

The pressure of the geodesic flow on a negatively curved manifold

David Ruelle

Manning [5] has identified the rate of exponential growth of the volume of a ball of radius r on the universal cover of a compact manifold M of negative curvature: it is the entropy of the geodesic flow on M. See also Sullivan [7], Chen [3], Chen and Manning [4]. Here we indicate an extension of Manning's result, where the entropy is replaced by the topological pressure P(A) associated with a function A on the tangent bundle. It turns out that the Riemann volume used by Manning plays no special role and may be replaced by many other measures.

Let M be a compact Riemann manifold with strictly negative sectional curvatures everywhere. We denote by M-tilde the universal cover of M (with the induced metric), by p: M-tilde -> M the canonical projection, and by N a fundamental domain of finite diameter a. We call B(x,r) the ball with center x and radius r in M-tilde. Let mu be a positive Radon measure on M-tilde, such that there are alpha, beta, b > 0 with

(1) alpha <= mu(B(x,b)) <= beta

for all x in M-tilde.

We denote by T^(1)M the unit tangent bundle and let

A.T^(1)M -> R

be a continuous function. For any pair x,y in M-tilde, let sigma(t) be the point of abscissa t in [0,d(x,y)] on the unique geodesic segment xy from x to y. We define

A_xy = integral from 0 to d(x,y) of A(T_t(p sigma(t))) dt

and, for 0 < r_1 < r_2,

Z(x,r_1,r_2) = integral over B(x,r_2) \ B(x,r_1) of mu(dy) exp A_x,y.

Theorem. Let $c \geq 2(a+b)$, then

$$(2) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log Z(x, r, r-c) = P(A)$$

uniformly with respect to x , where $P(A)$ is the pressure of A with respect to the geodesic flow (f^t) on $T^{(1)}M$.

Our proof will closely follow that of Manning for the case $A = 0$ (see [5]).

We shall use the formulae (cf. [2])

$$P(A) = \lim_{\delta \rightarrow 0} P^\pm(A, \delta)$$

$$P^\pm(A, \delta) = \limsup_{r \rightarrow \infty} \frac{1}{r} Z_r^\pm(A, \delta)$$

$$Z_r^+(A, \delta) = \sup \left\{ \sum_{\xi \in S} \exp \int_0^r A(f^t \xi) dt : S \text{ is } (r, \delta) \text{ separated} \right\}$$

$$Z_r^-(A, \delta) = \inf \left\{ \sum_{\xi \in S} \exp \int_0^r A(f^t \xi) dt : S \text{ is } (r, \delta) \text{ spanning} \right\}$$

These formulae are easily related to those for the time 1 map f^1 and the

$$\text{function } A^1 = \int_0^1 dt A \cdot f^t.$$

Lemma. Given $\delta, \Delta > 0$ there is R such that if $\sigma, \tau: [0, r] \rightarrow \bar{M}$ are two geodesics with $\sigma(0) = \tau(0)$, then $d(\sigma(r), \tau(r)) \leq \Delta$ and $r \geq R$ imply

$$d(T_t \sigma(t), T_t \tau(t)) \leq \delta$$

in $T^{(1)}\bar{M}$ for $t \in [0, r-R]$.

This is a form of Lemmas 1 and 2 of Manning corresponding to strictly negative curvature: geodesics diverge exponentially.

We shall use the fact, given $\epsilon > 0$, for $d(y, z) \leq \text{constant}$, and sufficiently large $d(x, y)$,

$$(3) \quad |A_{xy} - A_{xz}| < \frac{1}{2} \epsilon d(x, y)$$

This follows from the lemma and the uniform continuity of A .

To prove (2) we first show that, for $\delta \leq \frac{1}{2} b$ and all $\epsilon > 0$,

$$(4) \quad \limsup_{r \rightarrow \infty} \frac{1}{r} \log Z(x, r, r+\delta/2) \leq P^+(A, \delta) + \epsilon$$

(uniformly in x). Following Manning's proof of Theorem 1 in [5] we take a maximal 2δ -separated set $Q \subset B(x, r+\delta/2) \setminus B(x, r)$. Then, for r sufficiently large,

$$\begin{aligned} Z(x, r, r+\delta/2) &\leq \beta \sum_{y \in Q} \exp(A_{xy} + \frac{1}{2} \epsilon(r+\delta/2)) \leq \\ &\leq \beta e^{\epsilon r} \sum_{y \in Q} \exp A_{xy} \end{aligned}$$

where we have used (1) and the fact that (3) holds for $d(y, z) \leq 2\delta$. The estimate (4) follows from the remark that the unit initial vectors of the geodesics xy with $y \in Q$ are (r, δ) separated.

Now we show that

$$(5) \quad \liminf_{r \rightarrow \infty} \frac{1}{r} \log Z(x, r+a+b, r-a-b) \geq P(A)$$

(uniformly in x). Following Manning's proof of Theorem 2 in [5] we construct a maximal $2b$ -separated set $Q' \subset B(x, r+a) \setminus B(x, r-a)$. Using the lemma and the estimate (3) we see that, for given $\epsilon > 0$ and large r ,

$$(6) \quad Z(x, r+a+b, r-a-b) \geq \alpha \sum_{y \in Q'} \exp(A_{xy} - \frac{1}{2} \epsilon d(x, y)) \leq$$

An arbitrary geodesic segment of length r in M is the image by p of a geodesic segment uv in \bar{M} , with $u \in N$ and $v \in B(x, r+a) \setminus B(x, r-a)$. There is thus a geodesic segment xy with $y \in Q'$ such that $d(u, x) \leq a$, $d(v, y) \leq 2b$. Given $\delta > 0$, the lemma applied to uv , uy , then to yu , yx yields $R > 0$ such that $d(T_t \sigma(t), T_t \tau_y(t)) \leq 2\delta$ in $T^{(1)}\bar{M}$ for $t \in [R, r-R]$, where σ, τ_y denote the geodesics along uv , xy suitably parametrized. If ξ_y is the tangent vector to τ_y at R , the set $\{\xi_y : y \in Q'\}$ is a $(r-2R, 2\delta)$ spanning set for the geodesic flow. The right-hand side of (6) is, for large r ,

$$\geq \alpha \sum_{y \in Q'} \exp \left[\int_0^{r-2R} A(f^t \xi_y) dt - \epsilon r \right].$$

From this (5) follows.

Finally, (2) is a consequence of (4) and (5).

Remarks. (a) From (2) we also obtain

$$\lim_{r_2 \rightarrow \infty, r_1 \rightarrow 1} \frac{1}{r_2} \log Z(x, r_1, r_2) = P(A)$$

$$\lim_{r_2 \rightarrow \infty, r_2 - r_1 \geq c} \frac{1}{r_2} \log Z(x, r_1, r_2) = P(A), \text{ if } P(A) \geq 0$$

$$\lim_{r_1 \rightarrow \infty, r_2, r_1 \geq c} \frac{1}{r_1} \log Z(x, r_1, r_2) = P(A), \text{ if } P(A) < 0$$

In particular

$$(7) \quad \lim_{r \rightarrow \infty} \frac{1}{r} \log Z(x, 0, r) = P(A), \text{ if } P(A) \geq 0$$

$$\lim_{r \rightarrow \infty} \frac{1}{r} \log Z(x, r, \infty) = P(A), \text{ if } P(A) < 0$$

(b) The limit does not depend on the choice of μ . If one uses for μ the measure defined by the Riemann metric, and $A = 0$, then (7) is a theorem of Manning (Theorem 2 of [5]).

(c) One can take for μ the sum of a unit mass at each $y \in p^{-1}x$, so that

$$Z(x, r_1, r_2) = \sum_{y \in p^{-1}x \cap B(x, r_2) \setminus B(x, r_1)} \exp A_{xy}.$$

(d) Since the geodesic flow on M is Anosov, Bowen's specification property holds (see [1] Theorem (3.8)). Therefore (as in the proof of Lemma (4.10) of [1]) one can check that $P(A)$ is the abscissa of convergence of the product

$$\zeta_A(u) = \prod_{\gamma} [1 - \exp \int_0^{\lambda(\gamma)} (A(f^t \xi_{\gamma}) - u) dt]^{-1}.$$

This product is extended over oriented closed geodesics γ , $\lambda(\gamma)$ is the length of γ , and $\xi_{\gamma} \in T^{(1)}M$ is any properly oriented tangent unit vector to γ . In fact, if A is Hölder continuous, $P(A)$ is a simple pole of ζ_A , and ζ_A is analytic in some neighborhood of $P(A)$ (see [5]).

References

[1] R. Bowen, *Periodic orbits for hyperbolic flows*, Amer. J. Math. 94, 1-30 (1972).
 [2] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. 29, 181-202 (1975).

[3] S.-S. Chen, *Entropy of geodesic flow and exponent of convergence of some Dirichlet series*, To appear.
 [4] S.-S. Chen and A. Manning, *The convergence of zeta functions for certain geodesic flows depends on their pressure*, To appear.
 [5] A. Manning, *Topological entropy for geodesic flows*, Ann. of Math. 110, 567-573 (1979).
 [6] D. Ruelle, *Generalized zeta-functions for Axiom A basic sets*, Bull. Amer. Math. Soc. 82, 153-156 (1976).
 [7] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publ. Math. IHES 50, 171-202 (1979).

I.H.E.S.
 35 Route de Chartres
 91 Bures Sur Yvette
 France

In this paper the author proves that, on a compact connected and orientable two-dimensional C^{∞} manifold M , the gradient of a vector field has finite modulus of stability under conjugacy and topological equivalences. It is also proved that generalised zeta functions are stable under conjugacy of a graph of a C^{∞} vector field. The number of its saddles. Some new conjugacy invariants arise in the proof of these results.

Let $\mathcal{V}(M)$ be the Banach space of C^1 vector fields on M with the C^1 norm. We will consider the subspace $\text{Grad}^1(M)$ of the elements of $\mathcal{V}(M)$ which are gradients of C^{∞} real functions on M . Given $X \in \mathcal{V}(M)$ and $Y \in \mathcal{V}(M)$, we say that X and Y are conjugate if there is a homeomorphism $h: M \rightarrow M$ such that $h_* X = Y$, $h_* \in \mathcal{R}$. If there is a homeomorphism $h: M \rightarrow M$ taking orbits of X onto orbits of Y , preserving the orientation of the orbits, then we say that X and Y are topologically equivalent.

We say that a vector field is stable with respect to one of these relations if its equivalence class is open.

These equivalence relations can also be considered in $\text{Grad}^1(M)$.

Consider now an element unstable with respect to one of the relations above. If it is possible to describe all the equivalence classes in some neighbourhood of such an element by a finite number of real parameters, then we say that it has finite modulus of stability with respect to that relation. In this case the minimum number of such parameters is called the modulus of stability of the element given.

If in some neighbourhood of that element there are at most a denumerable number of equivalence classes, then we say that its modulus of stability is zero with respect to the relation considered.

If for that element some of the above conditions are satisfied then we say that its modulus of stability is infinite with respect to the relation considered.

(*) During the preparation of this paper the author was a visiting Professor at IMPA, and was partially supported by Financiadora de Estado e Projetos (FINEP).

Recebido em 11/10/79.