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The pressure of the geodesic flow on a negatively curved manifold

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Manning [5] has identified the rate of exponential growth of the volume of a ball of radius r on the universal cover of a compact manifold M of negative curvature: it is the entropy of the geodesic flow on M. See also Sullivan [7], Chen [3], Chen and Manning [4]. Here we indicate an extension of Manning's result, where the entropy is replaced by the topological pressure $P(A)$ associated with a function A on the tangent bundle. It turns out that the Riemann volume used by Manning plays no special role and may be replaced by many other measures.

Let M be a compact Riemann manifold with strictly negative sectional curvatures everywhere. We denote by $\tilde{M}$ the universal cover of M (with the induced metric), by $p:\tilde{M}\to M$ the canonical projection, and by N a fundamental domain of finite diameter a. We call $B(x,r)$ the ball with center x and radius r in $\tilde{M}$. Let μ be a positive Radon measure on $\tilde{M}$, such that there are $\alpha, \beta, b > 0$ with

$$\alpha \leqslant \mu(B(x,b)) \leqslant \beta \tag{1}$$

for all $x \in \tilde{M}$.

We denote by $T^{(1)}M$ the unit tangent bundle and let

$$A: T^{(1)}M \to \mathbb{R}$$

be a continuous function. For any pair $x,y \in \tilde{M}$, let $\sigma(t)$ be the point of abscissa $t \in [0, d(x,y)]$ on the unique geodesic segment xy from x to y. We define

$$A_{xy} = \int_0^{d(x,y)} A(T_t(p\sigma(t)))dt$$

and, for $0 < r_1 < r_2$,

$$Z(x,r_1,r_2) = \int_{B(x,r_2)\backslash B(x,r_1)} \mu(dy) \exp A_{x,y}.$$

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Theorem. *Let $c \geqslant 2(a+b)$, then*

(2) $$\lim_{r\to\infty} \frac{1}{r} \log Z(x,r,r-c) = P(A)$$

uniformly with respect to x, where P(A) is the pressure of A with respect to the geodesic flow (f^t) on $T^{(1)}M$.

Our proof will closely follow that of Manning for the case $A = 0$ (see [5]).

We shall use the formulae (cf. [2])

$$P(A) = \lim_{\delta\to 0} P^{\pm}(A,\delta)$$

$$P^{\pm}(A,\delta) = \lim_{r\to\infty} \sup \frac{1}{r} Z_r^{\pm}(A,\delta)$$

$$Z_r^{+}(A,\delta) = \sup\left\{ \sum_{\xi\in S} \exp \int_0^r A(f^t\xi)dt : S \text{ is } (r,\delta) \text{ separated}\right\}$$

$$Z_r^{-}(A,\delta) = \inf\left\{ \sum_{\xi\in S} \exp \int_0^r A(f^t\xi)dt : S \text{ is } (r,\delta) \text{ spanning}\right\}$$

These formulae are easily related to those for the time 1 map f^1 and the function $A^1 = \int_0^1 dt\, A\cdot f^t$.

Lemma. *Given $\delta, \Delta > 0$ there is R such that if $\sigma,\tau : [0,r] \to \tilde{M}$ are two geodesics with $\sigma(0) = \tau(0)$, then $d(\sigma(r),\tau(r)) \leqslant \Delta$ and $r \geqslant R$ imply*

$$d(T_t\sigma(t), T_t\tau(t)) \leqslant \delta$$

in $T^{(1)}\tilde{M}$ for $t \in [0,r-R]$.

This is a form of Lemmas 1 and 2 of Manning corresponding to strictly negative curvature: geodesics diverge exponentially.

We shall use the fact, given $\epsilon > 0$, for $d(y,z) \leqslant$ constant, and sufficiently large $d(x,y)$,

(3) $$|A_{xy} - A_{xz}| < \frac{1}{2}\epsilon\, d(x,y)$$

This follows from the lemma and the uniform continuity of A.

To prove (2) we first show that, for $\delta \leqslant \frac{1}{2} b$ and all $\epsilon > 0$,

(4) $$\lim_{r\to\infty} \sup \frac{1}{r} \log Z(x,r,r+\delta/2) \leqslant P^{+}(A,\delta) + \epsilon$$

(uniformly in x). Following Manning's proof of Theorem 1 in [5] we take a maximal 2δ-separated set $Q \subset B(x,r+\delta/2)\backslash B(x,r)$. Then, for r sufficiently large,

$$Z(x,r,r+\delta/2) \leqslant \beta \sum_{y\in Q} \exp(A_{xy} + \frac{1}{2}\epsilon(r+\delta/2)) \leqslant$$

$$\leqslant \beta\, e^{\epsilon r} \sum_{y\in Q} \exp A_{xy}$$

where we have used (1) and the fact that (3) holds for $d(y,z) \leqslant 2\delta$. The estimate (4) follows from the remark that the unit initial vectors of the geodesics xy with $y \in Q$ are (r,δ) separated.

Now we show that

(5) $$\lim_{r\to\infty} \inf \frac{1}{r} \log Z(x,r+a+b, r-a-b) \geqslant P(A)$$

(uniformly in x). Following Manning's proof of Theorem 2 in [5] we construct a maximal $2b$-separated set $Q' \subset B(x,r+a)\backslash B(x,r-a)$. Using the lemma and the estimate (3) we see that, for given $\epsilon > 0$ and large r,

(6) $$Z(x,r+a+b, r-a-b) \geqslant \alpha \sum_{y\in Q'} \exp(A_{xy} - \frac{1}{2}\epsilon\, d(x,y)) \leqslant$$

An arbitrary geodesic segment of length r in M is the image by p of a geodesic segment uv in $\tilde{M}$, with $n \in N$ and $v \in B(x,r+a)\backslash B(x,r-a)$. There is thus a geodesic segment xy with $y \in Q'$ such that $d(u,x) \leqslant a$, $d(v,y) \leqslant 2b$. Given $\delta > 0$, the lemma applied to uv, uy, then to yu, yx yields $R > 0$ such that $d(T_t\, \sigma(t), T_t\, \tau_y(t)) \subset 2\delta$ in $T^{(1)}\tilde{M}$ for $t \in [R,r-R]$, where σ,τ_y denote the geodesics along uv, xy suitably parametrized. If ξ_y is the tangent vector to τ_y at R, the set $\{\xi_y : y \in Q'\}$ is $a(r-2R,2\delta)$ spanning set for the geodesic flow. The right-hand side of (6) is, for large r,

$$\geqslant \alpha \sum_{y\in Q'} \exp\left[\int_0^{r-2R} A(f^t\xi_y)dt - \epsilon r\right].$$

From this (5) follows.

Finally, (2) is a consequence of (4) and (5).

Remarks. (a) From (2) we also obtain

$$\lim_{r_2-r_1\to\infty, r_2/r_1\to 1} \frac{1}{r_2} \log Z(x,r_1,r_2) = P(A)$$

$$\lim_{r_2\to\infty, r_2-r_1\geqslant c} \frac{1}{r_2} \log Z(x,r_1,r_2) = P(A), \text{ if } P(A) \geqslant 0$$

$$\lim_{r_1\to\infty, r_2, r_1\geqslant c} \frac{1}{r_1} \log Z(x,r_1,r_2) = P(A), \text{ if } P(A) < 0$$

In particular

$$\text{(7)} \qquad \lim_{r\to\infty} \frac{1}{r} \log Z(x,0,r) = P(A), \text{ if } P(A) \geqslant 0$$

$$\lim_{r\to\infty} \frac{1}{r} \log Z(x,r,\infty) = P(A), \text{ if } P(A) < 0$$

(b) The limit does not depend on the choice of μ. If one uses for μ the measure defined by the Riemann metric, and $A = 0$, then (7) is a theorem of Manning (Theorem 2 of [5]).

(c) One can take for μ the sum of a unit mass at each $y \in p^{-1}x$, so that

$$Z(x,r_1,r_2) = \sum_{y\in p^{-1}x\cap B(x,r_2)\setminus B(x,r_1)} \exp A_{xy}.$$

(d) Since the geodesic flow on M is Anosov, Bowen's specification property holds (see [1] Theorem (3.8)). Therefore (as in the proof of Lemma (4.10) of [1]) one can check that $P(A)$ is the abscissa of convergence of the product

$$\zeta_A(u) = \prod_\gamma [1 - \exp \int_0^{\lambda(\gamma)} (A(f^t\xi_\gamma)-u)dt]^{-1}.$$

This product is extended over oriented closed geodesics γ, $\lambda(\gamma)$ is the length of γ, and $\xi_\gamma \in T^{(1)}M$ is any properly oriented tangent unit vector to γ. In fact, if A is Hölder continuous, $P(A)$ is α simple pole of ζ_A, and ζ_A is analytic in some neighborhood of $P(A)$ (see [5]).

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