi_i^{-1}(pt \times R^k)$. Thus the leaves of $\mathscr{F}\,|_{U_i}$ and $\mathscr{G}_i$ intersect in single points and let $\mathscr{F} \cap \mathscr{G}_i$ denote the trivial "point-foliation" of U_i. We may assume $\mathscr{F}'$ to be transverse to all foliations $\pi^*\mathscr{G}_i$, $i = 1, \ldots, m$.

According to theorem 1.1, $\mathscr{F}' \cap \pi^*\mathscr{G}_i$ is C^r-close to $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}_i$, which is identical to $F \to F \times U_i \to U_i$. Now we look at the perturbed holonomy maps. We understand $\mathscr{F}' \cap \pi^*\mathscr{G}_i$ as the subfoliation of $\pi^*\mathscr{G}_i$, in other words the leaves of $\pi^*\mathscr{G}_i$ are foliated by the leaves of $\mathscr{F}' \cap \pi^*\mathscr{G}_i$.

Lemma 2.1. *Let $D^n(x)$ be a disc through the point $x \in F_{x_0}$, which is transverse to the fibres of $F \to E \xrightarrow{\pi} M$. Suppose $x_0 \in U_i$. Given $A > 0$, there exists a neighbourhood $\mathscr{N}$ of $\pi^*\mathscr{F}$ in the C^r-topology, and $\delta > 0$, such that any path $\alpha \subset F_{x_0}$ through x of the length $\leq A$ can be lifted into the leaf of $\mathscr{F}' \cap \pi^*\mathscr{G}_i$ starting with any point $y \in D\delta(x)$.*

This is lemma 2 from [7] modified by an application of theorem 1.1. It will be used in order to define a deformation of E, which carries the foliation $\mathscr{F}' \cap \pi^*\mathscr{G}_i$ onto $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}_i$ for all i. Let $x_1, \ldots, x_n$ be the vertices of Δ_1^n. For every x_i we choose a point $y_i \in F_{x_i}$ and suppose the conditions of the lemma hold. The assumption $H^1(F, R) = 0$ provides, the endpoint of the lift does not dependent on the connecting path, see [7]. By assigning the endpoint of the lift each endpoint of the path, we obtain the diffeomorphism of F_{x_i} onto $\mathscr{F}' \cap \pi^*\mathscr{G}_1(y_i)$. The same is done for all $i = 1, \ldots, n$, and this family of deformations may be extended to a single diffeomorphism h_0 of E having its support on a neighbourhood of $\pi^{-1}(\Delta_1^n)$. Now one considers $h_0^{-1}\mathscr{F}'$, which is still close by $\pi^*\mathscr{F}$ if h_0 is sufficiently near identity. $h_0^{-1}\mathscr{F}' \cap \pi^*\mathscr{G}_1$ and $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}_1$ possess the common leaves $F_{x_1}, \ldots, F_{x_n}$. Let $\Delta_i^1 \subset \Delta_1^n$ be the one dimensional sides. We choose $\Delta_i^1 \times \{z_i\} \subset \Delta_i^1 \times F$ to be a section of the trivialized fibration. The union of the leaves

$$A_i^1 = \bigcup_{y \in \Delta_i^1 \times \{z_i\}} (h_0^{-1}\mathscr{F} \cap \pi^*\mathscr{G}_1)(y)$$

is the manifold with boundary, C^r-close to $\Delta_i^1 \times F$. Thus, we may construct a diffeomorphism, by using the perturbed holonomy, which carries $\Delta_i^1 \times F$ onto A_i^1. Apparently, such diffeomorphisms, for various i, are C^r-close by id_E and equal id_E on the boundaries of the sets A_i^1. Hence, these maps can be extended to a single global diffeomorphism h_1 with support in the neighbourhood of $\pi^{-1}(U_1)$. Further, one considers $h_1^{-1} h_0^{-1}(\mathscr{F}')$ and proceeds by induction on the skeleton of Δ_1^n. In the last step we choose a section of $\overline{U}_1 \times F$, instead of the section over Δ_1^n. This leads to the foliation $\mathscr{F}^{(1)} = h^{(1)}(\mathscr{F}')$, where $h^{(1)}$ is the composite of all deformations h_i^{-1}, and $\mathscr{F}^{(1)} \cap \pi^*\mathscr{G}_1$ is equal to $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}_1$ on the set $\pi^{-1}(U_1)$. Notice that in order to get $\mathscr{F}^{(1)}$ F-saturated on $\pi^{-1}(U_1)$, we may diminish U_1 slightly. Suppose Δ_2^n is chosen such that $\Delta_2^n \cap \Delta_1^n \neq \emptyset$.

Considering $h^{(1)}\mathscr{F}' \cap \pi^*\mathscr{G}_2$, gives the diffeomorphism $h^{(2)}$ which transforms the foliation $h^{(1)}\mathscr{F}' \cap \pi^*\mathscr{G}_2$ into $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}_2$ on the set $\pi^{-1}(U_2)$. Notice that $h^{(2)}$ is identity on the set $\pi^{-1}(U_1)$. Now, by induction over the n-simplices $i = 1, \ldots, m$, one obtains the foliation $\mathscr{F}^{(m)} = h^{(m)} \ldots h^{(1)}\mathscr{F}'$ such that $\mathscr{F}^{(m)} \cap \pi^*\mathscr{G}_i = \pi^*\mathscr{F} \cap \pi^*\mathscr{G}_i$ for all i. Especially, $h^{(k)}$ is identity on the set $\bigcup_{i<k} \pi^{-1}(U_i)$ Each set $\pi^{-1}(U_i)$ can be associated a finite number of charts

$$\theta_{i,j} : V_{i,j} \to R^s \times R^{n-k} \times R^k,\ j = 1, \ldots, n_i$$

which are all simultaneosly distinguished for the foliations $\mathscr{F}^{(i-1)}$ and $\pi^*(\mathscr{G}_i)$ and such that the following is true: by applying the mapping $h^{(i)}$, the charts $\theta_{i,j}$ are transformed into $\theta_{i,j} h^{(i)-1} : h^{(i)} V_{i,j} \to R^s \times R^{n-k} \times R^k$ and $\pi^{-1}(U_i) = \bigcup_j h^{(i)}(V_{i,j})$.

Consider the system of charts $\theta_{i,j} h^{(i)-1}$ for all i, j. The sets $h^{(i)} V_{i,j}$ cover the whole manifold E. Since $\mathscr{F}^{(m)}$ is F-saturated, and each

$$(\theta_{i,j} h^{(i)-1})^{-1}(R^s \times pt \times pt)$$

is contained in some fibre F, we can project $\mathscr{F}^{(m)}$, by identifying the fibres to the points, and prove in this way that the foliation near $\mathscr{F}$ is gained: The identification gives the chart

$$\mathscr{V}_{i,j} = p_2\, \theta_{i,j} h^{(i)-1} \pi^{-1} : \pi(h^{(i)} V_{i,j}) \to R^{n-k} \times R^k$$

on M, where π^{-1} denotes an arbitrary section of the projection π and p_2 is the projection $R^s \times R^{n-k} \times R^k \to R^{n-k} \times R^k$. If $i \leq k$, then $\theta_{i,j}, \theta_{k,\ell}$ are the charts for $\mathscr{F}^{(k)}$ and therefore, any pair $\mathscr{V}_{i,j}$ and $\mathscr{V}_{k,\ell}$ belong to the same $P(k)$-structure on M.

The charts $\mathscr{V}_{i,j}$ remain unchanged by the application of $h^{(k)}$ for $k > i$ and this proves that the system of charts $\{\mathscr{V}_{i,j}\}$ defines the projected foliation $\pi_* \mathscr{F}^{(m)}$ on M. Let $h = h^{(m)} \ldots h^{(1)}$. Since $\mathscr{F}'$ is C^r-close by $\pi^*\mathscr{F}$ and h is the composite of deformations among which each is close to id_E, we conclude that $\mathscr{F}^{(m)} = h\mathscr{F}'$ will be close by $\pi^*\mathscr{F}$. Thus, also the projected foliations $\pi_*(\pi^*\mathscr{F}) = \mathscr{F}$ and $\pi_*\mathscr{F}^{(m)}$ are C^r-close together.

Now, the structural stability of $\mathscr{F}$ is used. There exists a topological equivalence $\gamma : M \to M$, $\gamma \sim id$, such that $\gamma(\pi_*\mathscr{F}^{(m)}) = \mathscr{F}$.

γ can be lifted to a C^0-bundle automorphism $\Gamma : E \to E$. We simply take Γ to be the pull back $\gamma^* : E \to E$, and since $\gamma \sim id$, $\gamma^*E = E$. Therefore,

$$\Gamma h(\mathscr{F}') = \Gamma\mathscr{F}^{(m)} = \Gamma(\pi^*\pi_*\mathscr{F}^{(m)}) = \pi^*\gamma\pi_*\mathscr{F}^{(m)} = \pi^*\mathscr{F}$$

3. Stability of a compact leaf.

Definition 3.1. The compact leaf L of the foliation $\mathscr{F}$ is called locally stable, if for every neighbourhood $\mathscr{U}$ of L, there is a neighbournood $\mathscr{V}$ of $\mathscr{F}$ in the space of all foliations, so that every foliation $\mathscr{G}$ in $\mathscr{V}$ possesses a compact leaf contained in the set $\mathscr{U}$.

The most general theorem providing the existence of a locally stable compact leaf, is the following:

Theorem (M. W. Hirsch [4]) *Let M be a C^∞-manifold without boundary and $\mathscr{F}$ a C^r-foliation on M, where $r \geq 1$. Let further*

1. *L be a compact leaf and $\alpha \in \pi_1(L)$ an element such that 1 does not belong to the spectrum of the linear holonomy $D\mathscr{H}(\alpha)$.*
2. *There exists a series of subgroups*

$$G_0 \lhd G_1 \ldots \lhd G_q \lhd \pi_1(L)$$

such that G_0 is cyclic generated by α, G_{i-1} is normal in G_i for every i and G_q has finite index in $\pi_1(L)$. Then, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the neighbourhood $\mathscr{N}$ of $\mathscr{F}$ exists, and every foliation $\mathscr{G} \in \mathscr{N}$ contains the compact leaf L' which is homotopy equivalent to L under the map $h : L \to L'$ with $d(x, h(x)) < \varepsilon$. Finally, L is the unique leaf with these properties.

First we will exhibit the proof of theorem 3.1. and then show the example of a locally stable leaf; which does not satisfy the conditions of Hirsch's theorem.

Proof of theorem 3.1. The argument is almost analogous to that of the proof of theorem 2.1. One takes the tubular neighbourhood of the leaf L, $D^k \to U \to L$ such that the fibres are transverse to the leaves of $\mathscr{F}$. Let us denote this fibration by $\mathscr{G}$. Then $\pi^{-1}(U)$ is a neighbourhood of the leaf $\pi^{-1}(L)$ and the intersection-foliation $\pi^*\mathscr{F} \cap \pi^*\mathscr{G}$ is identical to the fibering $F \to \pi^{-1}(U) \to U$.

According to theorem 1.1, $\mathscr{F}' \cap \pi^*\mathscr{G}$ is close to the fibration $F \to \pi^{-1}(U) \to U$. Now we use deformations of the same type as in theorem 2.1. with the purpose to carry $\mathscr{F}' \cap \pi^*\mathscr{G}$ in the foliation of $F \to \pi^{-1}(U) \to U$. These deformations are invariant on the leaves of $\pi^*\mathscr{G}$. Also, one obtains a foliation $h(\mathscr{F}')$ which is F-saturated on the set $\pi^{-1}(U)$. Also, $h(\mathscr{F}')$ is easily proved to be projectable and the projected foliation $\pi_* h(\mathscr{F}')$ is C^r-close to $\mathscr{F} = \pi_*\pi^*\mathscr{F}$. The foliation $\pi_* h(\mathscr{F}')$ must contain a compact leaf L' in U. Consequently, $h^{-1}\pi^{-1}(L)$ is the compact leaf of $\mathscr{F}'$ contained in $\pi^{-1}(U)$.

Let us next look at an example:

Consider the action of the cyclic group

$$Z_q \times S^1 \times D^2 \to S^1 \times D^2 \quad \text{defined by}$$

$$(1, e^{i\theta}, y) \mapsto (e^{i\theta}, y e^{i\frac{2\pi m}{q}})$$

where $(m, q) = 1$. This action gives the Seifert fibred neighbourhood $S^1 \times D^2 / Z_q$ with the exceptional fibre $S^1 \times \{0\} / Z_q$. This fibre is locally stable according to Hirsch's theorem. Let F be the q-fold connected sum of some lens space L_p with the fundamental group Z_p where p is an odd prime and $(p, q) = 1$. Thus $\pi_1(F) = Z_p(x_1) * \ldots * Z_p(x_q)$, where $x_1; \ldots, x_q$ are the cyclic generators of the lenscomponents of F. Obviously $H^1(F, R) = 0$. Let us define the Z_q-action on F by the cyclic permutation of the lens spaces in the sum. On the fundamental group this permutation is reflected as $x_i \mapsto x_{i+1} \pmod q$.

We have the product .action

$$Z_q \times S^1 \times D^2 \times F \to S^1 \times D^2 \times F$$

and we define the higher dimensional Seifert feibred neighbourhood $S^1 \times D^2 \times F / Z_q$ with the exceptional fibre $S^1 \times \{0\} \times F / Z_q$. The fundamental group of this fibre is the free product with amalgamations:

$$G = \pi_1(S^1 \times F / Z_q) = (x_1, \ldots, x_q, z : x_i^p = 1, z x_i z^{-1} = x_{i+1 (mod\ q)})$$

where z denotes the generating element of $\pi_1(S^1)$. Now we are looking for a series of groups $H \lhd G_1 \lhd \ldots \lhd G_k = G$ with properties as in Hirsch's theorem. The exceptional fibre $S^1 \times F / Z_q$ is the locally stable leaf of the Seifert foliation constructed above. This follows from theorem 3.1. The group H must be cyclic and generated by an element $\alpha = (nz)f$ where $n \not\equiv 0 \pmod q$ and $f \in Z_p(x_1) * \ldots * Z_p(x_q)$. The condition $n \not\equiv 0 \pmod q$ is necessary to provide that the holonomy of α has no eigenvalue equal to one. Thus, H is infinitely cyclic. Let us denote the conjugation in G with nz by $'$. More precisely, $'$ denotes the permutation of the alphabet $nzx_i(-nz) = x_i' = x_{i+n} (\mathrm{mod}\ q)$.

Further we want to prove:

Proposition. *Let G and H be the groups defined above. The normaliser $N(H)$ of H in G is equal to H.*

Proof. Consider $N(H)$ and take $u \in N(H)$, $u \notin H$. Let

1. u have the finite order m

If $u = (\ell z)g$ for $g \in Z_p * \ldots * Z_p$ it follows from $u^m = 1$ that $\ell = 0$ and $u \in Z_p * \ldots * Z_p$.

According to the theorem of Kuroš, u is conjugated to one among the generators x_i, e.g. $u = v x_1 v^{-1}$ and $m = p$. Since $u \in N(H)$ and p is an odd prime,

$$u(nzf)u^{-1} = nzf$$

Substituing for u yields, after the permutation and cancellation of nz, the formula

$$v' x_1' v'^{-1} f v x_1^{p-1} v^{-1} = f.$$

This is equivalent to

$$x_{1+n(q)} v'^{-1} f v x_1^{p-1} = v'^{-1} f v \quad \text{or}$$

$$x_{1+n(q)} w x_1^{p-1} = w \text{ for } w = v'^{-1} f v.$$

However, this is impossible in the free product $Z_p(x_1) * \ldots * Z_p(x_q)$ for all words w and all $n \not\equiv 0$ (mod q).

2. Let all elements $u \in N(H)$, $u \notin H$ have infinite order. Since $n \not\equiv 0$ (mod q) we may suppose there is $mzg \in N(H)$, $mzg \notin H$, where $m \not\equiv 0$ (mod q).

It holds $(mzg)^{-1} nzf\ mzg = nzf$, since the left hand side contains, after the permutation, the factor nz. It is also necessary that $g \neq 1$. The last equation can be written as

$$(*) \qquad g^{-1(n)} f^{(m)} g = f$$

after the cancellation of nz, and writing $\_^{(n)}$ for the n-th iterate of $'$. The equations show that on the left hand side $2|g|$ letters should be cancelled, where $|g|$ stands for the length of the word g. We may assume that one of the three cases

a) $g = u v w$, $f^{(m)} = u^{(n)} v^{(n)} h u^{-1}$
b) $f^{(m)} = h g^{-1}$
c) $f^{(m)} = g^{(n)} h$

is valid. In case a) the left hand side of (*) contains the words $u^{-1}u$, $v^{-1(n)} u^{-1(n)} u^{(n)} v^{(n)}$, which both cancel. In the rest, there is no further cancellation possible under the assumption, that f has been written in its shortest form. It follows that $4|u| + 2|v| = 2|g|$ and thus $|u| = |w|$. Substitution in (*) gives

$$(**) \qquad w = u^{-1(-m)} \quad \text{and} \quad h v = v^{(n-m)} h^{(-m)}$$

Further

$$(mzg)(mzg) = 2mzg^{(m)} g = 2mzu^{(n)} v^{(m)} v u^{-1(-m)}$$

and because

$$(mzg)^{-1} (mzg)^{-1}\ nzf\ mzg\ mzg = nzf$$

it follows that

$$u^{(n-m)} v^{-1(n)} h^{(m)} v^{(m)} v u^{-1(-m)} = u^{(n)} v^{(n)} h u^{-1}.$$

According to (**) $v^{-1(n)} h^{(m)} v^{(m)} = h$. Then, we obtain the equation

$$u^{(n-m)} h v u^{-1(-m)} = u^{(n)} v^{(n)} h u^{-1},$$

which cannot be cancelled any more. However, this is impossible if $m \not\equiv$ (mod q), since it implies that $u^{(m)} = u$.

In cases b) and c) we reason similarly.

There is another type of stability property of a compact leaf. Let us recall, that according to the theorem of Seifert [11], every C^0-sufficiently small perturbation of the vector field tangent to the fibres of the Hopf fibration $S^1 \to S^3 \to S^2$ has necessarily a compact orbit. This orbit, however, cannot be localised. The result about stability of that type, which we will prove next, uses the following fact proved first by Fuller [3]: If f is a diffeomorphism of a compact manifold M, with the Euler characteristic $\chi(M) \neq 0$, then, f necessarily possesses at least one periodic point.

Theorem 3.2. *Let $F \to E \to M$ be a C^r-fibre bundle $r \geq 1$, where F and M are closed manifolds, $\chi(M) \neq 0$ and $H^1(F, R) = 0$. Let further $g: E \to E$ be a C^r bundle isomorphism over a diffeomorphism $f: M \to M$. Consider the suspension*

$E x_g I = E \times I/(x, 1) \sim (gx, 0)$ with a fibre bundle structure

$$F \to E x_g I \xrightarrow{\pi} M x_f I$$

Let $\mathscr{F}$ denote the foliation of $E x_g I$ obtained by lifting the one dimensional foliation of $M x_f I$. Then, every C^r-close integrable perturbation of $\mathscr{F}$ possesses a compact leaf.

Proof. The foliation $\mathcal{F}$ is transverse to the fibration $E \to E\, x_g I \to S$ which is defined by suspension. Consider $\mathcal{F}'$ close to $\mathcal{F}$ and project the vector field $(0, \partial/\partial t)$, tangent to the suspension factor I, in some Riemannian metric, onto the leaves of $\mathcal{F}'$. If $\mathcal{F}'$ is close to $\mathcal{F}$, then the projection is a nonsingular vector field denoted by X'. According to theorem 1.1 the subfoliation $\mathcal{F}'' = \bigcup_{t \in S^1} (\mathcal{F}' \cap E \times \{t\})$ is close to the fibration $F \to E\, xg\, I \to M\, x_f I$. There exists a diffeomorphism $h : E\, x_g I \to E\, x_g I$, such that $h(\mathcal{F}'')$ is the fibration $F \to E\, x_g I \to M\, x_f I$. Moreover, h respects the manifolds $E \times \{t\}$ for all t. Also, h defines the bundle isomorphism $\psi = h\varphi' h^{-1} : E \times \{0\} \to E \times \{0\}$ where φ' is the first return map of $E \times \{0\}$ along the trajectories of X'. ψ is projected into the diffeomorphism $f' : M \to M$, and according to Fuller, there exists a periodic point of f', which corresponds to the compact leaf of $\mathcal{F}'$.

Remark. This can be viewed as a generalisation of theorem 2 from [8].

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