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On characteristic classes of compact homogeneous spaces and their applications in compact transformation groups I

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Introduction. Basically, this paper is a natural continuation of a series os papers of Borel-Hirzebruch on characteristic classes of homogeneous spaces [BH] which reduced the computation of characteristic classes of homogeneous spaces to their Lie group theoretical invariants, namely, roots and weights. However, in actually carrying out such computations, one finds that their method needs some basic modifications ir order to simplify the corresponding algebraic problem to a manageable level [Cf. Section 1]. As a direct application of such a modified method of Borel-Hirzebruch in the computation of characteristic classes of homogeneous spaces, we are able to solve the following problems which are the motivation of this paper to start with:

Problem 1. Let G be a classical compact simple Lie group, namely, $SU(n)$, $SO(n)$ or $Sp(n)$, and H be a connected closed subgroup of G. Suppose the first three Pontrjgin classes of the homogeneous space G/H vanish, i. e., $P_j(G/H)=0$, $j=1,2,3$. How to classify all such subgroups H of classical groups up to conjugacy classes?

Problem 2. Let M be a smooth manifold with $P_j(M)=0$, $j=1,2,3$, and Φ be a given smooth action of a classical compact simple Lie group G on M. Let (H_Φ^0) be the conjugacy class of the connected components of principal isotropy subgroups of the G-action Φ. What are the possibilities of such conjugacy classes (H_Φ^0) for *all* smooth actions Φ of classical groups G on *all* such manifolds M with $P_j(M)=0$, $j=1,2,3$?

Problem 3. Suppose M is a smooth manifold with $P_j(M)=0$, $j=1,2,3$, and Φ is a given G-action with dim $H_\Phi^0 \neq 0$. What are the possibilities of (G_x^0) for $x \in M$?

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A simple, basic idea introduces characteristic classes of equivariant vector bundles over homogeneous spaces as a useful tool of investigating orbit structures of compact transformation groups is the following basic fact about local structure of orbits:

Suppose G/H occurs as an orbit in a given G-manifold M. Then G/H imbeds in M with an *equivariant* normal bundle which is an associated bundle of the canonical $\dot{H}$-principal bundle over G/H, namely,

$$\nu(G/H)\colon \mathbb{R}^k_x \to G \times_H \mathbb{R}^k_x \to G/H$$

where $\mathbb{R}^k_x$ is the H-linear space of normal vectors of the orbit $G(x) = G/H$ at x with the induced H-action. Following the notation of Atiyah-Hirzebruch [AH], we shall denote the canonical homomorphism of $RO(H) \to \to KO(G/H)$ by $\alpha_{G,H}$. Then $\nu(G/H) = \alpha_{G,H}(\varphi_x)$ where φ_x is the induced linear representation of $H = G_x$ on the space of normal vectors $\mathbb{R}^k_x$. Therefore, the pullback of the tangent bundle of M, $\tau(M)$, with respect to the imbedding, $G/H \xrightarrow{i} M$, gives us the following equation:

$$i^!\tau(M) = \tau(G/H) + \nu(G/H) = \alpha_{G,H}(Ad_G|H - Ad_H) + \alpha_{G,H}(\varphi_x)$$

If one evaluates the above equation at the characteristic class level, then one gets a system of algebraic equations involving the following data:

(i) the homological data of the imbedding: $H^*(M) \xrightarrow{i^*} H^*(G/H)$,
(ii) the infinitesimal data of the location of H in G,
(iii) the weight system of the slice representation φ_x.

From the viewpoint of transformation groups, all the above three sets of data are "unknowns" that one would like to determine their possible solutions. As a simplified example, suppose M is a G-manifold with $P_j(M) = 0$, $j = 1, 2, 3$, and G/H is a principal bundle. Then φ_x is a trivial representation and one has the following system of equations:

$$P_j(G/H) = i^*P_j(M) = 0 \qquad j = 1, 2, 3.$$

Therefore, the solutions of problem 2 are exactly the same as the solutions of problem 1. Of course, it is rather remarkable that problem 1 actually consists of the following neat solutions:

Theorem 1. *Suppose H is a connected closed subgroup of $SU(m)$ such that $P_j(SU(m)/H) = 0$, $j = 1, 2, 3$. Then the possibilities of the representation $\psi\colon H \subset SU(m)$ are as follows modulo trivial representations:*

(i) *H is any given subtorus,*
(ii) *H is semi-simple and $\psi = Ad_H$,*
(iii) $H = SU(n) \times \tilde{H}$, $n/30$ *and* $\psi = \mu_n \otimes \mu_n + Ad_{\tilde{H}}$,

$$\text{(iv)}\quad H = \begin{cases} SU(n) \\ SO(n) \\ Sp(n) \\ G_2 \end{cases}, \quad \psi = \begin{cases} \mu_n \text{ or } 2\mu_n, \ \dim \mu_n = n, \\ \rho_n, \ \dim \rho_n = n, \\ \nu_n, \ \dim \nu_n = 2n, \\ \varphi_1 \text{ or } 2\varphi_1, \ \dim \varphi_1 = 7 \end{cases}$$

$$\text{(v)}\quad H = Sp(1)^\ell, \quad \psi = k(\nu_1^{(1)} + \dots + \nu_1^{(\ell)}), \quad k = 1, 2, 4,$$

$$H = \begin{cases} SU(3) \times SU(3) \\ G_2 \times G_2 \end{cases}, \quad \begin{cases} k(\mu_3 + \mu'_3) + \ell(\bar{\mu}_3 + \bar{\mu}'_3), \ k + \ell = 1,2, \\ (\varphi_1 + \varphi'_1) \text{ or } 2(\varphi_1 + \varphi'_1), \end{cases}$$

$$H = \begin{cases} SU(n), \ n = 3,4,5 \\ SU(3) \\ Sp(2) \\ Spin(8) \end{cases}, \psi = \begin{cases} \mu_n + \bar{\mu}_n \\ k\mu_3 + \ell\bar{\mu}_3, \ k + \ell = 3,6, \ (k, \ell) \neq (3,3) \\ \nu_2 + \Lambda^2\nu_2 \\ \Delta^+ + \Delta^- \end{cases}$$

Theorem 2. *Let $\psi\colon H \subset Sp(m)$ be a sympletic representation of a compact connected Lie group with $P_k(Sp(m)/\psi H) = 0$ for $k = 1,2,3$. Then the possibilities of all such pairs (H, ψ) are given by the following list modulo trivial representation:*

(i) *H is any given subtorus,*
(ii) $H = Sp(n)$, $\psi = \nu_n$,
(iii) $H = Sp(1)^\ell$, $\psi = k(\nu_1^{(1)} + \dots + \nu_1^{(\ell)})$, $k = 1,2,4$.

$$\text{(iv)}\quad \mathrm{H} = \begin{cases} SU(n), \ n = 3,4,5 \\ SU(3) \times SU(3) \\ G_2 \times G_2 \\ G_2 \end{cases} \quad \psi = \begin{cases} \mu_n + \bar{\mu}_n \\ (\mu_3 + \bar{\mu}_3) + (\mu'_3 + \mu'_3) \\ 2\varphi_1 + 2\varphi'_1 \\ 2\varphi_1 \end{cases}$$

Theorem 3. *Let $\psi\colon H \subset SO(m)$ be a compact connected Lie group with a given real representation ψ. If $P_k(SO(m)/\psi H) = 0$ for $k = 1,2,3$, then the possibilities of all such pairs (H, ψ) are given by the following list modulo trivial representations:*

(i) *H is any given subtorus,*
(ii) *H is semi-simple and $\psi = Ad_H$,*
(iii) *H is semi-simple withouth normal factors of B_n, C_{2n} or D_{2n} type and $\psi = 2Ad_H$,*

$$\text{(iv)}\quad H = [Sp(1)]^\ell, \ \psi = k(\nu_1^{(\ell)} + \dots + \nu_1^{(\ell)}), \ k/8,$$

$$H = \begin{cases} SO(n) \\ Sp(n) \end{cases}, \psi = \begin{cases} k\rho_n, \ k/(2, n) \\ 2\nu_n, \end{cases}$$

$$H = \begin{cases} SU(3) \times SU(3) \\ G_2 \times G_2 \end{cases} \quad \psi = \begin{cases} k \cdot [\mu_3 + \bar{\mu}_3 + \mu'_3 + \bar{\mu}'_3], \ k = 1,2, \\ k \cdot (\varphi_1 + \varphi'_1), \ k = 1,2, \end{cases}$$

$$\text{(vii) } H = \begin{cases} SU(n),\ n = 3,4,5 \\ G_2 \\ Sp(2) \\ Spin(8) \\ SU(4) \\ SU(2) \end{cases} \quad \psi = \begin{cases} k(\mu_n + \bar{\mu}_n),\ k = 1,2 \\ k\varphi_1,\ k = 1,2,4 \\ 2\Lambda^2\nu_2 + 2\nu_2 \\ k\cdot(\Delta^+ + \Delta^-),\ k = 1,2 \\ \Lambda^2\mu_4 + (\mu_4 + \bar{\mu}_4) \\ 4\mu_2 + Ad. \end{cases}$$

We shall prove the above theorems in Sections 3, 4, 5, and then apply them to further study the orbit type of actions of classical groups on manifold with vanishing first three Pontrjagin classes and with positive dimentional principal isotropy subgroups as determined by the above three theorems. Technically, this paper is a continuation as well as an improvement of the three papers of Borel-Hirzebruch [BH], and the result we obtain in this paper is a convincing indication of the computability and applicability of such a framework. From the viewpoint of transformation groups, this paper is a natural extension of the general ideas of [$HH1$]. One may further combine the result of this paper with cohomology theory of transformation groups to obtain deeper and more specific results for actions of classical groups on manifolds with given cohomology strucutres such as Stiefel manifolds and other homogeneous spaces.

Section 1.´ Basic reductions and the splitting principle of characteristic classes.

In this section, we shall recall some basic general facts about equivariant vector bundles over homogeneous spaces and the spliting principle of characteristic classes which will enable us to set up some fundamental reductions to reduce the computation of their characteristic classes to a manageable algebraic problem.

(A) *Equivariant Vector Bundles over Homogeneous Spaces.*

Let G be a compact connected Lie group and H be a closed subgroup of G. Then the equivariant KO-group of the homogeneous space G/H is simply the representation ring of H, namely

$$KO_G(G/H) \cong RO(H)$$

Let (G_1, H_1) and (G_2, H_2) be pairs of compact Lie groups and $\psi: (G_1, H_1) \to (G_2, H_2)$ be a Lie homomorphism of pairs, $\bar{\psi}: G_1/H_1 \to G_2/H_2$ be the induced equivariant map. Then, one has the following commutative diagram of induced morphisms:

$$\begin{array}{ccccc} RO(H_2) \cong KO_{G^2}(G_2/H_2) & \xrightarrow{\alpha} & KO(G_2/H_2) \\ \downarrow (\psi|H_1)^* \qquad \downarrow \bar{\psi}! & & \downarrow \bar{\psi}! \\ RO(H_1) \cong KO_{G_1}(G_1/H_1) & \xrightarrow{\alpha} & KO(G_1/H_1) \end{array}$$

Applying the above diagram to the special case of $(G, H) \to (G, G)$, it is then obvious that the composition of the following morphisms is equal to zero:

$$RO(G) \to RO(H) \xrightarrow{\alpha} KO(G/H) \to \widetilde{KO}(G/H).$$

Let $\tau(G/H)$ be the tangent bundle of G/H. Then it is well-known that $\tau(G/H) = \alpha_{G,H}(Ad_G|H - Ad_H)$. Therefore, one has the following reduction of basic importance:

Reduction 1. $\tau(G/H) = \alpha_{G,H}(Ad_G|H - Ad_H) = -\alpha_{G,H}(Ad_H)$ in $\widetilde{KO}(G/H)$ [$\because$ $\alpha_{G,H}(Ad_G|H) = 0$ in $\widetilde{KO}(G/H)$.]

As simple, direct applications of the above reduction, one has the following propositions:

Proposition 1.1. *If H is a torus subgroup of G, then G/H is stably parallelizable, i.e., $\tau(T/H) = -\alpha_{G,H}(Ad_H) = 0$ in $\widetilde{KO}(G/H)$.*

Proposition 1.2. *Let H be a semi-simple compact connected Lie group and $\psi: H \to SO(m)$ be a real representation with $\psi \equiv Ad_H$ modulo trivial representations. Then $SO(m)/\psi(H)$ is stably parallelizable.*

Reduction 2. Let $G \supset H \supset H_1$ be a triple of compact connected Lie groups and H_1 be *normal* in H. Then

$$p^!\tau(G/H) = \tau(G/H_1) \text{ in } \widetilde{KO}(G/H_1)$$

where $p: G/H_1 \to G/H$ is the induced projection.

Pf: Since H_1 is a connected normal subgroup of H, it is well-known that $Ad_H|H_1 = Ad_{H_1} +$ trivial representations. Hence

$$p^!\tau(G/H) \equiv -\alpha(Ad_H|H_1) \equiv -\alpha(Ad_{H_1}) \equiv \tau(G/H_1) \text{ in } \widetilde{KO}(G/H_1).$$

Reduction 3. Let $G \supset K \supset H$ be triple of compact Lie groups and $K/H \xrightarrow{i} G/H \to G/K$ be the associated fibration. Then

$$\tau(K/H) = i^!\tau(G/H) \text{ in } \cdot KO(K/H).$$

[$\because$ the local product structure of the fibration.]

(B) *Splitting principle of Borel-Hirzebruch.*

Following Borel-Hirzebruch [*BH*], let us formulate the splitting principle of characteristic classes of equivariant bundles over homogeneous spaces as follows:

Let Φ be a real representation of H on $\mathbb{R}^n$ and $\alpha(\Phi)$ be the associated equivariant $\mathbb{R}^n$-bundle over G/H. Let T, S be maximal tori of H and $SO(n)$ respectively, $\Phi(T) \subset S$, and let B_T, B_S, B_H and $B_{SO(n)}$ be their classifying spaces. Then, one has the following commutative diagram of maps:

$$\begin{array}{ccccc} G/T & \xrightarrow{i} & B_T & \xrightarrow{(\Phi \mid T)_*} & B_S \\ \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ G/H & \xrightarrow{i} & B_H & \xrightarrow{\Phi_*} & B_{SO(n)} \end{array}$$

Let ξ_n be the universal $\mathbb{R}^n$-bundle over $B_{SO(n)}$. Then

$$\alpha(\Phi) = i^! \cdot \Phi^!{}_*(\xi_n)$$

$$\pi^!\alpha(\Phi) = \alpha(\Phi \mid T) = i^!(\Phi \mid T)^!{}_* \cdot \pi^!(\xi_n).$$

Since the structural groups of the induced bundles over B_T and B_S are reducible to T and S respectively, it follows from the Schur lemma and the splitting principle that

$$\pi^* P_*(\alpha(\Phi)) = P_*(\pi^!\alpha(\Phi)) = P_*(\alpha(\Phi \mid T)) =$$

$$= i^* P_*(\alpha_{B_T}(\Phi \mid T)) = i^* \left\{ \prod_{w \in \Omega(\Phi)} (1 + w) \right\}$$

where $\Omega(\Phi)$ is the weight system of Φ and $w \in \Omega(\Phi)$ are considered as elements of $H^2(B_T; \mathbb{Z})$ via the following natural identifications:

$$H^2(B_T; \mathbb{Z}) \cong H^1(T; \mathbb{Z}) \cong \text{The group of characters of } T.$$

Summarizing the above discussion, one has the following special version of splitting principle for equivariant bundles over homogeneous spaces wich is the basic setting of the papers of Borel-Hirzebruch [*BH*]:

Reduction 4. Let $G \supset H \supset T$ be a triple of compact connected Lie groups and T be a maximsl torus of H. Let Φ be a real representation of H with weight system $\Omega(\Phi)$ and $\alpha(\Phi)$ be the associated equivariant bundle of Φ over G/H. Then

$$\pi^* P_*(\alpha(\Phi)) = i^* \left\{ \prod_{w \in \Omega(\Phi)} (1 + w) \right\}$$

where $\pi^*: H^*(G/H) \to H^*(G/T)$ $i^*: H^*(B_T) \to H^*(G/T)$ and P_* is the total Pontrjagin class.

In the special case of tangent bundle of G/H, one sets $\Phi = (Ad_G \mid H\text{-}Ad_H)$ and its weight system is called the *complementary root system of H in G*, namely,

$$\pi^*(P_*(G/H)) = i^*\{\Pi(1 + w)\}$$

where w runs through the complementary root system of H in G. This is the basic setting of Borel-Hirzebruch for the computation of characteristic classes of homogeneous spaces. However, in actual computations, the complementary root systems are usually rather complicated which make their symmetric products very difficult to deal with, especially in the case that the subgroup H is itself an unknown that one needs to solve, e.g. problem 1, 2 and 3. The following is a simple but significant modification of the setting of Borel-Hirzebruch which greatly simplifies the algebraic computations involved in the computations of characteristic classes of homogeneous spaces:

Reduction 4′. Let $G \supset H \supset T$ be as in reduction 4 and $\bar{P}_*$ be the total *dual* Pontrjagin class of G/H. Then

$$\pi^* \bar{P}_*(G/H) = i^* \left\{ \prod_{\alpha \in \Delta(H)} (1 + \alpha) \right\}$$

where $\Delta(H)$ is the root system of H.

[$\because$ $\tau(G/H) \equiv -\alpha(Ad_H)$ in $KO(G/H)$ and Pontrjagin classes are stable.]

Remark. The above symmetric product of root system $\{\Pi(1 + \alpha); \alpha \in \Delta(\mathrm{H})\}$ is not only much simpler than that of the complementary root system but is also *independent* of the imbedding of H in G, i.e., only dependent on the abstract Lie group structure of H. This simplification makes the solution os problems such as problem 1, 2 and 3 at all feasible.

(C) *Transgression theorem and the kernel of* i^*

In order to compute the homomorphism $i^*: H^*(B_T) \to H^*(G/T)$, let us consider the following commutative diagram:

$$\begin{array}{ccccc} & & G & \longrightarrow & G \\ & & \downarrow & & \downarrow \\ G \times_T G & \longrightarrow & E_G \times_T G & \xrightarrow{\bar{p}} & E_G \\ d \uparrow & \bar{d} \nearrow & \downarrow & & \downarrow \\ G/T & \xrightarrow{i} & B_T & \xrightarrow{p} & B_G \end{array}$$

where $G \to E_G \times_T G \to B_T$ is the induced G-bundle of $p: B_T \to B_G$ and d is the map induced from $\tilde{d}: G \to G \times G$ with $\tilde{d}(g) = (g, g^{-1})$. Observe that $\bar{d}$ is a cross-section of the fibration $E_G \to E_G \times_T G \to G/T$ with weakly contractible fibre E_G, it is clear that $\bar{d}$ is a weak homotopy equivalence. Therefore, one may identify $H^*(G/T)$ and $H^*(E_G \times_T G)$ via the isomorphism $\bar{d}^*$ and then it follows from the above diagram that $i^* = \pi_G^*$ where $\pi_G^*: H^*(B_T) \to H^*(E_G \times_T G)$ is the edge homomorphism of the Serre spectral sequence of the fibration $E_G \times_T G \to B_T$. The advantage of the above reformulation is that π_G^* is effectively computable in terms of the transgression theorem of A. Borel [B1]. Let us recall the concrete cases of $G = SU(m), Sp(m)$ and $SO(m)$ in the following:

(i) *The case* $G = SU(m)$. In this case, $H^*(SU(m); \mathbb{Z})$ is an exterior algebra generated by $(m-1)$ universally transgressive elements of degree $(2k+1)$, $1 \le k \le (m-1)$, whose transgressions are the universal Chern classes $\{C_{k+1}, 1 \le k \le (m-1)\}$. Therefore, it is not difficult to see that the kernel of $i^* = \pi_G^*$ is exactly the ideal (in $H^*(B_T; \mathbb{Z})$) generated by $\{p^*(C_{k+1})\}$. Moreover, it follows from the splitting principle that

$$p^*(C_*) = \prod_{w \in \Omega(\psi)} (1 + w) = 1 + P\psi^2 + P\psi^3 + \ldots + P\psi^m$$

where $\Omega(\psi)$ is the weight system of the representation of $\psi: H \subset SU(m)$ and $p^*C_k = P\psi^k =$ the homogeneous part of degreee k of the above symmetric product. Hence, one has the following reduction for the case $G = SU(m)$.

Reduction 5A. In the case $G = SU(m)$, $\psi: H \subset SU(m)$ can be considered as a complex representation of H with its weight systems $\Omega(\psi)$. Let $\prod_{w \in \Omega(\Phi)} (1 + w) = 1 + P\psi^2 + \ldots + P\psi^k + \ldots + P\psi^m$ be the symmetric product of weights of ψ. Then

$$Ker(i^*) = \langle P\psi^2, P\psi^3, \ldots, P\psi^m \rangle.$$

(ii) *The case* $G = Sp(m)$. Again, $H^*(Sp(m); \mathbb{Z})$ is an exterior algebra generated by m universally transgressive elements whose transgression are respectively the universal quaternionic classes $q_1, q_2, \ldots, q_m$. Therefore, one has the following reduction for the case $G = Sp(m)$.

Reduction 5C. In the case $G = Sp(m)$, $\psi: H \subset Sp(m)$ is a quaternionic representation of H. Then

$$Ker(i^*) = \langle P\psi^2, P\psi^4, \ldots, P\psi^{2m} \rangle$$

where $P\psi^{2k}$ is the homogeneous part of degree $2k$ in the following symmetric product:

$$\prod_{w \in \Omega(\psi)} (1 + w) = \prod_{w \in \Omega^+(\psi)} (1 - w^2) = 1 + P\psi^2 + P\psi^4 + \ldots + P\psi^{2m}.$$

(iii) *The case* $G = SO(2m+1)$. Since $H^*(B_T; \mathbb{Z})$ is torsion-free, no torsion elements can have a non-zero transgression in $H^*(B_T; \mathbb{Z})$. The torsion-free part of $H^*(SO(2m+1); \mathbb{Z})$ is again an exterior algebra generated by $\{x_k, 1 \le k \le m\}$ such that $\{2x_k\}$ are universally transgressive whose transgression are respectively the universal Pontrjagin classes $\{P_k\}$ in $H^*(B_{SO(2m+1)}; \mathbb{Z})$. Let $\psi: H \subset SO(2m+1)$ be the real representation and $\Omega(\psi)$ be its weight system. Then

$$\begin{aligned} p^*(1 + P_1 + \ldots + P_m) &= \prod_{w \in \Omega(\psi)} (1 + w) = \prod_{w \in \Omega^+(\psi)} (1 - w^2) = \\ &= 1 + P\psi^2 + P\psi^4 + \ldots + P\psi^{2m}. \end{aligned}$$

Furthermore, let us introduce the following notation, namely,

$$\overline{P\psi}^{2k} = \begin{cases} \frac{1}{2} P\psi^{2k} \text{ if } P\psi^{2k} \text{ is divisible by 2 in } H^{4k}(B_T, \mathbb{Z}) \\ P\psi^{2k} \text{ otherwise.} \end{cases}$$

Then, one may state thr result for this case as follows:

Reduction 5B. In the case $G = SO(2m+1)$,

$$\langle P\psi^2, P\psi^4, \ldots, P\psi^{2m} \rangle \subset Ker(i^*) \subset \langle \overline{P\psi}^2, \overline{P\psi}^4, \ldots, \overline{P\psi}^{2m} \rangle.$$

Remark. The case of $G = SO(2m)$ is essentially the same as that of $SO(2m+1)$, except that the m-th Pontrjagin class should be replaced by the Euler class. A precise statement about $Ker(i^*)$ will involve conditions on the $\mathbb{Z}_2$-weights of ψ, i.e., its Stiefel Whitney classes. We refer to $[BH]$ for similar results for the Stiefel-Whitney classes in terms of $\mathbb{Z}_2$-roots and $\mathbb{Z}_2$-weights.

Section 2. Basic Weyl invariants and algebraic computations of symetric sums and symmetric products.

The splitting principle of Section 1 reduces the computations of characteristic classes of homogeneous spaces into elementary algebraic problems involving symmetric products of roots and weights of the following type:

$$\prod_{w\in\Omega(\psi)}(1+w)=\Sigma\, P\psi^k$$

where $\Omega(\psi)$ is the weight system of a given representation ψ of H and $P\psi^k$ is the homogeneous part of degree k of the above product. We need to compute the explicit expression of $P\psi^k$ in terms of the *basic Weyl invariants* of invariant polynomials of the linear space of Cartan subalgebra of H with respect to the action of Weyl group of H. Straightforward computations of $P\psi^k$ are rather difficult to carry out and the idea to overcome such a computacional difficulty is to make use of the Newton's Formula which enables us to reduce the computation of $\{P\psi^k\}$ to that of the following symmetric sums:

$$S\psi^k=\sum_{w\in\Omega(\psi)} w^k;\ k=1,2,3,\ldots$$

(A) *Newton's Formula.*

Let $\{\theta_j, 1\le i\le\le n\}$ be a set of n indeterminants. Then

$$\prod_j(1+\theta_j)=1+\sigma_1(\theta)+\sigma_2(\theta)+\ldots+\sigma_n(\theta)$$

where the homogeneous part of degree k, $\sigma_k(\theta)$, is called the k-th elementary symmetric polynomial of $\{\theta_j\}$. On the other hand, let $s_k(\theta)=\Sigma\theta_j^k$ be the symmetric sum of k-th power of θ_j. Then the well-known Newton's Formula consists of the following identities relating $\{\sigma_k\}$ and $\{s_k\}$, namely,

$$s_k-\sigma s_{k-1}+\sigma_2 s_{k-2}-\ldots+(-1)^j\sigma_j s_{k-j}+\ldots+(-1)^k k\sigma_k=0$$

[adopting the convention that $\sigma_j=0$ for $j>n$]. From the above system of linear equations, one may solve s_k explicity in terms of σ_j and vis versa. In fact, one can use the formal power series of logarithm to get explicit general formulae for s_k in terms of σ_j, namely,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}s_k=\sum_{j=1}^{n}\log(1+\theta_j)=\log(1+\sigma_1+\sigma_2+\ldots+\sigma_n)=$$

$$=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}(\sigma_1+\sigma_2+\ldots+\sigma_n)^i.$$

If one expand $(\sigma_1+\sigma_2+\ldots+\sigma_n)^i$ by multi-nomial theorem and collect terms of degree k of the righthand side, one gets

$$(-1)^{k-1}\cdot\frac{s_k}{k}=\Sigma(-1)^{|\alpha|+1}\cdot\frac{(|\alpha|-1)!}{\alpha!}\sigma^\alpha$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ runs through all multi-indeces with $\Sigma i\alpha_i=k$ and $|\alpha|=\Sigma\alpha_i$, $\alpha!=\prod_{i=1}^{n}(\alpha_i!)$, $\sigma^\alpha=\prod_{i=1}^{n}\sigma_i^{\alpha_i}$.

Conversely, one can use the formal power series of exponential go get explicit general formula for σ_j in terms of s_k:

$$(1+\sigma_1+\sigma_2+\ldots+\sigma_n)=\exp\left[\sum_{k=1}^{\infty}(-1)^{k+1}\cdot\frac{s_k}{k}\right]=$$

$$=1+\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{s_k}{k}\right]+\ldots+\frac{1}{\ell!}\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{s_k}{k}\right]^{\ell}+\ldots$$

Again, by collecting terms of degree j in the right hand side, one gets

$$\sigma_j=\Sigma(-1)^{j+|\alpha|}\cdot\frac{s_k^\alpha}{\alpha!k^\alpha}$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_j)$ runs through all multi-indices with $\Sigma\, k\alpha_k=j$, $\alpha!=\prod_{k=1}^{j}\alpha_k!$, $s_k^\alpha=\prod_{k-1}^{j}s_k^{\alpha_k}$, $k^\alpha=\prod_{k=1}^{j}k^{\alpha_k}$

Remark. The above two explicit formulae are known as Waring formulae.

(B) *Basic Weyl invariants of simple Lie groups.*

Let G be a compact connected Lie group, T be a maximal torus of G and $N(T)$ be the normalizer of T in G. Then the Weyl group $W(G)=N(T)/T$ acts on T via conjugations and hence also acts on $H^*(B_T)$ via induced automorphisms. It is not difficult to show that $G/N(T)$ is Q-acyclic and $H^*(B_{N(T)};Q)\cong H^*(B_T;Q)^W$, the fixed elements of $H^*(B_T;Q)$ under the action of $W(G)$. Therefore, one has

$$H^*(B_G;Q)\cong H^*(B_{N(T)};Q)=H^*(B_T;Q)^W.$$

It is easy to see that $H^*(B_T;Q)^W$ can be canonically identified with the ring of invariant polynomials over Q of the Cartan subalgebra with respect to the action of W which acts as a finite group generated by reflections. It follows from a theorem of Coxeter and Chevalley [C1] that $H^*(B_T;Q)^W$ is then itself also a polynomial algebra generated r basic invariant polynomials, $r = rk(G)$. We shall such a set of basic invariant polynomials a set of *basic Weyl invariants* of G. For later use, we shall exhibit a set of basic Weyl invariants for each simple compact connected Lie group in the following:

(1) $G = A_{n-1}$, the Cartan subalgebra of A_{n-1} can be parametrized by $(\theta_1, \theta_2, \ldots, \theta_n)$ such that $\Sigma\, \theta_i = 0$ and $W(G)$ acts as the full permutation group of $\{\theta_i\}$. Hence a set of basic Weyl invariants of A_{n-1} are simply the elementary polynomials of θ_i, namely, $\sigma_2, \sigma_3, \ldots, \sigma_n$ (notice that $\sigma_1 = \Sigma\theta_i = 0$).

(2) $G = B_n$ or C_n: the Cartan subalgebra of B_n (or C_n) can be parametrized by $(\theta_1, \theta_2, \ldots, \theta_n)$ so that $W(G)$ acts as permutations of $\{\theta_i\}$ modulo an arbitrary number of changes of signs. Hence, a set of basic Weyl invariants of B_n (or C_n) are those elementary symmetric polynomials of $\{\theta_i^2\}$, say, denoted by $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n$.

(3) $G = D_n$: the Cartan subalgebra of D_n can be parametrized by $(\theta_1, \theta_2, \ldots, \theta_n)$ such that $W(G)$ acts as permutations of $\{\theta_i\}$ modulo an even number of changes of signs. Hence, a set of basic Weyl invariants of D_n are $\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n-1}$ and σ_n.

(4) $G = G_2$: the Cartan sualgebra of G_2 is parametrized by $(\theta_1, \theta_2, \theta_3)$ with $\theta_1 + \theta_2 + \theta_3 = 0$ and $W(G_2)$ acts as permutation of θ_i modulo a uniform change of signs of all θ_i. Therefore, σ_2 and σ_3^2 consists of a set os basic Weyl invariants of G_2.

(5) $G = F_4$: the weight system of the first basic representation ψ_1 of F_4, dim $\psi_1 = 26$, is given as follows:

$$\Omega(\psi_1) = \left\{\pm\, \theta_i;\ \frac{1}{2}(\pm\, \theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4);\ 2 \text{ zero weights}\right\}.$$

Let $\sigma_4(F_4) = S\psi_1^k = \sum_{w \in \Omega(\psi_1)} w^k$. Then $\sigma_2(F_4)$, $\sigma_6(F_4)$, $\sigma_8(F_4)$ and $\sigma_{12}(F_4)$ consists of a set of basic Weyl invariants of F_4.

(6) $G = E_6$: the weight system of the first basic representation ψ_1 of E_6 is given as follows:

$$\Omega(\psi_1) = \{\pm\, \lambda + \theta_i;\ -(\theta_i + \theta_j);\ 1 \le i < j \le 6\}, \dim \psi_1 = 27.$$

Again, let $\sigma_k(E_6) = S\psi_1^k = \Sigma\{w^k; w \in \Omega(\psi_1)\}$. Then

$$\{\sigma_k(E_6); k = 2,5,6,8,9,12\}$$

forms a basic system of Weyl invariants of E_6.

(7) $G = E_7$: the weight system of the first basic representation of E_7 is given as follows:

$$\Omega(\psi_1) = \{\pm\, (\theta_i + \theta_j); 1 \le i < j \le 8\}, \dim \psi_1 = 56$$

Let $\sigma_k(E_7) = \Sigma\{w^k; w \in \Omega(\psi_1)$. Then

$$\{\sigma_k(E_7); k = 2, 6, 8, 10, 12, 14, 18\}$$

forms a system of basic Weyl invariants of E_7.

(8) $G = E_8$: the first basic representation of E_8 is its adjoint representation and the root system of E_8 is as follows:

$$\Delta(E_8) = \{\pm\, (\theta_i - \theta_j); \theta_i + \theta_j + \theta_k; 1 \le i < j < k \le 9\}.$$

Let $SE_8^k = \Sigma\,\{\alpha^k; \alpha \in \Delta^+(E_8)$. Then

$$\{SE_8^k;\ k = 2,8,12,14,18,20,24,30\}$$

forms a system of basic Weyl invariants of E_8. Here, it is interesting to notice that the last seven exponents are exactly those $p + 1$ for the consecutive primes between 7 and 29 and

$$SE_8^{p+1} \equiv 30\, s_{p+1}(\theta_i) \pmod{p}.$$

In fact, one may use the above equations to give a simple proof of the above assertion that $\{SE_8^k; k = 2,\ p+1, 7 \le p \le 29\}$ forms a system of basic Weyl invariants of E_8.

(C) *Symmetric products and symmetric sums.*

In view of reduction 1, 4 and 5 of Section 1, the following symmetric products are of basic importance in the study of Pontrjagin classes of compact homogeneous spaces, namely,

$$\prod_{w \in \Omega(\psi)} (1 + w) = 1 + P\psi^1 + P\psi^2 + \ldots + P\psi^k + \ldots$$

especially when $\psi = Ad_H$ is the adjoint representation of H. We shall denote PAd_H^k simply PH^k, namely,

$$\prod_{\alpha \in \Delta(H)} (1 + \alpha) = \prod_{\alpha \in \Delta^+(H)} (1 - \alpha^2) = 1 - PH^2 + PH^4 + \ldots (-1)^k \cdot PH^{2k} + \ldots$$

i.e., $PH^{2k-1}=0$. One shall often need the explicit expression of PH^{2k} in terms of a chosen system of basic Weyl invariants of H. However, such computations are rather difficult to do directly. We shall first compute the corresponding symmetric sums $\{S\psi^k\}$ and $\{SH^k\}$ and then make use of the explicit Newton's Formula to compute $\{P\psi^k\}$ and $\{PH^k\}$ indirectly.

Lemma 2.1. *Let* $\{\theta_i; 1\le i\le n\}$ *be a set of* n *indeterminants and* $s_k=\Sigma\theta_i^k$. *Then*

$$\sum_{i<j}(\theta_i+\theta_j)^k=(n-1)s_k+\frac{1}{2}\cdot\sum_{\ell=1}^{(k-1)}\binom{k}{\ell}\cdot(s_\ell\cdot s_{(k-\ell)}-s_k)=$$

$$=[n-2^{(k-1)}]\cdot s_k+\frac{1}{2}\sum_{\ell=1}^{(k-1)}\binom{k}{\ell}s_\ell\cdot s_{(k-\ell)}.$$

Lemma 2.2. *With the same notation as in the above lemma,*

$$\sum_{i<j}(\theta_i-\theta_j)^{2k}=ns_{2k}+\frac{1}{2}\sum_{\ell=1}^{(2k-1)}(-1)^\ell\binom{2k}{\ell}s_\ell\cdot s_{(2k-\ell)}.$$

Notations. Let H be a compact connected Lie group, $\Delta(H)$ be the root system of H, $\Delta^+(H)$ be the system of positive roots and

$$SH^{2k}=\Sigma\{\alpha^{2k};\ \alpha\in\Delta^+(H)\},$$

$(-1)^k\cdot PH^{2k}=$ the homogeneous part of degree $2k$ in the symmetric product $\prod_{\alpha\in\Delta(H)}(1+\alpha)=\prod_{\alpha\in\Delta^+(H)}(1-\alpha^2)$.

Then, one may use the explicit Newton's Formula to express $\{PH^{2k}\}$ in terms of $\{SH^{2j}\}$, e.g.,

$$PH^2=SH^2,$$

$$PH^4=\frac{1}{2}\{-SH^4+[SH^2]^2\},$$

$$PH^6=\frac{1}{3}\left\{SH^6-\frac{3}{2}SH^2\cdot SH^4+\frac{1}{2}[SH^2]^3\right\},\ \text{etc}$$

Since we are going to deal with the first three Pontrjagin classes of various homogeneous spaces, we shall often need the explicit forms of PH^2. PH^4 and PH^6 for all simple compact Lie groups. We state the results of such computations as follows, [with the help of Newton's formula such computations can be reduced to that of symmetric sums SH^2, SH^4 and SH^6 which are rather straightforward.]:

Lemma A.
$$-PA_{n-1}^2=2n\sigma_2,$$
$$PA_{n-1}^4=2n\sigma_4+(2n^2-n-6)\sigma_2^2,$$
$$-PA_{n-1}^6=2n\sigma_6+(4n^2-2n-40)\sigma_2\sigma_4-(n-30)\sigma_3^2+\frac{2}{3}(2n^3-3n^2-17n+30)\cdot\sigma_2^3.$$

Lemma B.
$$PB_n^2=(2n-1)\bar\sigma_1,$$
$$PB_n^4=(2n-7)\bar\sigma_2+(n-1)(2n-1)\bar\sigma_1^2,$$
$$PB_n^6=(2n-31)\bar\sigma_3+(4n^2-18n+18)\bar\sigma_1\cdot\bar\sigma_2+\frac{1}{3}(n-1)(2n-3)(2n-1)\cdot\bar\sigma_1^3.$$

Lemma C.
$$PC_n^2=2(n+1)\bar\sigma_1,$$
$$PC_n^4=(2n+8)\bar\sigma_2+(n-1)(2n+5)\bar\sigma_1^2,$$
$$PC_n^6=(2n+32)\bar\sigma_3+(4n^2+18n-36)\bar\sigma_1\cdot\bar\sigma_2+\frac{2}{3}(n-1)(2n-3)(n+4)\bar\sigma_1^3.$$

Lemma D.
$$PD_n^2=2(n-1)\bar\sigma_1,$$
$$PD_n^4=(2n-8)\bar\sigma_2+(n-1)(2n-3)\bar\sigma_1^2,$$
$$PD_n^6=(2n-32)\bar\sigma_3+(4n^2-22n+28)\bar\sigma_1\cdot\bar\sigma_2+\frac{2}{3}(n-1)(n-2)(2n-3)\bar\sigma_1^3.$$

Lemma $\mathbf{E}_6$.
$$PE_6^2=-24(\lambda^2-\sigma_2),$$
$$PE_6^4=270(\lambda^2-\sigma_2)^2,$$
$$PE_6^6=1900\cdot(\lambda^2-\sigma_2)^3-6(24\sigma_6-4\lambda_4\sigma_2+4\lambda^2\sigma_2^2-20\lambda^2\sigma_4+3\sigma_3^2-4\sigma_2\sigma_4)$$

Lemma $\mathbf{E}_7$.
$$-PE_7^2=36\sigma_2,\ PE_7^4=624\sigma_2^2,$$
$$PE_7^6=4\cdot(24\sigma_6+3\sigma_3^2-4\sigma_2\sigma_4)+217\cdot32\cdot\sigma_2^3.$$

Lemma $\mathbf{E}_8$. $PE_8^2=-60\sigma_2$

[Since we shall only need PE_8^2 for later computations, we omit PE_8^4 and PE_8^6 here.]

Lemma $\mathbf{F}_4$.
$$PF_4^2=9\bar\sigma_1,$$
$$PF_4^4=\frac{147}{4}\bar\sigma_1^2,$$
$$PF_4^6=-7\left(3\bar\sigma_3-\frac{1}{2}\bar\sigma_1\cdot\bar\sigma_2\right)+\frac{721}{8}\bar\sigma_1^3.$$

Lemma G$_2$. $PG_2^2 = -8\sigma_2, PG_2^4 = 22\sigma_2^2,$
$-PG_2^6 = 26\sigma_3^2 + 28\sigma_2^3.$

Remark. (a) The computations involved in verifying the above lemmas consists of the following three steps:

(1) Express PH^{2k} in terms of $\{SH^{2j}; 1 \le j \le k\}$ by Newton's formula.
(2) Compute SH^{2j} in terms of $\{s_\ell\}$ by means of binominal or multinomial theorems (cf. Lemma (2.1), (2.2)).
(3) Use the Newton's formula again to replace those $\{s_\ell\}$ by their expressions in terms of $\{\sigma_i\}$ of $\{\bar{\sigma}_i\}$.

(b) In the above indirect method of computation, each step is rather simple and the advantage of such indirect method becomes more clear if one *tries* to compute PH^{2k} for slightly larger k, such as $k = 7$, by direct expansion of the symmetric product. [I think he will soon find out that it is inteolerably complicated and hence appreciate the above indirect method more.]

c) For a general compact connected Lie group H, then

$$\Delta(H) = \Delta(H_1) + \Delta(H_2) + \ldots + \Delta(H_s)$$

where H_j are the simple normal factors of H. Hence

$$SH^{2k} = \sum_{j=1}^{s} SH_j^{2k}$$

and it is easy to reduce the computation of PH to the case of simple Lie groups.

Section 3. Homogeneous spaces of SU(m) with vanishing characteristic classes.

Let H be a compact connected Lie group and $\psi: H \to SU(m)$ be an almost faithful unimodule complex representation of H. In this section we shall investigate the possibilities of such pairs (H, ψ) whose associated homogeneous spaces $SU(m)/\psi H$ satisfy various vanishing conditions on their characteristic classes. In particular, we shall prove Theorem 1 as stated in the introduction. First, let us prove some simple but basic lemmas:

Lemma 3.1. *Suppose $SU(m)/\psi H$ satisfies the following vanishing conditions, namely*

$$P_k(SU(m)/\psi H) = 0 \text{ for } j = 1, 2, \ldots, \ell.$$

Then the symmetric products of the roots of H and the weights of ψ satisfy the following systems os algebraic equations:

$$PH^2 \equiv 0 \pmod{P\psi^2},$$
$$PH^4 \equiv 0 \pmod{P\psi^2, P\psi^3, P\psi^4},$$
$$\text{-------------------------------------},$$
$$PH^{2\ell} \equiv 0 \pmod{P\psi^2, P\psi^3, \ldots, P\psi^{2\ell}}.$$

Proof. Let T be a maximal torus of H and $\pi: SU(m)/\psi T \to SU(m)/\psi H$. Then it follows from the duality of Pontrjagin classes (*) and reduction 1, reduction 4 that

$$P_k(\alpha(Ad_H | T)) = i^*(PH^{2k}) = 0 \text{ for } k = 1,2, \ldots, \ell$$

On the other hand, it follows from reduction 5A that

$$Ker(i^*) = \langle P\psi^2, P\psi^3, \ldots, P\psi^m \rangle.$$

Hence, $i^*(PH^{2k}) = 0$ if and only if

$$PH^{2k} \equiv 0 \pmod{P\psi^2, P\psi^3, \ldots, p\psi^{2k}}.$$

Lemma 3.2. *Let $H \subset SU(m)$ be a compact connected subgroup of $SU(m)$. If $P_1(SU(m)/H = 0$, then H is either abelian or semi-simple.*

Proof. Suppose the contrary that H is neither abelian nor semi-simple, namely, the Lie algebra of H is the sum of a non-trivial center and a non-trivial semi-simple part. Let $\tilde{H}$ be a finite covering group of H such that $\tilde{H} = H_0 \times H_1 \times \ldots \times H_k$ is the direct product of a torus group H_0 and k simple compact Lie groups $\{H_i; 1 \le i \le k\}$. Let $\psi: \tilde{H} \to H \subseteq SU(m)$, and

$$H^4(B_{\tilde{H}}) = H^4(B_{H_0}) \oplus H^4(B_{H_1}) \oplus \ldots \oplus H^4(B_{H_k}).$$

Since $\psi: \tilde{H} \to SU(m)$ is almost faithful, it is not difficult to see that $P\psi^2 \in H^4(B_{\tilde{H}})$ has non-zero component in $H^4(B_{H_0})$. On the other hand, it is obvious that the component of $PAd_{\tilde{H}}^2$ in $H^4(B_{H_0})$ is zero. Therefore

$$PAd_{\tilde{H}}^2 \equiv 0 \pmod{P\psi^2}$$

and it follows from Lemma (3.1) that $P_1(SU(m)/H) \neq 0$, which is a contradiction to the assumption. Hence, H is either abelian or semi-simple.

Remark. Proposition 1.1 asserts that G/H is stably parallelizable if H is abelian. Therefere, we shall from now concentrate in the case that H

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*P_{4j} together with $P_{4j+2} = W_{2j+1}^2$ satisfy duality formula. Since $H^*(B_T; \mathbb{Z})$ is torsion free, $W_{2j+1} = 0$ at $H^*(B_T; \mathbb{Z})$ level.

is *semi-simple*. Suppose H_1 is a connected normal subgroup of H and $p: SU(m)/H_1 \to SU(m)/H$. Then it follows from reduction 2 that

$$P_k(SU(m)/H_1) = p^* P_k(SU(m)/H),$$

and hence $SU(m)/H_1$ will satisfy the same kind of vanishing conditions as that of $SU(m)/H$. In view of the above fact, it is rather natural to first investigate the basic cases that H is *simple*.

For the study of first Pontrjagin class of homogeneous spaces, it is convenient to introduce the following definition:

Definition. Let H be a *semi-simple* compact connected Lie group and ψ be a representation of H with weight system $\Omega(\psi)$. Then the total length of ψ, denoted by $L(\psi)$, is defined to be: $L(\psi) = \Sigma\{|w|^2; w \in \Omega(\psi)\}$, where $|w|$ is the length of the weight vector w with respect to the usual Cartan-Killing inner product.

Lemma 3.3. *If H is a simple, compact, connected Lie group and $\psi: H \to SU(m)$ is a complex representation $P_1(SU(m)/\psi H) = 0$, then $L(\psi)$ divides $L(Ad_H)$.*

Proof. In the case that H is a simple, compact, connected Lie group, it is well-known that $H^4(B_T; \mathbb{Z})^W$ is an infinite cyclic group generated by the only basic Weyl invariant of formal degree 2 (dim 4) which is essentially the "*sum of squares*". Therefore the condition PH^2 is divisible by $P\psi^2$ is equivalent to the condition that $L(Ad_H)$ is divisible by $L(\psi)$. Hence Lemma (3.3) follows from Lemma 3.1.

Remarks. (1) It is not difficult to extend the above lemma to the general case that H is semi-simple. Suppose

$$H = H_1 \times \ldots \times H_k, H_i \text{ simple.}$$

Let $\psi_i = \psi | H_i$. Then $P_1(SU(m)/\psi H) = 0$ implies that

$$\frac{L(AdH_1)}{L(\psi_1)} = \frac{L(AdH_2)}{L(\psi_2)} = \cdots = \frac{L(AdH_k)}{L(\psi_k)}.$$

The above assertion follows from the following fact:

$$H^4(B_H; \mathbb{Z}) = \oplus \sum_{i=1}^{k} H^4(B_{H_i}; \mathbb{Z}) = \oplus \sum_{i=1}^{k} H^4(B_{T_i}; \mathbb{Z})^{W_i}$$

where $T_i \subset H_i$ are maximal tori of H_i and W_i are the Weyl groups of H_i respectively.

(i) Observe that, for a given simple Lie group H, $L(AdH)$ is rather small as compared with $L(\psi)$ for most representations ψ of H. In particular, there are only few representations of H (modulo trivial representations) satisfying the inequality $L(\psi) \leq L(AdH)$. Therefore, the above simple lemma already reduces the classification problem of $P_k(SU(m)/\psi H) = 0$, $k = 1,2,3$ to a manageable finite case.

(ii) Reduction 1 plays an important role in realizing as well as in proving the above lemma. If one uses Borel-Hirzebruch setting without the modification of reduction 1, then one would investigate the divisibility of $L(Ad_{SU(m)} | H - Ad_H)$ by $L(\psi)$ which is extremely complicated to compute or discuss with directly.

Next, let us investigate homogeneous spaces $SU(m)/H$ with simple subgroup H according to the classification of simple Lie groups:

(A) *The case* $H = A_{n-1}$.

It is not difficult to show that the following list exhausts all those *irreducible* (complex) representations of A_{n-1} with $L(\psi) \leq L(Ad)$;

$$L(Ad) = 2n \cdot (n-1), L(\mu_n) = L(\bar{\mu}_n) = (n-1),$$
$$L(\Lambda^2\mu_n) = L(\Lambda^2\bar{\mu}_n) = (n-2)(n-1), L(S^2\mu_n) = L(S^2\bar{\mu}_n) = (n+2)(n-1),$$
$$L(\Lambda^3\mu_n) = L(\Lambda^3\bar{\mu}_n) = \frac{1}{2}(n-2)(n-3)\cdot(n-1), \ 6 \leq n \leq 8,$$

where μ_n is the standard representation of $SU(n)$ on C^n. As a direct consequence of the above list and Lemma (3.3), one has the following proposition:

Proposition 3.1A. *If $P_1(SU(m)/\psi A_{n-1}) = 0$, then the possibilities of ψ are given by the following list modulo trivial representations and conjugations:*

(i) Ad,

(ii) $k \cdot \mu_n + \ell\bar{\mu}_n$ *with* $(k+\ell)/2n$,

(iii) $S^2\mu_n + \Lambda^2\mu_n$, *or* $S^2\mu_n + \Lambda^2\bar{\mu}_n$, *or* $S^2\mu_n + k\mu_n + \ell\bar{\mu}_n$ *with* $(k+\ell) = (n-2)$,

(iv) $\Lambda^2\mu_n + k\mu_n + \ell\bar{\mu}_n$ *with* $(k+\ell) = 2$ *or* $(n+2)$,
$2\Lambda^2\mu_n + k\mu_n + \ell\bar{\mu}_n$ *with* $(k+\ell) = 4$,
$\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n + k\mu_n + \ell\bar{\mu}_n$ *with* $(k+\ell) = 4$,

(v)
$$\begin{cases} n = 6: \Lambda^3\mu_6, 2\Lambda^3\mu_6, \Lambda^3\mu_6 + k\mu_6 + \ell\bar{\mu}_6, k+\ell = 6 \\ \quad\quad \Lambda^3\mu_6 + \Lambda^2\mu_6 + k\mu_6 + \ell\bar{\mu}_6 \text{ with } k+\ell = 2, \\ n = 7: \Lambda^3\mu_7 + k\mu_7 + \ell\bar{\mu}_7 \text{ with } k+\ell = 4, \\ n = 8: \Lambda^3\mu_8 + \mu_8 \text{ or } \Lambda^3\mu_8 + \bar{\mu}_8, \end{cases}$$

(7) Similar computation will show that $P_2(SU(m)/\psi A_{n-1}) \neq 0$ for the remaining cases of ψ listed in (vi) of Proposition 3.1A, namely $4\Delta^2\mu_4$, $3\Lambda^2\mu_4 + k\mu_4 + \ell\bar{\mu}_4$, $k + \ell = 2$ and $j\Lambda^2\mu_5 + k\Lambda^2\bar{\mu}_5 + \mu_5$ with $j + k = 3$. All of the above computations complete the proof of Proposition 3.2A.

Now, let us proceed to determine which of those representations listed in Proposition (3.2A) have vanishing third Pontrjagin class for $SU(m)/\psi A_{n-1}$. We shall prove the following proposition:

Proposition 3.3A. *If $P_k(SU(m)/\psi A_{n-1}) = 0$ for $k = 1,2,3$ then the possibilities of such complex representations ψ of A_{n-1} are given by the following list modulo trivial representations and conjugation:*

(i) Ad,
(ii) μ_n *or* $2\mu_n$ *or* $4\mu_2$
(iii) $\mu_n + \bar{\mu}_n$ *for the special cases* $n = 3,4,5$, *or* $k\mu_3 + \ell\bar{\mu}_3$ *with* $(k + \ell) = 3$ *or* 6, *but* $(k, \ell) \neq (3,3)$.
(iv) $\mu_n \otimes \mu_n$ *with* $n/30$,
(v) $\Lambda^2\mu_4 = p_6$.

Proof. (1) Let us first consider the case $\psi = \mathrm{k}\mu_n + \ell\bar{\mu}_n$ with $k + \ell = 3$ or 6 and $n \equiv 0 \pmod 3$, $n \geq 6$. In this case, the coefficients of σ_2, σ_4 and $\sigma_2 \cdot \sigma_4$ in $P\psi^2, P\psi^4$ and $P\psi^6$ respectively are all divisible by 3, but the coefficient of $\sigma_2 \cdot \sigma_4$ in PA^6_{n-1} is

$$-(4n^2 - 2n - 40) \equiv 40 \not\equiv 0 \pmod 3 \text{ for all } n \equiv 0 \pmod 3.$$

Hence $PA^6_{n-1} \not\equiv 0 \pmod{P\psi^2, P\psi^3, \dots, P\psi^6}$ and $P_3(SU(m)/\psi A_{n-1}) \neq 0$.

(2) Next let us consider the case $\psi = \mu_n + \bar{\mu}_n, n \geq 6$. In this case $P\psi^2 = 2\sigma_2$, $P\psi^3 = 0$, $P\psi^4 = 2\sigma_4 + \sigma_2^2$, $P\psi^6 = 2\sigma_6 - \sigma_3^2 + 2\sigma_2\sigma_4$. Therefore, $PA^6_{n-1} = 2n\sigma_6 - (n - 30)\sigma_3^2 + \dots \not\equiv 0 \pmod{P\psi^2, P\psi^3, \dots, P\psi^6}$.
Hence, again $P_3(SU(m)/\psi A_{n-1}) \neq 0$ for the case $\psi = \mu_n + \bar{\mu}_n$, $n \geq 6$.

(3) $\psi = \mu_n \otimes \mu_n = S^2\mu_n + \Lambda^2\mu_n$. Straightforward computation will show that $S\psi^2 = 2ns_2$, $S\psi^3 = 2ns_3$, $S\psi^4 = 2ns_4 + 6s_2^2$, $S\psi^6 = 2ns_6 + 30s_2s_4 + 20s_3^2$ and therefore, by Newton's formula, $P\psi^2 = 2n\sigma_2$, $P\psi^3 = 2n\sigma_3$, $P\psi^4 = 2n\sigma_4 + (2n^2 - n - 6)\sigma_2^2$ and $P\psi^6 = 2n\sigma_6 + (2n^2 - n - 30)\sigma_3^2 + (4n^2 - 2n - 40)\sigma_2\sigma_4 + \frac{2}{3}(2n^3 - 3n^2 - 17n + 30)\sigma_2^3$. Hence $PA^6_{n-1} \equiv 0$ $(\mathrm{mod}\ P\psi^2, P\psi^3, P\psi^4, P\psi^6)$ implies that $60 \equiv (\mathrm{mod}\ 2n) \Rightarrow n/30$.

(3′) $\psi = S^2{}_n + \Lambda^2\bar{\mu}_n$. Similar computation will show that $P\psi^2 = 2n\sigma_2$, $P\psi^3 = 8\sigma_3$, $P\psi^4 = 2n\sigma_4 + (2n^2 - n - 6)\sigma_2^2$ and $P\psi^6 = 2n\sigma_6 - (n - 2)\sigma_3^2 + (4n^2 - 2n - 40)\sigma_2\sigma_4 + \frac{2}{3}(2n^3 - 3n^2 - 17n + 30)\sigma_2^3$. Since $28 \not\equiv 0$ (mod

8), it is not difficult to see that $PA^6_{n-1} \not\equiv 0 \pmod{P\psi^2, P\psi^3, P\psi^4, P\psi^6}$ for this case.

(4) Similar computations will also show that $P_3(SU(m)/\psi A_{n-1}) \neq 0$ for the following $\psi: \Lambda^3\mu_6$; $\Lambda^3\mu_6 + \Lambda^2\mu_6 + k\mu_6 + \ell\mu_6$, $k + \ell = 2$; and $\Lambda^2\mu_6 + k\mu_6 + \ell\bar{\mu}_6, k + \ell = 8$.

(B) *The case* $H = B_n$, $n \geq 3$.

The following are those complex irreducible representations ψ: $B_n \to SU(m)$ with $L(\psi) \leq L(Ad_{B_n})$:

$$L(Ad) = 2n \cdot (2n - 1),$$
$$L(\rho) = 2n,$$
$$L(\Delta) = 2^{(n-2)} \cdot n \leq L(Ad) \text{ for } 3 \leq n \leq 6,$$

where ρ is the first basic representation of dimension $(2n + 1)$ and Δ is the spinor representation of dimension 2^n.

As a direct consequence of the above list and Lemma 3.3, one has the following proposition:

Proposition 3.1B. *If $P_1(SU(m)/\psi B_n) = 0$, then the possibilities of such ψ are given in the following list modulo trivial representations*:

(i) Ad,
(ii) $k\rho$ *with* $k/(2n - 1)$ *and the following extra cases for* $3 \leq n \leq 6$:
(iii)
$$\begin{cases} n = 6: \ \Delta + 3\rho, \\ n = 5: \ \Delta + 5\rho \text{ or } 2\Delta + \rho, \\ n = 4: \ k\Delta + \ell\rho \text{ with } (2k + \ell) = 7, \\ n = 3: \ k\Delta + \ell\rho \text{ with } (k + \ell)/5. \end{cases}$$

Next let us study which ψ among the above list also satisfy the condition $\pi^*P_2(SU(m)/\psi B_n) = 0$. We shall prove the following proposition:

Proposition 3.2B. *If $P_k(SU(m)/\psi B_n) = 0$ for $k = 1,2$, then the possibilities of such representations $\psi: B_n \to SU(m)$ are given by the following simple list modulo trivial ones*:

(i) $Ad = \Lambda^2\rho$, (ii) $k\rho$ *with* $k/(2n - 1, 3)$.

Proof. Let us first consider the case $\psi = k\rho$. Then,

$$P\psi^2 = k\bar{\sigma}_1, P\psi^4 = k\bar{\sigma}_2 + \binom{k}{2}\bar{\sigma}^2$$

and hence $PB_n^2 = (2n - 1)\sigma_1 \equiv 0 \pmod{P\psi^2}$ and

$$PB_n^4 = (2n-7)\bar{\sigma}_2 + (n-1)(2n-1)\bar{\sigma}_1^2 \equiv 0 \pmod{P\psi^2, P\psi^4})$$

imply that $k/(2n-1,3)$, the g.c.d. of $(2n-1)$ and 3.

Next let us consider those cases listed in (iii) of Proposition 3.1B case by case:

(1) $n=6$, $\psi = \Delta + 3\rho$. In this case one has

$$S\psi^2 = 22\bar{\sigma}_1,\ S\psi^4 = 4\bar{\sigma}_2 + 10\bar{\sigma}_1^2,\ P\psi^2 = 11\bar{\sigma}_1,\ P\psi^4 = -\bar{\sigma}_2 + 58\bar{\sigma}_1^2.$$

Therefore, it is easy to see that $P_2(SU(m)/\psi B_6) \neq 0$.

(2) $n=5$, $\psi = \Delta + 5\rho$. In this case, one has

$$S\psi^2 = 18\bar{\sigma}_1,\ S\psi^4 = -12\bar{\sigma}_2 + 12\bar{\sigma}_1^2,\ P\psi^2 = 9\bar{\sigma}_1,\ P\psi^4 = 3\bar{\sigma}_2 + \frac{75}{2}\bar{\sigma}_1^2.$$

Hence $PB_5^4 = 3\bar{\sigma}_2 + 4.9\bar{\sigma}_1^2 \not\equiv 0 \pmod{P\psi^2, P\psi^4}$ and $P_2(SU(m)/\psi B_5) \neq 0$. Similar computation will also show that $P_2(SU(m)/\psi B_5) \neq 0$ for $\psi = 2\Delta + \rho$.

(3) $n=4$, $\delta\psi = k\Delta + \ell\rho$ with $2k + \ell = 7$. Similar computation will show that $P\psi^2 = 7\bar{\sigma}_1$ and $P\psi^4 = (\ell - k)\bar{\sigma}_2 + \frac{1}{4}(98 - k - 2\ell)\bar{\sigma}_1^2$. Then, it is not difficult to check that in all the above cases, $P^2(SU(m)/\psi B_4) \neq 0$.

(4) $n=3$, $\psi = k\Delta + \ell\rho$ with $k + \ell = 5$ or $\psi = \Delta$. Similar computation will show that $P\psi^2 = 10\bar{\sigma}_1$ or $2\bar{\sigma}_1$ and

$$P\psi^4 = \left(\ell - \frac{k}{2}\right)\bar{\sigma}_2 + \frac{1}{8}(100 - 4\ell - k)\bar{\sigma}_1^2 \text{ or } \left(-\frac{1}{2}\bar{\sigma}_2 + \frac{3}{8}\bar{\sigma}_1^2\right).$$

In all the above cases, it is not difficult to show that $P_2(SU(m)/\psi B_3) \neq 0$. This completes the proof Proposition 3.2B.

Proposition 3.3B. *If $P_k(SU(m)/\psi B_n) = 0$ for $k = 1,2,3$ then either $\psi = Ad$ or $\psi = \rho$ modulo trivial representations.*

Proof. We need only to show that

$$P_3(SU(m)/\psi B_n) \neq 0 \text{ for } \psi = 3\rho,\ (2n-1) \equiv 0 \pmod 3.$$

Since $P\psi^2 = 3\bar{\sigma}_1$, $P\psi^4 = 3\bar{\sigma}_2 + 3\bar{\sigma}_1^2$, $P\psi^6 = 3\bar{\sigma}_3 + 6\bar{\sigma}_1\bar{\sigma}_2 + \bar{\sigma}_1^3$, one has

$$PB_n^6 = (2n-31)\bar{\sigma}_3 + 2(n-3)(2n-3)\bar{\sigma}_1\bar{\sigma}_2 + \frac{1}{3}(n-1)(2n-3)(2n-1)\bar{\sigma}_1^3$$

$$\equiv \bar{\sigma}_1\bar{\sigma}_2 + \frac{1}{3}(n-1)(n-2)(2n-1)\bar{\sigma}_1^3 \not\equiv 0 \pmod{3, P\psi^2, P\psi^4, P\psi^6})$$

Hence, $P_3(SU(m)/3\rho B_n) \neq 0$.

(C) *The case* $H = C_n$, $n \geq 2$.

The following are those irreducible complex representations

$$\psi: C_n \to SU(m) \text{ with } L(\psi) \leq L(Ad_{C_n}) = 4(n+1)n.$$

$$\begin{cases} L(Ad) = 4(n+1)n \\ L(v_n) = 2n \\ L(\Lambda^2 v_n) = 4(n-1)n \\ n = 3,\ L(\Lambda^3 v_3 - v_3) = 30 = 10 \cdot n \end{cases}$$

where v_n is the standard 2n-dimensional representation of Sp(n). Hence it follows from Lemma 3.3 that

Proposition 3.1C. *If $P_1(SU(m)/\psi C_n) = 0$, then the possibilities of such representation ψ are given as follows modulo trivial representations:*

(i) *Ad*, (ii) *kv with $k/2(n+1)$*, (iii) *$\Lambda^2 v + 4v$ and*

(iv) $\begin{cases} n = 3,\ \Lambda^3 v_3 + 2v_3, \Lambda^2 v_3, 2\Lambda^2 v_3 \\ n = 2\text{: } k \cdot v_2 + \ell \cdot \Lambda^2 v_2 \text{ with } (k + 2\ell)/6. \end{cases}$

Next, let us investigate which ψ among the above list also have vanishing 2nd Pontrjagin class for $SU(m)/\psi C_n$. We shall prove that following result:

Proposition 3.2C. *If $P_k(SU(m)/\psi C_n) = 0$ for $k = 1,2$, then ψ is one of the following representations modulo trivial ones:*

(i) *Ad*, (ii) *kv with $k/(3, n+1)$*

(iii) $\begin{cases} n = 3\text{: } \Lambda^2 v_3 \\ n = 2\text{: } \Lambda^2 v_2 = \rho_5, 3\Lambda^2 v_2 = 3\rho_5, v_2 + \Lambda^2 v_2. \end{cases}$

Proof. (1) First, let us consider the case $\psi = kv$, $k/2(n+1)$.

$$P\psi^2 = k\bar{\sigma}_1,\ P\psi^4 = k\bar{\sigma}_2 + \binom{k}{2}\bar{\sigma}_1^2.$$

Therefore, $PC_n^4 = 2(n+4)\bar{\sigma}_2 + (n-1)(2n+5)\bar{\sigma}_1^2 \pmod{P\psi^2, P\psi^4}$ implies that

$$2(n+1) \equiv 6 \equiv (n-1)(2n+5) - (n+4)(k-1) \equiv 4n + 1 \equiv 0 \pmod k$$

which is equivalent to $k/(n+1, 3)$.

(2) $\psi = \Lambda^2 v + 4v$. $P\psi^2 = 2(n+1)\bar{\sigma}_1$, $P\psi^4 = (2n-4)\bar{\sigma}_2 + (2n+1)(n+1)\bar{\sigma}_1^2$. Therefore $PC_n^4 = (2n+8)\bar{\sigma}_2 + (n-1)((2n+5)\bar{\sigma}_1^2 \pmod{P\psi^2, P\psi^4}$ implies that $(n-2)/6$ and $\frac{n+4}{n-2} \cdot (2n+1)(n+1) - (n-1)(2n+5) \equiv 0 \pmod{2(n+1)}$.

There are the following 4 cases of $n = 8,5,4,3$ to consider and it is not difficult to see that none of them satisfy the above equation. Hence

$$P_2(SU(m)/\psi C_n) \neq 0 \text{ for } \psi = \Lambda^2 v + 4v.$$

(3) $\Lambda^3 v_3 + 2v_3 \cdot S\psi^2 = 16\bar{\sigma}_1, S\psi^4 = 16\bar{\sigma}_2 + 16\bar{\sigma}_1^2;$

$P\psi^2 = 8\bar{\sigma}_1, P\psi^4 = -4\bar{\sigma}_2 + 28\bar{\sigma}_1^2$. Hence $P_2(SU(m)/\psi C_3) \neq 0$.

(3′) $\psi = 2\Lambda^2 v_3$. Similar computation will show that

$$P\psi^2 = 8\bar{\sigma}_1, P\psi^4 = -4\bar{\sigma}_2 + 28\bar{\sigma}_1^2$$

and hence again $P_2(SU(m)/\psi C_3) \neq 0$ in this case.

(4) $n = 2, \psi = kv_2 + \ell\Lambda^2 v_2, (k + 2\ell)/6$.

$$S\psi^2 = (2k + 4\ell)\bar{\sigma}_1, S\psi^4 = 2k(-2\bar{\sigma}_2 + \bar{\sigma}_1^2) + 4\ell(4\bar{\sigma}_2 + \bar{\sigma}_1^2)$$

$$P\psi^2 = (k + 2\ell)\bar{\sigma}_1, P\psi^4 = (k - 4\ell)\bar{\sigma}_2 + \binom{k+2\ell}{2}\bar{\sigma}_1^2.$$

From the above result, it is not difficult to check that

$$P_2(SU(m)/\psi C_2) \neq 0 \text{ for } \psi = 2v + 2\Lambda^2 v \text{ or } 4v + \Lambda^2 v.$$

This completes the proof of Proposition 3.2C.

Proposition 3.3C. *If $P_k(SU(m)/\psi C_n) = 0$ for $k = 1,2,3$, then ψ is one of the following representations modulo trivial ones:*

(i) *Ad*, (ii) *v, or the special cases of*
(iii) $n = 2$: $\Lambda^2 v_2 = \rho_5$, $v_2 + \Lambda v_2$.

Proof. (1) $\psi = 3v$, $(n + 1) \equiv 0 \pmod 3$.

$$P\psi^2 = 3\bar{\sigma}_1, P\psi^4 = 3\bar{\sigma}_2 + 3\bar{\sigma}_1^2, P\psi^6 = 3\bar{\sigma}_3 + 6\bar{\sigma}_1 \cdot \bar{\sigma}_2 + \bar{\sigma}_1^3$$

Since $PC_n^6 \equiv \bar{\sigma}_1 \cdot \bar{\sigma}_2 \pmod{3, \bar{\sigma}_1^3}$ but $P\psi^2$, $P\psi^4$, $P\psi 6 \equiv 0 \pmod{3, \bar{\sigma}_1^3}$, therefore $P_3(SU(m)/\psi C_n) \neq 0$ for this case.

(2) $\psi = \Lambda^2 v^3$. $P\psi^2 = 4\bar{\sigma}_1$, $P\psi^4 = -2\bar{\sigma}_2 + 6\bar{\sigma}_1^2$, $P\psi^6 = -26\bar{\sigma}_3 - 2\bar{\sigma}_1\bar{\sigma}_2 + 4\bar{\sigma}_1^3$. Hence $PC_3^6 = 2.19\bar{\sigma}_3 + 54\bar{\sigma}_1 \cdot \bar{\sigma}_2 + 28\bar{\sigma}_1^3 \not\equiv 0 \pmod{P\psi^2, P\psi^4, P\psi^6}$ and

$$P_3(SU(m)/\psi C_3) \neq 0 \text{ for } \psi = \Lambda^2 v_3.$$

Similar computation will show that $P_3(SU(m)/\psi C_2) \neq 0$ for the case $\psi = 3\Lambda^2 v_2$.

(D) *The case $H = D_n$, $n \geq 4$.*

The following are those irreducible complex representations $\psi: D_n \to SU(m)$ with $L(\psi) \leq L(Ad^D{}_n) = 4(n - 1) \cdot n$.

$$L(Ad) = 4(n-1)n$$
$$L(\rho) = 2n$$
$$L(\Delta^{\pm}) = 2^{n-3} \cdot n \leq 4(n-1) \cdot n \text{ for } 4 \leq n \leq 7.$$

Therefore, the following proposition is a direct consequence of the above computation and Lemma 3.3.

Proposition 3.1D. *If $P_1(SU(m)/\psi D_n) = 0$, $n \leq 4$, then the possibilities of such complex representations ψ are as follows:*

(i) *Ad*, (ii) *$k\rho$ with $k/2(n-1)$ and*

$$\begin{cases} n = 7: \Delta_7^{\pm} + 4\rho \\ n = 6: k_1\Delta^+ + k_2\Delta^- + \ell\rho,\ 4(k_1 + k_2) + \ell/10 \\ n = 5: k_1\Delta^+ + k_2\Delta^- + \ell\rho,\ 2(k_1 + k_2) + \ell/8 \\ n = 4: k_1\Delta^+ + k_2\Delta^- + \ell\rho,\ k_1 + k_2 + \ell/6. \end{cases}$$

Next, let us study the 2nd Pontrjagin class of D_n case.

Proposition 3.2D. *If $P_k(SU(m)/\psi D_n) = 0$ for $k = 1,2$ and $n \geq 4$, then the possibilities of such complex representations ψ are as follows:*

(i) *Ad*, (ii) *$k\rho$ with $k/(n - 1,3)$ and the special cases*
(iii) *$k_1\Delta^+ + k_1\Delta^- + \ell\rho$ with $(k_1 + k_2 + \ell)/6$.*

Proof. (1) First let us consider $\psi = k\rho$, $k/(2n - 1)$.

$P\psi^2 = k\bar{\sigma}_1, P\psi^4 = k\bar{\sigma}_2 + \binom{k}{2}\bar{\sigma}_1^2$. Hence

$PD_n^4 = (2n - 8)\bar{\sigma}_2 + (n - 1)(2n - 3)\bar{\sigma}_1^2 \equiv 0 \pmod{k\bar{\sigma}_1, (k\bar{\sigma}_2 + \binom{k}{2}\bar{\sigma}_1^2)}$

implies that $6 \equiv 0$ and $-3 \equiv 0 \pmod k$, i.e., $k/(n - 1, 3)$.

(2) $\psi = \Delta_7^{\pm} + 4\rho$. In this case, $S\psi^2 = 24\bar{\sigma}_1$, $S\psi^4 = 12\bar{\sigma}_1^2$ and hence, it is easy to see that $PD_7^4 \not\equiv 0 \pmod{P\psi^2, P\psi^4}$. Therefore $P_2 \neq 0$.

(3) $\psi = \Delta_6^{\pm} + \rho$, or $\Delta_6^{\pm} + 6\rho$, or $\Delta_6^+ + \Delta_6^- + 2\rho$, or $2\Delta_6^{\pm} + 2\rho$. Straightforward computations will show the following results:

ψ	$S\psi^2$	$S\psi^4$	$P\psi^2$	$P\psi^4$
$\Delta_6^{\pm} + \rho$	$10\bar\sigma_1$	$4\bar\sigma_2 + 4\bar\sigma_1^2$	$5\bar\sigma_1$	$-\bar\sigma_2 + \frac{23}{2}\bar\sigma_1^2$
$\Delta_6^{\pm} + 6\rho$	$20\bar\sigma_1$	$-16\bar\sigma_2 + 14\bar\sigma_1^2$	$10\bar\sigma_1$	$4\bar\sigma_2 + \frac{93}{2}\bar\sigma_1^2$
$2\Delta_6^{\pm} + 2\rho$ $\Delta_6^{+} + \Delta_6^{-} + 2\rho$	$20\bar\sigma_1$	$8\bar\sigma_2 + 8\bar\sigma_1^2$	$10\bar\sigma_1$	$-\bar\sigma_2 + 48\bar\sigma_1^2$

From the above results on $P\psi^2$, $P\psi^4$, it is easy to check that

$$PD_6^4 = 4\bar\sigma_2 + 45\bar\sigma_1^2 \not\equiv 0 \pmod{P\psi^2, P\psi^4}$$

therefore $P_2 \neq 0$ for these cases.

(4) $\psi = k_1\Delta_5^+ + k_2\Delta_5^- + \ell\rho$, $2(k_1+k_2)+\ell/8$. Let $k = k_1 + k_2$. Then it is easy to check [assume $k > 0$.]:

$$S\psi^2 = (4k+2\ell)\bar\sigma_1,\; S\psi^4 = (k+2\ell)\bar\sigma_1^2 + 4(k-\ell)\bar\sigma_2$$

$$P\psi^2 = (2k+\ell)\bar\sigma_1, (2k+\ell)/8;\; P\psi^4 = (\ell - k)\bar\sigma_2 + \left\{\frac{1}{2}(2k+\ell)^2 - \frac{1}{4}(k+2\ell)\right\}\bar\sigma_1^2.$$

Observe that $\frac{\bar\sigma_1}{2} \in H^4(B_T;\mathbb{Z})^W$ but $\frac{\bar\sigma_1}{4} \notin H^4(B_T;\mathbb{Z})^W$ for the case of Spin(10).

It is not difficult to check that

$$P_2(SU(m)/\psi D_5 \neq 0$$

for $\psi = k_1\Delta_5^+ + k_2\Delta_5^- + \ell\rho$ with $2k_1 + 2k_2 + \ell/8$ and $k_1 + k_2 \neq 0$.

The above computations complete the proof of Proposition 3.2D.

Proposition 3.3D. *If $P_k(SU(m)/\psi D_n) = 0$ for $k = 1,2,3$ and $n \geqslant 4$, then $\psi = Ad$ or ρ or $(\Delta_4^+ + \Delta_4^-)$ for the special case of Spin (8).*

Proof. (1) First let us consider the easy case $\psi = 3\sigma$ and $(n-1) \equiv 0 \pmod 3$. In this case, $P\psi^2 = 3\sigma_1, P\psi^4 = 3\bar\sigma_2 + 3\bar\sigma_1^2$ and $P\psi^6 = 3\bar\sigma_3 + 6\bar\sigma_1\bar\sigma_2 + \bar\sigma_1^3$. Therefore

$$PD_n^6 = (2n-32)\bar\sigma_3 + 2(2n-7)(n-2)\bar\sigma_1\bar\sigma_2 + \frac{2}{3}(n-1)(n-2)(2n-3)\bar\sigma_1^3$$

$$\equiv \bar\sigma_1\bar\sigma_2 + \frac{2}{3}(n-1)(n-2)(2n-3)\bar\sigma_1^3 \not\equiv 0 \pmod{3, P\psi^2, P\psi^4, P\psi^6}.$$

Hence $P_3(SU(m)/3\rho D_n) \neq 0$.

(2) The only remaining cases are those cases of $D_4 = $ Spin (8). Spin(8) is rather special here and deserves more careful treatment. [For one thing, the coefficient of $\bar\sigma_2$ in PD_4^4 is zero.] The outer automorphism group of Spin (8) is a permutation group of three objects which permutes $\{\Delta^+, \Delta^-, \rho\}$. One may imbed Spin(8) as a subgroup of F_4 and realize the above outer automorphism group as $N(\text{Spin}(8))/\text{Spin}(8)$. Let $\alpha = \frac{1}{2}(\theta_1 - \theta_2 - \theta_3 - \theta_4)$ be one of the simple root of F_4. It is not difficult to check that

$$r_\alpha(\bar\sigma_2) = -\frac{1}{2}\bar\sigma_2 - 3\chi + \frac{3}{8}\bar\sigma_1^2$$

where r_α is the reflection of $\alpha, \chi = \theta_1\cdot\theta_2\cdot\theta_3\cdot\theta_4$, and

$$r_\alpha(\bar\sigma_3) = \bar\sigma_3 - \frac{1}{4}\bar\sigma_1\cdot\bar\sigma_2 + \frac{1}{16}\bar\sigma_1^3 - \frac{1}{2}\bar\sigma_1\cdot\chi.$$

Therefore one has the following table for the symmetric products of Δ^+ and Δ^-:

$$\prod_{w\in\Omega(\Delta^+)}(1+w) = 1 - \bar\sigma_1 + \left(-\frac{1}{2}\bar\sigma_2 - 3\chi + \frac{3}{8}\bar\sigma_1^2\right) - \left(\bar\sigma_3 - \frac{1}{4}\bar\sigma_1\bar\sigma_2 + \frac{1}{16}\bar\sigma_1^3 - \frac{1}{2}\bar\sigma_1\chi\right).$$

$$\prod_{w\in\Omega(\Delta^-)}(1+w) = -\bar\sigma_1 + -\frac{1}{2}\bar\sigma_2 + 3\chi + \frac{3}{8}\bar\sigma_1^2 - \left(\bar\sigma_3 - \frac{1}{4}\bar\sigma_1\bar\sigma_2 + \frac{1}{16}\bar\sigma_1^3 + \frac{1}{2}\bar\sigma_1\chi\right).$$

In order to complete the proof of Proposition 3.3D, we need to show that $P_3(SU(m)/\psi D_4) \neq 0$ for the following cases:

$$\Delta^+ + \Delta^- + \rho, 2\rho + \Delta^+, 5\rho + \Delta^+, 4\rho + \Delta^+ + \Delta^-,$$
$$3\rho + 2\Delta^+ + \Delta^-, 4\rho + 2\Delta^+.$$

Remark. The other cases are either conjugate to one of the above cases or already covered in (1).

Based on the above result of the symmetric products of Δ^+ and Δ^-, it is easy to compute $P\psi^2$, $P\psi^4$ and $P\psi^6$ for all the above representations. For example, in the case $\psi = \Delta^+ + \Delta^- + \rho$, $P\psi^2 = 3\bar\sigma_1$, $P\psi^4 = \frac{15}{4}\bar\sigma_1^2$,

$P\psi^6 = 3\bar{\sigma}_3 - \frac{1}{2}\bar{\sigma}_1\bar{\sigma}_2 + \frac{21}{8}\bar{\sigma}_1^3$. Then, it is rather straightforward to check that $P_3 \neq 0$ for all the above cases. For example

$$PD_4^6 = -24\bar{\sigma}_3 + 4\sigma_1\sigma_2 + 20\bar{\sigma}_1^3 \equiv 41\bar{\sigma}_1^3 \not\equiv 0 \pmod{P\psi^2, P\psi^4, P\psi^6}.$$

(E) *The exceptional cases.*

Finally, let us consider the remaining cases that H is an exceptional compact connected Lie group. Let φ be an irreducible complex representation of H with $L(\varphi) \le L(Ad_H)$. Then, it is not difficult to check that $\varphi = Ad$ or dim $\varphi = 7, 26, 27, 56$ for G_2, F_4, E_6, E_7 respectively, Therefore, the following proposition follows easily from Lemma 3.3.

Proposition 3.1E. *If H is an exceptional compact connected Lie group and $\psi: H \to SU(m)$ is a complex representation of H with $P_1(SU(m)/\psi H) = 0$, then the possibilities of such pairs (H, ψ) are as follows modulo trivial representations*:

(i) *The adjoint representation of H.*

(ii) $\begin{cases} H = G_2, k\varphi \text{ with dim } \varphi = 7, k/4 \\ H = F_4, k\varphi \text{ with dim } \varphi = 26, k = 1,3 \\ H = E_6, k\varphi + \ell\varphi^*, \text{dim } \varphi = 27, (k+\ell)/4 \\ H = E_7, k\varphi, \text{dim } \varphi = 56, k = 1,3. \end{cases}$

Next let us compute the second and third Pontrjagin classes of the above cases. We shall prove the following result.

Proposition 3.2E. *If H is an exceptional compact connected Lie group and $\psi: H \to SU(m)$ with $P_k(SU(m)/\psi H) = 0$ for $k = 1,2,3$, the possibilities of such pairs (H, ψ) are as follows*:

(i) $\psi = Ad_H$ *for all the five exceptional groups*

(ii) $H = G_2$, $\psi = \varphi$ *or* 2φ, *dim* $\varphi = 7$,

Proof. (1) $H = G_2$, $\psi = 4\varphi$, *dim* $\varphi = 7$. In this case,

$$P\psi^2 = -8\sigma_2,\ P\psi^4 = 28\sigma_2^2,\ P\psi^6 \equiv 4\sigma_3^2 \pmod{\sigma_2}$$

Hence $PG_2^6 = 26\sigma_3^2 + 28\sigma_2^3 \not\equiv 0 \mod (\sigma_2,\ P\psi^2,\ P\psi^4,\ P\psi^6)$ and $P_3(SU(m)/4\varphi \cdot G_2) \neq 0$.

(2) $H = F_4$, $\psi = \varphi$, dim $\varphi = 26$. Since $\varphi \mid \text{Spin}(8) = \Delta^+ + \Delta^- + \rho$, it is not difficult to use computation of Proposition 3.3D that

$$P\varphi^2 = 3\bar{\sigma}_1,\ P\varphi^4 = \frac{15}{4}\bar{\sigma}_1^2,\ P\psi^6 = \left(3\bar{\sigma}_1 - \frac{1}{2}\bar{\sigma}_1 \cdot \bar{\sigma}_2\right) + \frac{21}{8}\bar{\sigma}_1^3$$

Therefore

$$PF_4^6 = 7 \cdot \left(3\bar{\sigma}_3 - \frac{1}{2}\bar{\sigma}_1 \cdot \bar{\sigma}_2\right) - \frac{721}{8}\bar{\sigma}_1^3 \equiv \left(-\frac{721}{8} - 7 \cdot \frac{21}{8}\right)\bar{\sigma}_1^3 =$$

$= -\frac{217}{2}\bar{\sigma}_1^3 \not\equiv 0 \pmod{P\psi^2, P\psi^4, P\psi^6}$, hence $P_3(SU(m)/\varphi F_4) \neq 0$.

Similar computation will also show $P_3(SU(m)/3\varphi F_4) \neq 0$.

(3) $H = E_6$, $\psi = k\varphi + \ell\varphi^*$, dim $\varphi = 27$, $(k+\ell)/4$. In this case, let $I = (\lambda^2 - \sigma_2)$ and $J = (24\sigma_6 - 4\sigma_2\sigma_4 + 3\sigma_3^2 - 4\lambda^4\sigma_2 + 4\lambda^2\sigma_2^2 - 20\lambda^2\sigma_4)$.

$$P\varphi^2 = -6I,\ P\varphi^4 = 15I^2,\ P\varphi^6 = -J - 20I^3,\ P\varphi^3 = 0$$

Therefore $PE_6^6 = 6J - 1900I^3 \equiv 2020I^3 \not\equiv 0 \pmod{P\varphi^2, P\varphi^4, P\varphi^6}$. Hence $P_3(SU(m)/E_6) \neq 0$. Similar computation will also show that $P_3(SU(m)/(k\varphi + \ell\varphi^*)E_6) \neq 0$, $(k+\ell)/4$.

4) $H = E_7$, $\psi = \varphi$, dim $\varphi = 56$. In this case,

$$P\varphi^2 = -12\sigma_2,\ P\varphi^4 = 66\sigma_2^2,\ P\varphi^6 = 2(24\sigma_6 - 4\sigma_2\sigma_4 + 3\sigma_3^2) - 220\sigma_2^3.$$

Hence, $PE_7^6 = 4 \cdot (24\sigma_6 - 4\sigma_2\sigma_4 + 3\sigma_3^2) + 217 \cdot 32\sigma_2^3 \equiv 7384\sigma_2^3 \not\equiv 0 \pmod{3, P\varphi^2, P\varphi^4, P\varphi^6}$.

Therefore $P_3(SU(m)/\varphi E_7) \neq 0$, and similar computation will also show $P_3(SU(m)/3\varphi E_7) \neq 0$.

(F) *The case that H is semi-simple but non-simple.*

Based on the above detail understanding of the cases that H are simple, we shall finally proceed to prove the main theorem of this section.

Theorem 1. *Let $\psi: H \subset SU(m)$ be a compact connected Lie group with a given almost faithful complex representation ψ. If $P_k(SU(m)/\psi H) = 0$ for $k = 1,2,3$, then the possibilities of all such pairs (H, ψ) are given by the following list modulo trivial representations*:

(i) *H is any given subtorus,*

(ii) *H is semi-simple and $\psi = Ad_H$,*

(iii) $H = SU(n) \times \tilde{H}$, $n/30$ *and* $\psi = \mu_n \otimes \mu_n + Ad_{\tilde{H}}$,

$$\left\{\begin{matrix} SU(n) \\ SO(n) \\ Sp(n) \\ G_2 \end{matrix}\right., \quad \psi = \begin{cases} \mu_n \text{ or } 2\mu_n, \text{ dim } \mu_n = n \\ \rho_n \text{ dim } \rho_n = n \\ \nu_n, \text{ dim } \nu_n = 2n \\ \varphi_1 \text{ or } 2\varphi_1, \text{ dim } \varphi_1 = 7, \end{cases}$$

(v) $H = Sp(1)^\ell$, $\ell \ge 1$, $\psi = k \cdot (\nu_1^{(1)} + \nu_1^{(2)} + \ldots + \nu_1^{(\ell)})$, $k = 1,2,4$,

(vi) $H = \begin{cases} SU(3) \times SU(3) \\ G_2 \times G_2 \end{cases}, \quad \psi = \begin{cases} k(\mu_3 + \mu_3') + \ell(\bar{\mu}_3 + \bar{\mu}_3'), k + \ell = 1 \text{ or } 2, \\ (\varphi_1 + \varphi_1' \text{ or } 2(\varphi_1 + \varphi_1'), \end{cases}$

(vii) $H = \begin{cases} SU(n), n = 3,4,5 \\ SU(3) \\ Sp(2) \\ Spin(8) \end{cases}, \quad \psi = \begin{cases} \mu_n + \bar{\mu}_n \\ k\mu_3 + \ell\bar{\mu}_3, k + \ell = 3,6 \\ \nu_2 + \Lambda^2\nu_2 \\ \Delta^+ + \Delta^- \end{cases}$

Proof. In view of the results of Lemma 3.2, Proposition 3.3A, 3.3B, 3.3C, 3.3D and 3.3E, what remains to be investigated is the case that H is semi-simple but non-simple. The case (H, Ad_H) is obviously always such a possibility, let us study the other possibilities:

(1) Suppose (H, ψ) is a semi-simple subgroup of $SU(m)$ with $P_k(SU(m)/\psi H) = 0$ for $k = 1,2,3$, and H_1 is a simple normal subgroup of H, $\psi_1 = \psi | H_1$. Then, it follows from Reduction 2 of Section 1 that $P_k(SU(m)/\psi_1 H_1) = 0$. for $k = 1,2,3$. Hence the possibilities of (H_1, ψ_1) are given by those lists of Proposition (3.3A,B,C,D,E). Furthermore, suppose $H = H_1 \times \ldots \times H_a$, H_i simple and $\psi_i = \psi | H_i$. Then $P_1(SU(m)/\psi H) = 0$ implies that

$$\frac{L(AdH_1)}{L(\psi_1)} = \frac{L(AdH_2)}{L(\psi_2)} = \ldots = \frac{L(AdH_a)}{L(\psi_a)},$$

(cf. remark following Lemma 3.3, which puts a strong restriction on the possible pairings of (H_i, ψ_i).

(2) Suppose $H = SU(n) \times \tilde{H}$, $n \geq 3$, $n/30$ and $\psi | SU(n) = \mu_n \otimes \mu_n$. Let H_2 be an arbitrary simple normal subgroup of $\tilde{H}$, $\psi_2 = \psi | H_2$. Then, it follows from (1) that $L(\psi_2) = L(Ad_{H_2})$. Therefore, it is not difficult to see from the lists of Proposition (3.3A,B,C,D,E) that either $\psi_2 = Ad_{H_2}$ or (H_2, ψ_2) are among the following special cases:

$$H_2 = \begin{cases} SU(m), m \geq 3, m/30 \\ SU(3) \\ SU(2) \end{cases}, \quad \psi_2 = \begin{cases} \mu_m \otimes \mu_m \\ k\mu_3 + \ell\bar{\mu}_3, k + \ell = 6 \\ 4\mu_2. \end{cases}$$

We shall show that all the above three special cases are impossible and hence the only remaining possibility is $\psi = \mu_n \otimes \mu_n + Ad_{\tilde{H}}$. The computations involved in such demonstrations are quite similar, we shall only exhibit the case $(SU(m), \mu_m \otimes \mu_m)$ as follows. In view of reduction 2 of Section 1, we may assume that $H = SU(m) \times SU(m)$, $\psi = \mu_n \otimes \mu_n + \mu_m \otimes \mu_m$, n, $m/30$. Let

$$(\theta_1, \ldots, \theta_n) \text{ and } (\lambda_1, \ldots, \lambda_m) \text{ with } \Sigma\theta_i = 0, \Sigma\lambda_j = 0$$

be the usual coordinates of the maximal tori of $SU(n)$ and $SU(m)$ respectively, and $\{\sigma_2, \ldots, \sigma_n\}$, $\{\tau_2, \ldots, \tau_m\}$ be their elementary symmetric polynomials. Then straightforward computation will show that

$$P\psi^2 = PH^2 = 2n\sigma_2 + 2m\tau_2,$$
$$P\psi^4 = PH^4 = 2n\sigma_4 + (n^2 - n - 6)\sigma_2^2 + 2m\tau_4 + (2m^2 - m - 6)\tau_2^2 + 4mn\sigma_2\tau_2,$$
$$P\psi^3 = 2n\sigma_3 + 2m\tau_3,$$
$$PH^6 - P\psi^6 = (60 - 2n^2)\sigma_3^2 + (60 - 2m^2)\tau_3^2 - 4mn\sigma_3\tau_3 = 60(\sigma_3^2 + \tau_3^2) - (2n\tau_3 + 2m\tau_3)(n\sigma_3 + m\tau_3).$$

From the above result, it is easy to show that $P_3(SU(m)/\psi H) \neq 0$, which contradicts to the assumption. Hence, the case $\psi_2 = \mu_m \otimes \mu_m$ is in fact impossible.

(3) Suppose $H = H_1 \times \tilde{H}$, H_1 is a simple Lie group of rank ≥ 3 and $L(\psi_1) < L(Ad_{H_1})$, $\psi_1 = \psi | H_1$. Then, it follows from reduction 2 and the results of Proposition 3.3A,B,C,D,E that

$$H_1 = \begin{cases} SU(n) \\ SO(n) \\ Sp(n) \\ Spin(8) \\ SU(4) \text{ or } SU(5) \end{cases} \cdot \quad \begin{cases} \mu_n \text{ or } 2\mu_n \\ \rho_n \\ \nu_n \\ \Delta^+ + \Delta^- \\ (\mu_4 + \bar{\mu}_4) \text{ or } (\mu_5 + \bar{\mu}_5). \end{cases}$$

In this case, we claim that H must be in fact equal to H_1. For otherwise, $\tilde{H}$ contains at least one simple normal factor say H_2 and we may reduce to consider the case $H = H_1 \times H_2$ by reduction 2. Again, it follows from the remark following Lemma (3.3) that

$$\psi = \psi_1 + \psi_2 \text{ and } \frac{L(AdH_1)}{L(\psi_1)} = \frac{L(AdH_2)}{L(\psi_2)}.$$

Then it is tedious but rather straightforward to check that $P_2(SU(m)/\psi H) \neq 0$ for all such cases. Therefore one must have $H = H_1$ for this case.

(4) Suppose $H = H_1 \times \ldots \times H_a$, H_i are simple Lie groups and $\psi | H_1 = Ad_{H_1}$. Then it follows from reduction 2, the remark of Lemma 3.3 and the results of Proposition 3.3A,B,C,D,E that either $\psi_i = \psi | H_i = AdH_i$ or

$$H_i = \begin{cases} SU(n), n \geq 3, n/30 \\ SU(3), \\ SU(2) \end{cases}, \quad \psi = \begin{cases} \mu_n \otimes \mu_n \\ k\mu_3 + \ell\bar{\mu}_3, k + \ell = 6 \\ 4\mu_2 \end{cases}$$

Since the case $(H_i, \psi_i) = (SU(n), \mu_n \otimes \mu_n)$, $n/30$ has already been discussed in (2), we shall show that the following two cases: $(SU(3), k\mu_3 + \ell\bar{\mu}_3)$ and $(SU(2), 4\mu_2)$ are both impossible. Again, we may reduce the proof to the case that $H = H_1 \times H_2$. We shall show that $P_2(SU(m)/(H_1 \times H_2)) \neq 0$ for the following cases:

$$\begin{cases} H_2 = SU(3) \\ H_2 = SU(2)' \end{cases} \quad \begin{cases} \psi = Ad_{H_1} + k\mu_3 + \ell\bar{\mu}_3, \ k + \ell = 6, \\ \psi = Ad_{H_1} + 4\mu_2. \end{cases}$$

In the case $\psi = Ad_{H_1} + k\mu_3 + \ell\bar{\mu}_3$, $P\psi^2 = PH^2 = PH_1^2 + PA_2^2$, $P\psi^4 = PH_1^4 + PH_1^2 \cdot PA_2^2 + 15\sigma_2^2$, $PH^4 = PH_1^4 + PH_1^2 \cdot PA_2^2 + 9\sigma_2^2$. Therefore

$$PH^4 \equiv -6\sigma_2^2 \not\equiv 0 \ (\text{mod } P\psi^2, P\psi^4), P_2(SU(m)/\psi H) \neq 0.$$

In the case $\psi = Ad_{H_1} + 4\mu_2$, $P\psi^2 = PH^2 = PH_1^2 + PA_1^2$, $PH^4 = PH_1^4 + PH_1^2 \cdot PA_1^2$ $P\psi^4 = PH_1^4 + PH_1^2 \cdot PA_1^2 + 6\tau^2$. Hence $PH^4 \equiv -6\tau^2$ (mod $P\psi^2, P\psi^4$). We shall show that $-6\tau^2 \not\equiv 0$ (mod $P\psi^2$, $P\psi^4$) for the cases $H_1 = SU(2)$ or $SU(3)$, and all the other cases are even simpler.

(i) $H_1 = SU(2)$. $P\psi^2 = 4\sigma + 4\tau$, $P\psi^4 = 16\sigma\tau + 6\tau^2$. Therefore the relations are $4\sigma = -4\tau$, $-10\tau^2 = 0$, $4\sigma^2 = 4\tau^2$ and $-6\tau^2 \not\equiv 0$.

(ii) $H_1 = SU(3)$. $P\psi^2 = 6\sigma_2 + 4\tau$, $P\psi^4 = 9\sigma_2^2 + 24\sigma_2 + 6\tau^2$. Therefore, the relations are $6\sigma_2 = -4\tau$, $18\sigma_2^2 = 8\tau^2$, $9\sigma_2^2 = 10\tau^2$ and $12\tau^2 = 0$, hence $6\tau^2 \not\equiv 0$ (mod $P\psi^2$, $P\psi^4$). This proves that $\psi_i = Ad_{H_i}$ or $\mu_n \otimes \mu_n$, $n/30$ and *at most* one of them can be different from Ad_{H_i}.

(5) Suppose $H = SU(3) \times \tilde{H}$ and $\psi | SU(3) = k\mu_3 + \ell\bar{\mu}_3$, $k + \ell = 6$. Then, it follows from the results of Proposition 3.3A,B,C,D,E and the above discussion of (2) and (4) that all normal factors of H must be of $SU(3)$ or $SU(2)$ type and $\psi | SU(3) = k'\mu_3 + \ell'\mu_3'$ or $\psi | SU(2) = 4\mu_2$. Again, one may reduce to investigate the cases $H = SU(3) \times SU(3)$ or $SU(3) \times SU(2)$ with $\psi = k\mu_3 + \ell\bar{\mu}_3 + k'\mu_3' + \ell'\mu_3'$ or $\psi = k\mu_3 + \ell\bar{\mu}_3 + 4\mu_2$, with $k + \ell = 6$, $k' + \ell' = 6$. Direct computations will show that all such cases are impossible. For example, if $H = SU(3) \times SU(3)$, $\psi = 6\mu_3 + 6\mu_3'$, then $P\psi^2 = 6\sigma_2 + 6\tau_2$, $P\psi^3 = 6\sigma_3 + 6\tau_3$, $P\psi^4 = 15(\sigma_2^2 + \tau_2^2) + 36\sigma_2\tau_2$ and

$$P\psi^6 = 15(\sigma_3^2 + \tau_3^2) + 20(\sigma_2^3 + \tau_2^3) + 36\sigma_3\tau_3 + 90\sigma_2\tau_2(\sigma_2 + \tau_2),$$
$$PH^6 = 27(\sigma_3^2 + \tau_3^2) + 4(\sigma_2^3 + \tau_2^3) + 54\sigma_2\tau_2(\sigma_2 + \tau_2).$$

Let $I_1 = \sigma_2 + \tau_2$, $I_2 = \sigma_2\tau_2$, $J_1 = \sigma_3 + \tau_3$, $J_2 = \sigma_3\tau_3$. Then

$$PH^6 = 27J_1^2 - 54J_2 + 4I_1^3 + 42I_1 \cdot I_2$$

subject to relations: $6I_1 = 0$, $15I^2 + 6I_2 = 3I_1^2 + 6I_2 = 0, 6J_1 = 0$ and $15J_1^2 + 6J_2 + 20I_1^3 + 30I_1I_2 = 3J_1^2 + 6J_2 + 2I_1^3 = 0$. Therefore $PH^6 \equiv 4I_1^3 \not\equiv 0$ (mod $P\psi^2, P\psi^3, P\psi^4, P\psi^6$) and hence $P_3(SU(m)/\psi H) \neq 0$ for this case.

(6) Suppose $H = SU(2) \times H_2 \times \ldots \times H_a$, and $\psi | SU(2) = k\mu_2$, $k = 1,2,4$. Then we claim that $P_j(SU(m)/\psi H) = 0$ for $j = 1,2,3$ implies that $H = [SU(2)]^a$, $\psi = k \cdot (\mu_2^{(1)} + \ldots + \mu_2^{(a)})$. Again, we may reduce the proof to the special case that $H = SU(2) \times H_2$. If $\psi | SU(2) = \mu_2$, then it follows from the remark of Lemma 3.3 and Proposition 3.3A,B,C,D,E as well as the results of (3), that either $(H_2, \psi_2) = (G_2, \varphi_1)$ or $(H_2, \psi_2) = (H_2, \psi_2) = (SU(2), \mu_2)$. Therefore, we need only to show that the case (G_2, φ_1) is impossible. Suppose the contrary that $H = SU(2) \times G_2$, $\psi = \mu_2 + \varphi_1$, dim $\varphi_1 = 7$. Then

$$P\psi^2 = \tau + 2\sigma_2, P\psi^4 = 2\sigma_2\tau + \sigma_2^2, P\psi^6 = t\sigma_2^2 - \sigma_3^2$$

which implies $\tau = -2\sigma_2$, $3\sigma_2^2 = 0$ and $\sigma_3^2 = -2\sigma_2^3$. Therefore

$$PH^6 = 26\sigma_3^2 + 28\sigma_2^3 + 88t\sigma_2^2 \equiv (-52 + 28 - 176)\sigma_2^3 \equiv 2\sigma_2^3 \not\equiv 0$$

and hence $P_3(SU(m)/\psi H) \neq 0$ for this case. Similar computation will show that $P_3(SU(m)/\psi H) \neq 0$ for the case $\psi = 2(\mu_2 + \varphi_1)$. Next, let us show that $\psi = 2\mu_2 + k\mu_3 + \ell\bar{\mu}_3$, $k + \ell = 3$ is also impossible. For this case, $5\tau^2 = 0$. Therefore

$$PH^4 = 9\sigma_2^2 + 4\tau \cdot 6\sigma_2 = 9\tau^2 - 16\tau^2 = -7\tau^2 \neq 0.$$

Similar computation will show that the case $\psi = 2\mu_2 + \nu_2 + \Lambda^2\nu_2$ is also impossible. This proves that $\psi | SU(2) = 2 \cdot \mu_2$ implies that $(H_2, \psi_2) = (SU(2), 2\mu_2)$.

Finally, let us remark that the case $\psi | SU(2) = 4\mu_2$ is essentially included in the discussions of (2), (4), and (5). Because it follows from the remark of Lemma 3.3 and the results of Proposition 3.3A,B,C,D,E that the possibilities for (H_2, ψ_2) are follows:

$$(H_2, Ad_{H_2}), (SU(n), \mu_n \otimes \mu_n), n/30, (SU(3), k\mu_3 + \ell\bar{\mu}_3)$$
$$k + \ell = 6 \text{ and } (SU(2), 4\mu_2),$$

and it follows from the results of (2), (4) and (5) that only $(SU(2), 4\mu_2)$ is still possible.

(7) Suppose $H = SO(4) \times \tilde{H}$ and $\psi | SO(4) = \rho_4$. Straightforward computation will show that $P_2(SU(m)/\psi H) \neq 0$ for $H = SO(4) \times SU(2)$ $\psi = \rho_4 + 2\mu_2$. Therefore the only possible case is $H = SO(4)$. $\tilde{H} = \{\text{id}\}$ (Cf. (6) and reduction 2). Similar computation will show that $P_2(SU(m)/2\rho_4 \cdot SO(4)) \neq 0$ (Cf. Proposition 3.2D).

(8) Direct computation will also show that $P_3(SU(m)/\psi H) \neq 0$ for the case $H = SU(3) \times SU(3)$, $\psi = \mu_3 \otimes u_3'$ or $2(\mu_3 \otimes \mu_3')$.

(9) Finally, what remains to be investigated are those cases of $H = H_1 \times \ldots \times H_a$, $\psi = \psi_1 + \ldots + \psi_a$ with (H_i, ψ_i) as follows:

$$H_i = \begin{cases} SU(3) \\ Sp(2), \\ G_2 \end{cases} \quad \psi_i = \begin{cases} k\mu_3 + \ell\bar{\mu}_3, k + \ell = 2 \text{ or } 3 \\ \nu_2 + \Lambda^2\nu_2 \\ \varphi_1 \text{ or } 2\varphi_1 \end{cases}$$

Straightforward computations will show that if H is *non-simple*, then there are only the following such possibilities, namely

$$H = \begin{cases} SU(3) \times SU(3) \\ G_2 \times G_2 \end{cases}, \quad \psi = \begin{cases} k(\mu_3 + \mu'_3) + \ell(\bar{\mu}_3 + \bar{\mu}'_3), k + \ell = 1,2, \\ (\varphi_1 + \varphi'_1 \text{ or } 2(\varphi_1 + \varphi'_1). \end{cases}$$

All the above computations complete the proof of Theorem 1.

Remarks.

(i) Of course, one may further reduce the above list of possibilities by stronger vanishing condition on its Pontrjagin classes or Stiefel-Whitney classes. For example, in the case H semi-simple, $SU(m)/Ad\cdot H$ always has some non-vanishing Stiefel-Whitney classes.

(ii) Suppose $H \subset SU(m)$ is a disconnected, positive dimensional subgroup with $P_k(SU(m)/H) = 0$ for $k = 1,2,3$ and H^0 is its connected component identity. Then $\pi: SU(m)/H^0 \to SU(m)/H$ is a covering map and hence $\tau(SU(m)/H^0) = \pi!\,\tau(SU(m)/H)$. Therefore, it follows that $P_k(SU(m)/H^0) = 0$ for $k = 1,2,3$ and consequently H^0 is one of the possibilities listed in the above theorem, which already strongly restricts the possibilities of H itself.

Section 4. Homogeneous spaces of Sp(m) with vanishing characteristic classes.

In this section, we proceed to determine those homogeneous spaces of symplectic groups, $Sp(m)/H$, with vanishing first three Pontrjagin classes. Let H be a compact connected Lie group and $\psi: H \to Sp(m)$ be an almost faithful sympletic representation of $H, \psi': H \to Sp(m) \subset SU(2m)$ be the associated complex representation of ψ. Then the weight system of ψ is by definition the weight system of ψ', ie., $\Omega(\psi) = \Omega(\psi')$. Since $\psi' = \bar{\psi}'$, it is clear that $-\Omega(\psi) = \Omega(\psi)$ and hence

$$\prod_{w \in \Omega(\psi)} (1 + w) = 1 + P\psi^2 + P\psi^4 + P\psi^6 + \ldots + P\psi^{2k} + \ldots$$

where the odd degree symmetric products $P\psi^{2k-1}$ are automatically zero. Similar to the case of homogeneous spaces of $SU(m)$. One may state the splitting principle of Borel-Hirzebruch as follows:

Lemma 4.1. *Suppose $Sp(m)/\psi H$ satisfies the vanishing conditions*

$$P_k(Sp(m)/\psi H) = 0, \qquad k = 1,2,\ldots,\ell.$$

Then the symmetric products of roots of H and the weights of ψ satisfy the following system of algebraic equations:

$$PH^2 \equiv 0 \ (mod\ P\psi^2)$$
$$PH^4 \equiv 0 \ (mod\ P\psi^2,\ P\psi^4)$$
$$\text{--------------------------}$$
$$PH^{2\ell} \equiv 0 \ (mod\ P\psi^2,\ P\psi^4, \ldots, P\psi^{2\ell}).$$

Proof. It follows from $P_k(Sp(m)/\psi H) = 0$, $k = 1,2,\ldots,\ell$, that

$$P_k(\alpha(Ad_H|T)) = i^*(PH^{2k}) = 0 \text{ for } k = 1,2,\ldots,\ell.$$

On the other hand, it follows from reduction 5C that

$$Ker(i^*) = \langle P\psi^2, P\psi^4, \ldots, P\psi^{2m} \rangle.$$

Hence $i^*(PH^{2k}) = 0$ if and only if $PH^{2k} \equiv 0$ (mod $P\psi^2$, $P\psi^4$, ..., $P\psi^{2k}$).

If one compares the above Lemma 4.1 with Lemma 3.1 and observes that $P\psi^{2k-1} = P\psi'^{2k-1} = 0$ for the symplectic representation ψ, then it is clear that the conditions of Lemma 4.1 and Lemma 3.1 are in fact identical. Therefore, one may simply select those symplectic representations of listed in Theorema 1, one obtains the following main result for the case of symplectic groups.

Theorem 2. *Let $\psi: H \subset Sp(m)$ be a symplectic representation of a compact connected Lie group with $P_k(Sp(m)/\psi H) = 0$ for $k = 1,2,3$. Then the possibilities of all such pairs (H, ψ) are given by the following list modulo trivial representations:*

(i) *H is any given subtorus*

(ii) $H = Sp(n)$, $\psi = \nu_n$, $n \geq 1$.

(iii) $H = [Sp(1)]^\ell$, $\psi = k\cdot(\nu_1^{(1)} + \nu_1^{(2)} + \ldots + \nu_1^{(\ell)})$, $k = 1,2,4$,

(iv) $H = SU(n)$, $n = 3,4,5$, $\psi = \mu_n + \bar{\mu}_n$

(v) $$H = \begin{cases} SU(3) \times SU(3) \\ G_2 \times G_2 \\ G_2 \end{cases} \qquad \psi = \begin{cases} (\mu_3 + \bar{\mu}_3) + (\mu'_3 + \bar{\mu}'_3) \\ 2\varphi_1 + 2\varphi'_1 \\ 2\varphi_1. \end{cases}$$

Remark. Observe that $Sp(m)/Sp(1)^m$ is the principal orbit type of the linear $Sp(m)$-action via $\Lambda^2\nu_m$. Therefore $Sp(m)/Sp(1)^m$ is stably parallelizable and consequently $Sp(m)/Sp(1)^\ell$ is also stably parallelizable.

Section 5. Homogeneous spaces of SO(m) with vanishing characteristic classes.

In this section, we shall study homogeneous spaces of $SO(m)$, $SO(m)/H$, with vanishing first three Pontrjagin classes. Let $\psi: H \subset SO(m)$ be a given real representation of H and $\psi_{\mathbb{C}}: H \subset SO(m) \subset SU(m)$ be its complexification, $T \subset H$ be a maximal torus of H. Then the weight system of ψ, $\Omega(\psi)$, is by definition the weight system of $\psi_{\mathbb{C}}|T$, which is even in the sense $-\Omega(\psi) = \Omega(\psi)$. Hence the odd degree symmetric products $P\psi^{2k-1}$ vanish automatically, namely

$$\prod_{w \in \Omega} (1 + w) = 1 + P\psi^2 + P\psi^4 + \ldots + P\psi^{2k} + \ldots$$

In view of reduction 5B, we define $\overline{P\psi}^{2k}$ as follows:

$$\overline{P\psi}^{2k} = \begin{cases} P\psi^{2k} \text{ if it is not divisible by 2} \\ \frac{1}{2} P\psi^{2k} \text{ if it is divisible by 2.} \end{cases}$$

Then one combines Reductions 1, 4 and 5B into the following lemmas:

Lemma 5.1. *Let $\psi: H \subset SO(m)$ be a given real representation of a compact connected Lie group H. If $P_k(SO(m)/\psi H) = 0$ for $k = 1, 2, \ldots, \ell$, then the symmetric products of the weights of ψ and the roots of H satisfy the following algebraic equations:*

$$PH^2 \equiv 0 \,(mod\ \overline{P\psi}^2)$$
$$PH^4 \equiv 0 \,(mod\ \overline{P\psi}^2, \overline{P\psi}^4)$$
$$\text{- - - - - - - - - - - - - - - - - - - -}$$
$$PH^{2\ell} \equiv 0 \,(mod\ \overline{P\psi}^2, \overline{P\psi}^4, \ldots, \overline{P\psi}^{2\ell}).$$

Lemma 5.2. *If $P_1(SO(m)/\psi H) = 0$, $m \geqslant 5$, then H is either abelian or semi-simple.*

Lemma 5.3. *Let $H = H_1 \times \ldots \times H_a$ be a product of simple compact connected Lie groups, $\psi: H \to SO(m)$ be an almost faithful real representation, $\psi_i = \psi | H_i$ and $L(\psi_i)$ be the total length of weights of ψ_i defined in Section 3. Then $P_1(SO(m)/\psi H) = 0$ implies that*

$$\frac{L(AdH_1)}{L(\psi_1)} = \frac{L(AdH_2)}{L(\psi_2)} = \ldots\ldots = \frac{L(AdH_a)}{L(\psi_a)} \equiv 0 \left(mod\ \frac{1}{2} \right)$$

The proofs of the above lemmas are essentially the same as that of Lemmas 3.1, 3.2 and 3.3. However, because of the possible factor of $\frac{1}{2}$ involved in $\overline{P\psi}^{2k}$, we can not directly reduce the case of $SO(m)$ to their complexifications as we did for the $Sp(m)$ case. Therefore, we shall again proceed to investigate the case of simple Lie groups via classification.

(A) *The case $H = A_{n-1}$.*

$L(Ad_{A_{n-1}}) = 2n(n-1)$ and simple computation will show the following are those irreducible *real* representations of A_{n-1} with $L(\psi) \leq 2L(Ad) = 4n \cdot (n-1)$.

$$L(\mu_n + \bar{\mu}_n) = 2(n-1)$$
$$L(\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n) = 2(n-2)(n-1), \text{ and the special case } \Lambda^2\mu_4 = \rho_6,\ L(\psi) = 6$$
$$L(Ad) = 2n(n-1)$$
$$L(S^2\mu_n + S^2\bar{\mu}_n) = 2(n+2)(n-1)$$
$$L(\Lambda^3\mu_n + \Lambda^3\bar{\mu}_n) = (n-2)(n-3)(n-1) < 4n(n-1) \text{ for } n = 6,7,8$$
$$L(\Lambda^4\mu_8) = 20 \cdot 7$$

Based on the above results and Lemma 5.3, one has the following:

Proposition 5.1A. *If ψ is a real representation of A_{n-1}, $n \neq 4$, such that $P_1(SO(m)/\psi A_{n-1}) = 0$, then the possibilities of ψ are given by the following list modulo trivial representations:*

(i) $Ad, 2Ad, Ad + n(\mu_n + \bar{\mu}_n), Ad + (\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n) + 2(\mu_n + \bar{\mu}_n)$,
(ii) $k(\mu_n + \bar{\mu}_n), k/2n$,
(iii) $(S^2\mu_n + S^2\bar{\mu}_n) + (n-2)(\mu_n + \bar{\mu}_n), \mu_n \otimes \mu_n + \bar{\mu}_n \otimes \bar{\mu}_n$,
(iv) $(\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n) + 2(\mu_n + \bar{\mu}_n), 2(\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n) + 4(\mu_n + \bar{\mu}_n)$, $(\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n) + (n+2)(\mu_n + \bar{\mu}_n)$,

$$\begin{cases} n = 6:\ 2\Lambda^3\mu_6, 4\Lambda^3\mu_6, 2\Lambda^3\mu_6 + 6(\mu_6 + \bar{\mu}_6), \\ \qquad 2\Lambda^3\mu_6 + \Lambda^2\mu_6 + \Lambda^2\bar{\mu}_6 + 2(\mu_6 + \bar{\mu}_6) \\ n = 7:\ \Lambda^3\mu_7 + \Lambda^3\bar{\mu}_7 + 4(\mu_7 + \bar{\mu}_7) \\ n = 8:\ \Lambda^3\mu_8 + \Lambda^3\bar{\mu}_8 + \mu_8 + \bar{\mu}_8, \Lambda^4\mu_8 + 6(\mu_8 + \bar{\mu}_8), \\ \qquad \Lambda^4\mu_8 + \Lambda^2\mu_8 + \Lambda^2\bar{\mu}_8. \end{cases}$$

Next let us investigate which ψ among the above list also have vanishing second Pontrjagin class. Observe that if $\psi(A_{n-1}) \subset SU\left[\frac{m}{2}\right] \subset SO(m)$, then it follows from reduction 3 that $P_k(SO(m)/\psi A_{n-1}) = 0$ implies

$$P_k(SU\left[\frac{m}{2}\right]/\psi A_{n-1}) = 0.$$

Therefore, one may simply apply Proposition 3.2A to conclude that $P_2(SO(m)/\psi A_{n-1}) \neq 0$ for the following cases:

(1) $k(\mu_n + \bar{\mu}_n)$ with $k \nmid (2n, 6)$;
(2) $S^2\mu_n + S^2\bar{\mu}_n + (n-2)(\mu_n + \bar{\mu}_n); \Lambda^2\mu_n + \Lambda^2\bar{\mu}_n + (n+2)(\mu_n + \bar{\mu}_n), n \neq 6$, or $k \cdot \{\Lambda^2\mu_n + \Lambda^2\bar{\mu}_n + 2\mu_n + 2\bar{\mu}_n\}$, $k = 1, 2$:

(3) $\Lambda^3\mu_8 + \Lambda^3\bar{\mu}_8 + \mu_8 + \bar{\mu}_8$; $\Lambda^3\mu_7 + 4(\mu_7 + \bar{\mu}_7)$;
$2\Lambda^3\mu_6 + 6(\mu_6 + \bar{\mu}_6)$; $4\Lambda^3\mu_6$

Proposition 5.2A. *If $P_k(SO(m)/\psi A_{n-1}) = 0$ for $k = 1,2$ and $n \neq 4$, then the possibilities of ψ are given by the following list modulo trivial representations:*

(i) *Ad, or 2Ad,* (ii) $k(\mu_n + \bar{\mu}_n)$ *with* $k(2n, 6)$
(iii) $\mu_n \otimes \mu_n + \bar{\mu}_n \otimes \bar{\mu}_n$

$$\text{(iv)} \begin{cases} n = 6\colon\ 2\Lambda^3\mu_6, 2\Lambda^3\mu_6 + \Lambda^2\mu_6 + \Lambda^2\bar{\mu}_6 + 2(\mu_6 + \bar{\mu}_6), \\ \qquad \Lambda^2\mu_6 + \Lambda^2\bar{\mu}_6 + 8(\mu_6 + \bar{\mu}_6) \\ n = 2\colon\ 4\mu_2 + S^2\mu_2, 8\mu_2 \end{cases}$$

Proof. If one compares the above list with the lists of Proposition 3.2A and 5.1A, it is clear that one only needs to show $P_2(SO(m)/\psi A_{n-1}) \neq 0$ for the following cases:

(1) $Ad + n(\mu_n + \bar{\mu}_n)$, or $Ad + \Lambda^2\mu_n + \Lambda^2\bar{\mu}_n + 2(\mu_n + \bar{\mu}_n)$, $n \geq 3$
(2) $\Lambda^4\mu_8 + 6\equiv(\mu_8 + \bar{\mu}_8)$, or $\Lambda^4\mu_8 + \Lambda^2\mu_8 + \Lambda^2\bar{\mu}_8$.

The computations involved are rather simple. For example, in the case $\psi = Ad + n(\mu_n + \bar{\mu}_n)$, $\overline{P\psi}^2 = 2n\sigma_2$, $\overline{P\psi}^4 = 2n\sigma_4 + (4n^2 - n - 3)\bar{\sigma}_2^2$ and in the case $\psi = Ad + \Lambda^2\mu_n + \Lambda^2\bar{\mu}_n + 2(\mu_n + \bar{\mu}_n)$, $\overline{P\psi}^2 = 2n\sigma_2$, $\overline{P\psi}^4 = (2n-6)\bar{\sigma}_4 + (4n^2 - n - 3)\sigma_2^2$. In both cases, it is easy to check that

$$PH^4 \not\equiv (\mathrm{mod}\ \overline{P\psi}^2, \overline{P\psi}^4),$$

hence $P_2(SO(m)/\psi A_{n-1}) \neq 0$. The other two cases involve $\Lambda^4\mu_8$ are in fact easier.

Proposition 5.3A. *If $P_k(SO(m)/\psi A_{n-1}) = 0$ for $k = 1,2,3$ and $n \neq 4$, then the possibilities of such ψ are as follows modulo trivial representations:*

(i) *Ad or 2Ad*
(ii) $n = 3,4,5$, $\mu_n + \bar{\mu}_n$ *or* $2(\mu_n + \bar{\mu}_n)$
(iii) $n = 2$, $2\mu_2, 4\mu_2, 8\mu_2$ *or* $Ad + 4\mu_2$.

Proof. In view of the results of Proposition 5.2A and 3.3A, we need only to show that $P_3(SO(m)/\psi A_{n-1}) \neq 0$ for the following cases, namely,

(1) $(\mu_n + \bar{\mu}_n)$ or $2(\mu_n + \bar{\mu}_n)$ for $n \geq 6$,
(2) $3(\mu_n + \bar{\mu}_n)$ or $6(\mu_n + \bar{\mu}_n)$ for $n \geq 3$,
(3) $\mu_n \otimes \mu_n + \bar{\mu}_n \otimes \bar{\mu}_n$ for $n \geq 3$.

Let us verify the above facts according to the three cases:

(i) Suppose $H = SU(n)$, $n \geq 6$ and $\psi = \mu_n + \bar{\mu}_n$. Then

$$\overline{P\psi}^6 = (2\sigma_6 + 2\sigma_2\bar{\sigma}_4 - \sigma_3^2)$$

and hence

$$PH^6 \equiv 2n\sigma_6 - (n - 30)\sigma_3^2 \not\equiv 0 \pmod{\sigma_2, \sigma_4, \overline{P\psi}^6}$$

Therefore $P_3(SO(m)\psi H) \neq 0$ for the case $\psi = \mu_n + \bar{\nu}_n$, $n \geq 6$. The case $\psi = 2(\mu_n + \bar{\mu}_n)$, $n \geq 6$, is essentially the same.

(ii) Suppose $H = SU(n)$, $n \geq 3$ and $\psi = k(\mu_n + \mu_n)$, $k = 3,6$. Then $\overline{P\psi}^2 \equiv 0 \pmod{3\sigma_2}$, $P\psi^4 \equiv 0 \pmod{3\sigma_2, 3\sigma_4}$. However $PH^6 \not\equiv 0 \pmod{3\sigma_2, 3\sigma_4, \overline{P\psi}^6}$, therefore $P_3(SO(m)/\psi H) \neq 0$.

(iii) $\psi = \mu_n \otimes \mu_n + \bar{\mu}_n \otimes \bar{\mu}_n$. Then, the computation of Section 3 show that

$$\overline{P\psi}^2 = PH^2 = 2n\sigma_2, \overline{P\psi}^4 = PH^4 + 2n^2\sigma_2^2,$$
$$\overline{P\psi}^6 = PH^6 + PH^2 \cdot PH^4 - 60\sigma_3^2$$

and it is difficult to see that $PH^6 \not\equiv (\mathrm{mod}\ \overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6)$, for example, in the case $n = 3$, $PH^6 \equiv 27\sigma_3^2 \pmod{\sigma_2, \sigma_4}$ and $\overline{P\psi}^6 \equiv 33\sigma_3^2 \pmod{\sigma_2, \sigma_4}$.

(B) *The case $H = B_n$, $n \geq 3$.*

$L(Ad_{B_n}) = (2n - 1)2n$ and the following is the list of those irreducible real representations, ψ, of B_n with $L(\psi) \leq 2L(Ad_{B_n})$.

(i) $L(Ad) = (2n - 1)\cdot 2n$, $L(\rho) = 2n$, $L(S^2\rho - 1) = (2n + 3)\cdot 2n$

$$\text{(ii)} \begin{cases} n = 3,4,7, L(\Delta_n) = 2^{n-2}\cdot n \leq 2\cdot L(Ad) = 4(2n-1)\cdot n \\ n = 5,6\ L(2\Delta_n) = 2^{n-1}\cdot n \leq 2L(Ad) \\ n = 3, \Lambda^3\rho_7 = 10\cdot 6 = 2L(Ad_{B_3}) \end{cases}$$

where Δ_n is the spin representation of $\mathrm{Spin}(2n+1)$. It follows directly from the above result and Lemma 5.3 that

Proposition 5.1B. *If $P_1(SO(m)/\psi B_n) = 0$, then the possibilities of ψ are given by the following list modulo trivial representations:*

(i) *Ad, 2Ad,* $Ad + (2n - 1)\rho$, (ii) $k\rho$ *wth* $k/s\cdot(2n - 1)$,
(iii) $S^2\rho + (2n - 5)\rho$, *and the following additional cases for small n:*

$$\text{(iv)} \begin{cases} n = 7\colon \Delta_7 + 10\rho, n = 6\colon 2\Delta_6 + 6\rho, \\ n = 5\colon 2\Delta_5 + \rho, 2\Delta_5 + 10\rho, 4\Delta_5 + 2\rho, 2\Delta_5 + Ad + \rho; \\ n = 4\colon k\Delta_4 + \ell\rho, (2k + \ell)/14; Ad + k\Delta_4 + \ell\rho, 2k + \rho = 7; S^2\rho + \Delta_4 + \rho; \\ n = 3\colon \Lambda^3\rho, k\Delta_3 + \ell\rho, (k + \ell)/10; S^2\rho + \Delta_3; Ad + k\Delta_3 + \ell\rho, k + \ell = 5. \end{cases}$$

Next let us investigate the second Pontrjagin class of $SO(m)/\psi B_n$ with ψ in the above list.

Proposition 5.2B. *If* $P_k(SO(m)/\psi Bn) = 0$ *for* $k = 1,2,\ n \geq 3$, *then the possibilities of such* ψ *are given as follows*:
(i) Ad, (ii) $k\rho$ *with* $k/(3,2n-1)$, (iii) $n = 3,\ \psi = \Delta_3$.

Proof.
(1) Suppose $\psi = 2 \cdot Ad$. Then $P\psi^2 = 2PH^2 = 2(2n-1)\bar\sigma_1$ and $P\psi^4 = 2PH^4 + (PH^2)^2 = 2 \cdot PH^4 + (2n-1)^2\bar\sigma_1^2$. Therefore $\overline{P\psi}^2 = \frac{1}{2} P\psi^2 = (2n-1)\bar\sigma_1$ and $\overline{P\psi}^4 = P\psi^4$, consequently, $PH^4 \not\equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^2}$ which implies $P_2(SO(m)/\psi B_n) \neq 0$.

Similar computations will show that $P_2(SO(m)/\psi B_n) \neq 0$ for the case $\psi = Ad + (2n-1)\rho$ and $\psi = S^2\rho + (2n-5)\rho$.

(2) Suppose $\psi = k\rho$. Then $P\psi^2 = k\bar\sigma_1$, $P\psi^4 = k\bar\sigma_2 + \binom{k}{2}\bar\sigma_1^2$. Therefore $\overline{P\psi}^4 = P\psi^4$ if $k \not\equiv 0 \bmod 4$ and $\overline{P\psi}^4 = \frac{1}{2}P\psi^4 = \frac{k}{2}\bar\sigma_2 + \frac{k(k-1)}{4}\bar\sigma_1^2$ if $k \equiv 0 \pmod 4$. Hence

$$PH^4 = (2n-7)\bar\sigma_2 + (n-1)(2n-1)\bar\sigma_1^2 \equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^4)}$$

implies that $k/(3,2n-1)$.

(3) Next let us consider those special cases for $n = 7,6,5,4,3$ listed in (iv) of Proposition 5.1B. Observe that

$$\Omega(\Delta_n) = \left\{\frac{1}{2}(\pm\theta_1 \pm \theta_2 \pm \ldots \pm \theta_n)\right\}, \text{ and for } \varphi = \Delta_n$$

$$S\varphi^2 = 2^{(n-2)} \cdot \bar\sigma_1,\quad S\varphi^4 = 2^{(n-4)} \cdot (\bar\sigma_1^2 + 4\bar\sigma_2).$$

Moreover, in the case of Spin$(2n+1)$, $H^4(B_W;\mathbb{Z})^W$ contains $\frac{1}{2}\bar\sigma_1$ but not $\frac{1}{4}\bar\sigma_1$. Based on the above results, it is not difficult to verify that $P_2(SO(m)/\psi B_n) \neq 0$ for all those special cases listed in (iv) of Proposition 5.1B except $\psi = \Delta_3$. This completes the proof of Proposition 5.2B.

Proposition 5.3B. *If* $P_k(SO(m)/\psi Bn) = 0$ *for* $k = 1,2,3$ *and* $n \geq 3$, $\psi = Ad$ *or* ρ *modulo trivial representations.*

Proof. Suppose $\psi = 3\rho$ and $3/(2n-1)$. Then $P\psi^2 = 3\bar\sigma$, $P\psi^4 = 3\bar\sigma_2 + 3\bar\sigma_1^2$, $P\psi^6 = 3\bar\sigma_3 + 6\bar\sigma_1\bar\sigma_2 + \bar\sigma_1^3$. Therefore

$$PH^6 = (2n-31)\bar\sigma_3 + 2(n-3)(2n-3)\bar\sigma_1\bar\sigma_2 + \frac{1}{3}(n-1)(2n-3)(2n-1)\bar\sigma_1^3 \equiv$$

$$\equiv \bar\sigma_1\bar\sigma_2 + \frac{1}{3}(n-1)(n-2)(2n-1)\bar\sigma_1^3 \not\equiv 0 \pmod{3, \overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6}.$$

Straightforward computation will also show that $P_3(SO(m)/\Delta_3 B_3) \neq 0$. This completes the proof of Proposition 5.3B.

(C) *The case* $H = C_n,\ n \geq 2$.

$L(AdC_n) = 4(n+1)n$ and the following are those irreducible real representations of C_n with $L(\psi) \leq 2 \cdot L(Ad_{C_n})$.
(i) $L(Ad) = 4(n+1) \cdot n$, $L(2v) = 4n$, $L(\Lambda^2 v - 1) = 4(n-1)n$,
(ii) $n = 3$: $L(2(\Lambda^3 v_3 - v_3)) = 20 \cdot 3$; $n = 4$: $L(\Lambda^4 v_4 - \Lambda^2 v_4) = 28 \cdot 4$.

Proposition 5.1C. *If* $P_1(SO(m)/\psi C) = 0$, *then the possibilities of such* ψ *are given by the following*:
(i) $Ad, 2Ad, Ad + (n+1) \cdot 2v$, (ii) $2kv$ *with* $k/(n+1)$,
(iii) $\Delta^2 v + 4v, \Delta^2 v + 2(n+3)v, 2\Lambda^2 v + 8v, \Lambda^2 v + Ad + 4v$,
(iv) $\begin{cases} n = 4: \Lambda^4 v_4, (\Lambda^4 v_4 - \Lambda^2 v_4) + 6v, \\ n = 3: 2\Lambda^3 v_3 + 4v_3, 2\Lambda^3 v_3 + \Lambda^2 v_3, k\Lambda^2 v, k = 1,2,4 \\ n = 2: Ad + 2\Lambda^2 v_2 + 2v_2, Ad + 3\Lambda^2 v_2, k\Lambda^2 v_2 + \ell \cdot 2v_2, (k+\ell)/6. \end{cases}$

Proposition 5.2C. *If* $P_k(SO(m)/\psi C_n) = 0$ *for* $k = 1,2$, *and* $n \geq 2$, *then the possibilities of such representations are as follows*:
(i) Ad *or* $2Ad$ *for* n *odd*, (ii) $k \cdot v$, $k/(3,n+1)$
(iii) $\begin{cases} n = 5: \Lambda^2 v_5 + 4v_5, \text{ or } \Lambda^2 v_5 + 16v_5, \\ n = 3: \Lambda^2 v_3 \text{ or } 2\Lambda^2 v_3, \\ n = 2: 3\Lambda^2 v_2 \text{ or } 2\Lambda^2 v_2 + 2v_2. \end{cases}$

Proof. In view of the list of Proposition 5.1C, the proof of the above proposition consists of a series of case by case elimination. For example, in the case $\psi = 2Ad$, $H = Sp(n)/\mathbb{Z}_2$, PH^2 is not divisible by 2 in $H^4(B_T;\mathbb{Z})$ for the case n even. Therefore $\overline{P\psi}^4 = P\psi^4 = 2PH^4 + PH^2 \cdot PH^2$ and $PH^4 \not\equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^4}$.

(1) It is easy to show that $P_2(SO(m)/\psi C_n) \neq 0$ for the cases $\psi = Ad + (n+1) \cdot 2v$ and $k \cdot 2v$ with $k \neq 1$ or $(3, n+1)$. If $\psi = \Lambda^2 v + 4v$, then $P\psi^2 = 2(n+1)\bar\sigma_1$, $P\psi^4 = (2n-4)\bar\sigma_2 + (2n+1)(n+1)\bar\sigma_1^2$, $\overline{P\psi}^2 = (n+1)\bar\sigma_1$ and $\overline{P\psi}^4 = \frac{1}{2}P\psi^4$ if n is odd, $P\psi^4$ if n is even. It is then easy to check that $PH^4 \equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^4}$ *only when* $n = 5$. Similarly, in the case $\psi = \Lambda^2 v + (n+3) \cdot 2v$, $P\psi^2 = 4(n+1)\bar\sigma_1$, $P\psi^4 = 2 \cdot (2n-1)\bar\sigma_2 + [8(n+1)^2 - 2(n+1)]\bar\sigma_1^2$, $\overline{P\psi}^2 = 2(n+1)\bar\sigma_1$, $\overline{P\psi}^4 = (2n-1)\bar\sigma_2 + [4(n+1)^2 - (n+1)]\bar\sigma_1^2$. Hence $PH^4 \equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^4}$ implies that $2n + 8 \equiv 9 \equiv 0 \pmod{2n-1}$, i.e., $n = 5$.

(2) The computation of Proposition 3.2C applies to show that $P_2(SO(m)/\psi C_n) \neq 0$ for the case $\psi = 2\Lambda^2 v + 8v$. As for the case $\psi = Ad + \Lambda^2 v + 4v$, it follows easily from $P\psi^4 \equiv 2(2n+2)\bar\sigma_2 \pmod{\bar\sigma_1^2}$ that $PH^4 \not\equiv 0$ (mod $P\psi^2, P\psi^4$), if $n \neq 2$.

(3) In the cases of $n = 4,3,2$ listed in (iv) of Proposition 5.1C, straightforward computations will show that $P_2(SO(m)/\psi C_n) \neq 0$ except the following possibilities:

$$n = 3\colon\ \Lambda^2 v_3 \text{ or } 2\Lambda^2 v_3;\ n = 2\colon 3\Lambda^2 v_2, 2\Lambda^2 v_2 + 2v_2.$$

Proposition 5.3C. *If $P_k(SO(m)/\psi C_n) = 0$ for $k = 1,2,3$ then the possibilities of such ψ are given as follows:*
(i) *Ad or $2Ad$ for n odd,*
(ii) *$2v$, and the special case $2\Lambda^2 v_2 + 2v_2$ for $n = 2$.*

Proof. In view of the list of Proposition 5.2C, we need only to show that $P_3(SO(m)/\psi C_n) \neq 0$ for the following cases, namely $3 \cdot 2v$ for $n + 1 \equiv 0$ mod 3, $\Lambda^2 v_5 + 4v_5$, $\Lambda^2 v_5 + 16 v_5$, $\Lambda^2 v_3$, $2\Lambda^2 v_3$ and $3\Lambda^2 v_2 = 3\rho_5$.

(1) $\psi = 6v$, $3/(n+1)$. Then $\overline{P\psi}^6 = 3\bar\sigma_3 + 15\bar\sigma_1\bar\sigma_2 + 10\bar\sigma_1^3$ and $PH^6 \equiv \bar\sigma_1\bar\sigma_2 \not\equiv 0$ (mod $\overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6, 3$). Hence $P_3 \neq 0$. Similar computation will show $P_3 \neq 0$ for the case $3\rho_5$.

(2) In the case $\psi = \Lambda^2 v_5 + 4v_5$. $P\psi^6 \equiv -18\bar\sigma_3$ and $PH^6 \equiv 42\bar\sigma_3$ (mod $\bar\sigma_1, \bar\sigma_2$) and hence $P_3 \neq 0$ for this case. In the case $\psi = \Lambda^2 v_5 + 16 v_5$: $\overline{P\psi}^2 = \bar\sigma_1$, $\overline{P\psi}^4 = 9\bar\sigma_2 + 138\bar\sigma_1^2$, $P\psi^6 \equiv -6\bar\sigma_3 + 418\bar\sigma_1 \cdot \bar\sigma_2$ (mod $12\sigma_1$). Therefore

$$PH^6 + 7P\psi^6 \equiv -\bar\sigma_1\bar\sigma_2 \not\equiv 0 \ (\text{mod } \overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6, 3).$$

Hence again $P_3 \neq 0$.

(3) For the case $\Lambda^2 v_3$ or $2 \cdot \Lambda^2 v_3$, $P\psi^6 \equiv -26\bar\sigma_3$ or $-52\bar\sigma_3$ (mod $\bar\sigma_1, \bar\sigma_2$) and hence $PH^6 = 38\bar\sigma_3 + \ldots \not\equiv 0$ (mod $\overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6$). This completes the proof of Proposition 5.3C.

(D) *The case $H = D_n$, $n \geq 3$.*

The following is a list of irreducible real representations ψ with $L(\psi) \leq 2 \cdot L(Ad) = 4 \cdot (n-1) \cdot 2n$.

(i) $L(Ad) = 2(n-1) \cdot 2n$, $L(\rho) = 2n$, $L(S^2\rho - 1) = 4n(n+1)$,

(ii) $\begin{cases} n = 4,8\colon & L(\Delta^\pm) = 2^{n-3} \cdot n \\ n = 6\colon & L(2\Delta^\pm) = 2^{n-2} \cdot n = 16 \cdot n \\ n = 3,5,7\colon & L(\Delta^+ + \Delta^-) = 2^{n-2} \cdot n \end{cases}$

Based on the above result, it is immediate that one has the following proposition.

Proposition 5.1D. *If $P_1(SO(m)/\psi D_n) = 0$, $n \geq 3$, then the possibilities of such real representation ψ are as follows:*
(i) *$Ad, 2Ad, Ad + 2(n-1)\rho$,* (ii) *$S^2\rho + (2n-6)\rho$*
(iii) *$k\rho$ with $k/4(n-1)$ and the following special cases involving spin representations:*

$$\text{(iv)} \begin{cases} n = 8\colon & \Delta^\pm + 12\rho \\ n = 7\colon & (\Delta^+ + \Delta^-) + 8\rho \\ n = 6\colon & 2\Delta^\pm + 2\rho, 4\Delta^\pm + 4\rho, 2\Delta^\pm + 12\rho, 2\Delta^\pm + Ad + 2\rho \\ n = 5\colon & k(\Delta^+ + \Delta^-) + \ell\rho, (4k+\ell)/16, S^2\rho + (\Delta^+ + \Delta^-), \\ & Ad + 2(\Delta^+ + \Delta^-), Ad + (\Delta^+ + \Delta^-) + 4\rho \\ n = 4\colon & j\rho + k\Delta^+ + \ell\Delta^-, (j+k+\ell)/12 \\ & Ad + j\rho + k\Delta^+ + \ell\Delta^-, j + k + \ell = 6, S^2\rho + \Delta^+ + \Delta^- \\ n = 3\colon & k\rho + \ell(\Delta^+ + \Delta^-), (k+\ell)/8, \\ & Ad + k\rho + \ell(\Delta^+ + \Delta^-), k + \ell = 4 \end{cases}$$

Next let us investigate which of the above representations has $P_2(SO(m)/\psi D_n) = 0$ and $P_3(SO(m)/\psi D_n) = 0$.

Proposition 5.3D. *If $P_k(SO(m)/\psi D_n) = 0$ for $k = 1,2,3$ and $n \geq 3$, then the possibilities of such representations ψ are as follows:*
(i) *Ad, $2Ad$ for n odd,* (ii) *ρ or 2ρ*

$$\text{(iii)} \begin{cases} n = 4\colon & \Delta^+ + \Delta^- \text{ or } 2(\Delta^+ + \Delta^-) \text{ modulo conjugations} \\ n = 3\colon & \rho + (\Delta^+ + \Delta^-), (\Delta^+ + \Delta^-), 2(\Delta^+ + \Delta^-). \end{cases}$$

Proof. We start with the list of Proposition 5.1D and compute their second or third Pontrjagin classes to see if they are still vanishing. The following is a summary of such computations:

(1) In the case $\psi = 2Ad$ and n even, $P\psi^4 = 2PH^4 + PH^2 \cdot PH^2$ where $PH^2 = 2(n-1)\sigma_1$ is *not divisible* by 2 in $H^4(B_T; \mathbb{Z})$ because $AdD_n = SO(2n)/\mathbb{Z}_2$, $2(\theta_1 + \ldots + \theta_n) \in H^2(B_T; \mathbb{Z})$ but $\theta_1 + \ldots + \theta_n \notin H^2(B_T; \mathbb{Z})$. Therefore $\overline{P\psi}^4 = P\psi^4$ and $PH^4 \not\equiv 0$ (mod $\overline{P\psi}^2, \overline{P\psi}^4$).

(2) $\psi = Ad + 2(n-1)\rho$. Then $\overline{P\psi}^2 = 2(n-1)\bar\sigma_1$ and $\overline{P\psi}^4 = (2n-5)\bar\sigma_2 + [(n-1)(2n-3) + 2(n-1)^2]\bar\sigma_1^2$. Since $(2n-5) \nmid (2n-8)$ except for $n = 3,4$, it is easy to see that $P_2 \neq 0$ for $n \geq 5$. For the special cases $n = 3,4$, $P_2 \neq 0$ because of the following facts:

$$PD_4^4 = 3 \cdot 5\sigma_1^2 \not\equiv 0 \ (\text{mod } 6\bar\sigma_1)$$
$$PD_3^4 + 2\overline{P\psi}^4 \equiv 6\sigma_1^2 \not\equiv 0 \ (\text{mod } 4\bar\sigma_1).$$

(3) $\psi = S^2\rho + (2n-6)$. In this case, $P\psi^4 = 2 \cdot (2n+1)\bar\sigma_2$ modulo $\bar\sigma_1$ and hence $PH^4 = (2n-8)\bar\sigma_2 \not\equiv 0$ (mod $\overline{P\psi}^2, \overline{P\psi}^4$).

(4) $\psi = k\rho$. $P\psi^2 = k\bar\sigma_1$, $P\psi^4 = k\bar\sigma_2 + \binom{k}{2}\bar\sigma_1^2$ and

$$P\psi^6 = k\bar\sigma_3 + k(k-1)\bar\sigma_1\bar\sigma_2 + \binom{k}{3}\bar\sigma_1^3.$$

Based on the above result, it is not difficult to check that $P_2 = 0$, $P_3 = 0$ imply that $k = 1,2$. In fact $P_2 = 0$ implies that $k/(6, 2(n-1))$ and in the case $\psi = 3\rho$, $PH^6 \equiv \bar{\sigma}_1\bar{\sigma}_2 \not\equiv 0 \pmod{\overline{P\psi}^2, \overline{P\psi}^4, \overline{P\psi}^6}$.

(5) $n = 8$, $\psi = \Delta^{\pm} + 12\rho$. $\overline{P\psi}^2 = 14\bar{\sigma}_1$, $P\psi^4 = 4\bar{\sigma}_2 + 384\bar{\sigma}_1^2$, $PH^4 = = 8\bar{\sigma}_2 + 7 \cdot 13\bar{\sigma}_1^2$, $PH^4 - 2P\psi^4 = 7 \cdot 13\bar{\sigma}_1^2 - 2 \cdot 384\bar{\sigma}_1^2 \equiv 2\bar{\sigma}_1^2 \pmod 7$. Therefore, $P_2 \neq 0$.

(6) $n = 7$, $\psi = (\Delta^+ + \Delta^-) + 8\rho$; $n = 6$, $\psi = 2\Delta^{\pm} + 2\rho$, $4\Delta^{\pm} + 4\rho$, $2\Delta^{\pm} + 12\rho$ or $n = 5$, $k(\Delta^+ + \Delta^-) + \ell\rho$, $(4k + \ell)/16$ and ℓ even. In all the above cases, the representation factor through $SU\left[\dfrac{m}{2}\right]$. Therefore, it follows from reduction 3 and the computation of Proposition (3.2D) that $P_2 \neq 0$.

(7) $n = 6$. $\psi = 2\Delta^{\pm} + Ad + 2\rho$: Then $\overline{P\psi}^2 = 10\bar{\sigma}_1$ and $P\psi^4 = 2\bar{\sigma}_2 + + 193\bar{\sigma}_1^2$, $PH^4 - 2P\psi^4 \equiv 0 \pmod{10\bar{\sigma}_1}$ and hence $P_2 \neq 0$.

(8) $n = 5$. $\psi = S^2\rho + (\Delta^+ + \Delta^-)$, $Ad + 2(\Delta^+ + \Delta^-)$ or $Ad + (\Delta^+ + \Delta^-) + + 4\rho$. In the above three cases, straightforward computations will show that $P_2 \neq 0$.

(9) $n = 4$. This case is rather special because Spin(8) has an outer automorphism group of S_3-type and the coefficient of $\bar{\sigma}_2$ in PD_4^4 is zero. Recall that (cf. Proposition 3.3D).

$$\prod_{w \in \Omega(\Delta^+)} (1 + w) = 1 - \bar{\sigma}_1 + \left(-\frac{1}{2}\bar{\sigma}_2 + 3\chi + \frac{3}{8}\bar{\sigma}_1^2\right) - \left(\bar{\sigma}_3 - \frac{1}{4}\bar{\sigma}_1\bar{\sigma}_2 + \frac{1}{16}\bar{\sigma}_1^3 + \frac{1}{2}\sigma_1\chi\right)$$

$$\prod_{w \in \Omega(\Delta^-)} (1 + w) = 1 - \bar{\sigma}_1 + \left(-\frac{1}{2}\bar{\sigma}_2 - 3\chi + \frac{3}{8}\bar{\sigma}_1^2\right) - \left(\bar{\sigma}_3 - \frac{1}{4}\bar{\sigma}_1\bar{\sigma}_2 + \frac{1}{16}\bar{\sigma}_1^3 - \frac{1}{2}\bar{\sigma}_1\chi\right)$$

Based on the above result, it is tedious but not difficult to check that $P_2 = 0$, $P_3 = 0$ imply that $\psi = \Delta^+ + \Delta^-$ or $2(\Delta^+ + \Delta^-)$ modulo conjugations of outer automorphism (of course, also $\Delta^{\pm}$ or $2\Delta^{\pm}$ which conjugate to ρ or 2ρ).

(10) $n = 3$. $\mathrm{Spin}(6) = SU(4)$, $\rho = \Lambda^2\mu_4$, $\Delta^+ = \mu_4$, $\Delta^- = \bar{\mu}_4$. Direct computation will show that, $\rho, 2\rho, (\Delta^+ + \Delta^-), 2(\Delta^+ + \Delta^-)$ and $\rho + (\Delta^+ + \Delta^-)$ are the only possibilities with $P_2 = 0$, $P_3 = 0$.

(E) *The case H is exceptional.*

Proposition 5.3E. *If H is an exceptional compact connected Lie group and $\psi: H \to SO(m)$ be a real representation with $P_k(SO(m)/\psi H) = 0$ for $k = 1,2,3$, then the possibilities of such pairs (H, ψ) are as follows modulo trivial representations:*

(i) $Ad, 2Ad$, (ii) $H = G_2$, $\psi = k\varphi_1$, $k = 1,2,4$.

The proof of the above proposition is essentially the same as that of Proposition 3.3E.

(F) Based on all the above results for the case H is simple, we are, finally, ready to prove the following main theorem of this section.

Theorem 3. *Let $\psi: H \to SO(m)$ be a compact connected Lie group with a given real representation ψ. If $P_k(SO(m)/\psi H) = 0$ for $k = 1,2,3$, then the possibilities of all such pairs (H, ψ) are given by the following list modulo trivial representations:*

(i) *H is any given subtorus,*
(ii) *H is semi-simple and $\psi = Ad_H$,*
(iii) *H is semi-simple without simple normal factors of B_n, C_{2n} or D_{2n} type, $\psi = 2Ad_H$,*
(iv) $H = [Sp(1)]^r$, $\psi = k(v_1^{(1)} + \ldots + v_1^{(r)})$, $k/8$

(v) $\begin{cases} H = SO(n), & \psi = k \cdot \rho_n\, k/(2,n) \\ H = Sp(n), & \psi = 2v_n \end{cases}$

(vi) $\begin{cases} H = SU(3) \times SU(3), & \psi = k \cdot [\mu_3 + \bar{\mu}_3 + \mu_3' + \bar{\mu}_3'],\ k = 1,2 \\ H = G_2 \times G_2, & \psi = k(\varphi_1 + \varphi_1'),\ k = 1,2 \end{cases}$

(vii) $\begin{cases} H = SU(n),\ n = 3,4,5,\ \psi = k(\mu_n + \bar{\mu}_n),\ k = 1,2 \\ H = G_2,\ \psi = k\varphi_1,\ k/4. \\ H = Sp(2),\ \psi = 2\Lambda^2 v_2 + 2v_2 \\ H = Spin(8),\ \psi = k(\Delta^+ + \Delta^-),\ k = 1,2 \\ H = SU(4),\ \psi = \Lambda^2\mu_4 + (\mu_4 + \bar{\mu}_4) \\ H = SU(2),\ \psi = 4\mu_2 + Ad \end{cases}$

Proof. The proof of the above theorem is quite similar to that of Theorem 1. It is easy to show that H is either abelian or semi-simple. The cases of simple Lie groups have already been treated separately in Proposition 5.3A,B,C,D,E. Therefore, one need only to investigate the cases that H are semi-simple but non simple.

(1) Suppose $H = H_1 \times \ldots \times H_a$ is a product of simple Lie groups and $\psi_1 = \psi \mid H_1 = 2Ad_{H_1}$. Then it follows from Lemma 5.3 that

$$L(\psi \mid H_i) = 2L(Ad_{H_i}), \text{ for } i \leq i \leq a.$$

On the other hand, it follows from Reduction 2 and the lists of Proposition 5.3A,B,C,D,E that either $\psi_i = 2Ad_{H_i}$ or $H_i = SU(2)$ and $\psi_i = 8\mu_2$ or $Ad + + 4\mu_2$. We claim that it is impossible to have any $\psi_i = 8\mu_2$ or $Ad + 4\mu_2$. Suppose the contrary that $(H_2, \psi_2) = (SU(2), 8\mu_2)$ or $(SU(2), Ad + 4\mu_2)$. Then, by reduction 2, we may reduce to the case $H = H_1 \times SU(2)$ and $\psi = 2Ad_{H_1} + 8\mu_2$ or $2Ad_{H_1} + S^2\mu_2 + 4\mu_2$. Simple computations will show that, in both of the above cases, $P_2(SO(m)/\psi H) \neq 0$, which is a contradiction to the assumption. Therefore, in case $\psi_1 \mid H_1 = 2 \cdot Ad_{H_1}$, $\psi = 2 \cdot Ad_H$, and it follows from Proposition 5.3B,C,D that no H_i can be of B_m, C_{2n} or D_{2n} type.

(2) Suppose $H = H_1 \times \ldots \times H_a$ is a product of simple Lie groups and $\psi_1 = \psi \mid H_1 = Ad_{H_1}$. Then, again it follows from Lemma 5.3 that $L(\psi \mid H_i) = L(Ad_{H_i})$ for $1 \leq i \leq a$. On the other hand, it follows from Reduction 2 and the lists of Proposition 5.3A,B,C,D,E that either $\psi_i = = Ad_{H_i}$ or $\psi_i = 2\Lambda^2 v_2 + 2v_2$ when $H_i = Sp(2)$, or $\psi_i = 4\varphi_1$ when $H_i = G_2$, or $\psi_i = 4\mu_2$ when $H_i = SU(2)$. Again, it is not difficult to show that ψ_i must be Ad_{H_i}. For otherwise, one would have $P_2(SO(m)/\psi H) \neq 0$.

The rest of the proof is essentially parallel to that of Theorem 1.

Section 6. Characteristic class and local orbit structure.

The characteristic class theory of equivalent bundles over homogeneous spaces provides a powerful tool in the study of local orbit structure, especially in the setting of differentiable compact transformation groups. Suppose G/H occurs as an orbit in a given G-manifold M. Then G/H imbeds in M with an *equivariant* normal bundle which is associated with the slice representation, Φ_x, of G_x on the normal vectors of $G(x)$ at x. Therefore, the restriction of the tangent bundle of M to the orbit $G(x)$ is given by the following equation:

$$i^! \tau(M) = \tau(G(x)) + v(G(x)) = \alpha(Ad_G \mid H - Ad_H) + \alpha(\Phi_x).$$

Hence, it is rather natural to apply the splitting principle of Borel-Hirzebruch to evaluate the above equation at characteristic class level which provides strong restrictions on the possibilities of the orbit type as well as the slice representation.

(A) *Slice representation and principal orbit type.*

Let G be a compact Lie group and M be a *connected* manifold with given differentiable G-action. The well-known principal orbit type theorem of Montgomery-Samelson-Yang proves that there exists a *unique minimal conjugacy class of isotropy subgroups* in $\{G_x; x \in M\}$ (with respect to the natural partial ordering induced by inclusion), which corresponds to the unique maximal orbit type in M called the *principal orbit type* of M; the totality of principle orbits consists of an open dense submanifold with connected image in the orbit space M/G. Therefore, the principal orbit type of a given G-manifold M is an invariant of dominant importance. We shall denote the conjugacy class of principal isotropy subgroups by (H_M) and that of their connected component of identity by (H_M^0). It follows from the slice structure that $G(x) = G/G_x \subset M$ is a principal orbit if and only if the slice representation Φ_x is a trivial representation. Therefore, a principal orbit, G/H_M, imbeds in M with *trivial normal* bundle. In particular, if $P_k(M) = 0$ for $k = 1,2,3$, then it follows that $P_k(G/H_M) = 0$ and hence also $P_k(G/H_M^0) = 0$ for $k = 1,2,3$. Hence, as an immediate consequence of Theorems 1, 2, 3 of §3, 4, 5, one has the following theorem on the possibilities of connected principal isotropy subgroup type for actions of classical groups on manifolds with vanishing first three Pontrjagin classes.

Theorem 4. *Let G be $SU(m)$ (resp. $Sp(m)$, $SO(m)$) and M be a given G-manifold with $P_k(M) = 0$, $k = 1,2,3$. Then the possibilities of connected principal isotropy subgroups (H_M^0) is either (Id) or those listed in Theorem 1 (resp. Theorem 2, Theorem 3).*

Remark. In the generic case of $H_M = Id$, examples of various orthogonal linear G-spaces already show that any homogeneous space of G may occur as an orbit type of a suitable linear G-space. However, in case $H_M^0 \neq Id$ and $P_k(M) = 0$ for $k = 1,2,3$, then we shall show that the principal orbit type imposes a strong restriction on the possibilities of all other orbit types of such a G-manifold M.

(B) *Local orbit structure of G-manifolds with non-trivial principal isotropy subgroup type.*

Let M be a G-manifold with $(H_M) \neq Id$, and $G(x) = G/G_x$ be an arbitrary orbit in M. Then, it follows from the slice structure around the orbit $G(x)$, one has the following restrictions on the possibilities of (G_x, Φ_x):

(i) up to conjugation, $G \supset G_x \supset H_M$,
(ii) the principal isotropy subgroup type of Φ_x is conjugate to H_M,
(iii) $i^!\tau(M) = \tau(G(x)) + \nu(Gx)) = \alpha(Ad_G | G_x - Ad_{G_x}) + \alpha(\Phi_x)$.

As one shall see later, in the case (H_M) is non-trivial and given, the conditions (i) and (ii) are rather restrictive on the possibilities of (G_x, Φ_x). Therefore, we shall first apply Lie group representation theory to conditions (i) and (ii) to control the possibilities of such (G_x, Φ_x), and then apply characteristic class theory to equation (iii) to decide the possibilities of orbit types as well as slice representations. For this purpose, the following lemmas are rather useful.

Lemma 6.1. *Let G be a compact Lie group, Φ be an orthogonal representation of G and (H_Φ) be the conjugacy class of principal isotropies of the linear G-action given by Φ. Then, one has the following equation of restricted representations, namely,*

(1.1) $\quad \Phi | H_\Phi = (Ad_G | H_\Phi - Ad_{H_\Phi}) +$ *trivial representations.*

Proof. Let x be a point in the representation space $V(\Phi)$ with $G_x = H_\Phi$. In restricting the given G-action to H_Φ, the local representation of H_Φ at the origin 0 is by definition $\Phi | H_\Phi$. On the other hand, the local representations of H_Φ at x splits into the sum of the tangent part and the normal part, namely, $(Ad_G | H_\Phi - Ad_{H_\Phi})$ and Φ_x which is trivial because $G_x = H_\Phi$ is principal. Now, the *linearity* of H_Φ-action clearly guarantees that the translation by $\overrightarrow{0x}$ is H_Φ-equivariant. Hence the local representations of H_Φ at 0 and x are equal.

Remark. (i) Lemma 6.1 is a special case of the following equation which relates the slice representation at x to the restricted representation, namely,

(1.2) $\quad \Phi | G_x = (Ad_G | G_x - Ad_{G_x}) + \Phi_x$

where Φ_x is the slice representation of G_x on normal vectors to the orbit $G(x)$ at x.

(ii) For a given simple group G, equation (1.1) is extremely restrictive that there are only few representations (modulo trivial reps) which are possible to satisfy equation (1.1) with *non-trivial H*.

Technically, it is convenient to combine equation (1.2) with Schur lemma to formulate the following lemma in terms of weight system.

Definition. Let Φ be a representation of G and T be a sub-p-torus of G ($p = 0$ if T is a connected torus). Then $\Phi | T$ splits into direct sum of complex (real if $p = 2$) one dimensional representations. The system of weights of such representations is called the weight system of Φ with respect to T, denoted by $\Omega(\Phi; T)$.

Lemma 6.2. *Suppose $\Omega(\Phi; T)/\Omega(Ad_G; T)$ is non-empty and*

$$w \in \{\Omega(\Phi;\ T)/\Omega(Ad_G;\ T)\}.$$

Then there exist $x \in V(\Phi)$ such that $\ker w = G_x \cap T$ and $rkp(G_{gx} \cap T) < rkpT$ for all $g \in G$.

Proof. If there exists $y_0 \in V(\Phi)$ such that $T \subseteq G_{y_0} \subsetneq G$, then it follows from equation (1.2) that

$$\Omega(\Phi;\ T)\backslash\Omega(Ad_G;\ T) = \Omega(\Phi_{y_0};\ T)\backslash\Omega(Ad_{G_{y_0}};\ T).$$

Therefore, one may reduce the proof of the above lemma to the case that $rk_p(G_x \cap T) < rk_p T$ for all $0 \neq x \in V(\Phi)$. In this case, the assertion of the above lemma clearly follows from the definition of weight system.

Lemma 6.2. provides an effective inductive procedure of finding isotropy subgroups of lower p-rank if $\{\Omega(\Phi;\ T)\backslash\Omega(Ad_G;\ T)\}$ contains non-zero elements. Combining this with Lemma 1.1, it is not difficult to prove that $\Omega'(\Phi)$ must be rather simple in order to have *non-trivial* principal isotropy subgroups. We refer to $[HH2]$ for actual proofs of the following final results of classification.

Theorem 5 $[HH2]$. *Let G be a simple compact connected Lie group and Φ be an orthogonal G-action on $V(\Phi)$ with non-trivial principal isotropy subgroups (H_Φ). Then Φ is one of the following four types of representations modulo trivial representations:*

Type I – *Regular actions* (*Table* I)
Type II – *Actions of adjoint type* (*Table* II)
Type III – *Actions of near adjoint type* (*Table* III)
Type IV – *Actions of mixed type* (*Table* IV)

Table I Regular Actions

G	φ	$(H\varphi)$	*Remark*
$SO(n)$	$k\rho_n$	$SO(n-k)$	$k \le (n-2)$, $\dim \rho_n = n$
$SU(n)$	$k[\mu_n]_{\mathbb{R}}$	$SU(n-k)$	$k \le (n-2)$; $\dim_{\mathbb{C}} \mu_n = n$
$Sp(n)$	$k[\nu_n]_{\mathbb{R}}$	$Sp(n-k)$	$k \le (n-1)$; $\dim_{\mathbb{C}} \nu_n = 2n$
G_2	$k\varphi_1$	$SU(3), SU(2)$	$k = 1, 2$; $\dim \varphi_1 = 7$
$Spin(7)$	$k \cdot \Delta_7$	$G_2, SU(3), SU(2)$	$k = 1, 2, 3$; $\dim \Delta_7 = 8$
$Spin(9)$	$k \cdot \Delta_9$	$Spin(7)^{\pm}$, $SU(3)$	$k = 1, 2$; $\dim \Delta_9 = 16$
$Sp(3)\mathbb{Z}_2$	$k(\Lambda^2\nu_3 - 1)$	$Sp(1)^3, S^1$	$k = 1, 2$; $\dim(\Lambda^2\nu_3 - 1) = 14$
F_4	$k \cdot \varphi_1$	$Spin(8)$, $SU(3)$	$k = 1, 2$; $\dim \varphi_1 = 26$

Table II Actions of Adjoint Type

G	φ	$H\varphi$	$\dim \varphi$
$G/Z(G)$	Ad_G	T	$\dim G$
$SO(n)$	$(S^2\rho_n - 1)$	$\mathbb{Z}_2^{(n-1)}$	$\frac{n(n+1)}{2} - 1$
$Sp(n)/\mathbb{Z}_2$	$(\Lambda^2\nu_n - 1)$	$Sp(1)^n/\mathbb{Z}_2$	$n(2n-1) - 1$
$Sp(4)/\mathbb{Z}_2$	$(\Lambda^4\nu_4 - \Lambda^2\nu_4)$	$\mathbb{Z}_2^6$	42
$SU(8)/\mathbb{Z}_4$	$\Lambda^4\mu_8$	$\mathbb{Z}_2^7$	70
$Spin(16)/\mathbb{Z}_2$	$\Delta_{16}^{\pm}$	$\mathbb{Z}_2^8$	128

Table III Actions of Near Adjoint Type

G	φ	$(H\varphi)$	$\dim \varphi$
$SU(n)$	$[\Lambda^2\mu_n]_{\mathbb{R}}$	$SU(2)^{n/2}$	$n(n-1)$
$SU(n)$	$[S^2\mu_n]_{\mathbb{R}}$	$\mathbb{Z}_2^{(n-1)}$	$n(n+1)$
$SU(6)$	$[\Lambda^3\mu_6]_{\mathbb{R}}$	T^2	40
$SP(3)$	$[\Lambda^3\nu_3 - \nu_3]_{\mathbb{R}}$	$\mathbb{Z}_2^2$	28
$Spin(10)$	$(\Delta_{10} + \bar{\Delta}_{10})$	$SU(4)$	32
$Spin(12)/\mathbb{Z}_2$	$[\Delta_{12}^{\pm}]_{\mathbb{R}}$	$SU(2)^3$	64
E_6	$[\varphi_1]_{\mathbb{R}}$	$Spin(8)$	54
E_7	$[\varphi_1]_{\mathbb{R}}$	$Spin(8)$	112

Table IV Actions of Mixed Type

G	φ	$H\varphi$	$\dim \varphi$
$SU(4)$	$[\mu_4]_{\mathbb{R}} + \Lambda^2\mu_4$	$SU(2)$	$8 + 6$
$Spin(7)$	$k\Delta_7 + \ell\rho_7$ $k + \ell = 2, 3$	$SU(3)$ $SU(2)$	$8 + 7$ $8k + 7\ell$, $k + \ell = 3$
$Spin(8)$	$k\Delta_8^+ + \ell\Delta_8^- + m\rho_8$ $k + \ell + m = 2, 3, 4$	G_2 $SU(3)$ $SU(2)$	16 24 32
$Spin(9)$	$\Delta_9 + \rho_9$ $\Delta_9 + 2\rho_9$ $\Delta_9 + 3\rho_9$ $2\Delta 9 + \rho_9$	G_2 $SU(3)$ $SU(2)$ $SU(2)$	25 34 43 41
$Spin(9)$	$[\Delta_{10}^{\pm}]_{\mathbb{R}} + \rho_{10}$ $[\Delta_{10}^{\pm}]_{\mathbb{R}} + 2\rho_{10}$	$SU(3)$ $SU(2)$	42 52

Lemma 6.3. *Let G be a compact Lie group and M be a given connected G-manifold. Then the ranks of isotropy subgroups $\{rk(G_x);\ x \in M\}$ have no gap as a set of integers.*

Proof. Let (H_M) be the principal orbit type. Suppose $rk\, G_x = rk\, H_\phi$ for all $x \in M$. Then there is nothing to prove. Suppose x is such a point with $rk\, G_x > rk\, H_\phi$. We claim that there always exists y with $rk\, G_y = rk\, G_x - 1$. We may assume that G_x is already a *local minimal* among isotropy subgroups of that given rank. Then, by restricting the G_x-action on the slice S_x to a maximal torus T of G_x, it is not difficult to see (cf. Lemma 6.2) that there exist $y \in S_x$ with $rk\, G_y = rk\, G_x - 1$.

Remark. In fact, the above lemma still holds for topological actions on connected rational cohomology manifold; one simply applies the Borel formula [*ST*, p. 175] to the above T-action on the slice S_x.

(C) *Reduction to connected isotropy subgroup type.*

In order to simplify some computations, we shall first study the local orbit structure only up to the connected isotropy subgroup type, namely, G_x^0 and $\Phi_x^0 = \Phi_x \mid G_x^0$. Let π: $G/G_x^0 \to G/G_x = G(x) \subset M$. Then one may pull back the tangent bundle and normal bundle of the orbit $G(x)$ up to G/G_x^0, namely

$$\pi^! i^! \tau M = \pi^!(\tau G(x) + \alpha(\Phi_x)) = \alpha_{G,G_x{}^0}(Ad_G \mid G_x^0 - Ad_{G_x{}^0} + \Phi_x^0).$$

(D) In the case that we shall consider in this paper, the manifold M is assumed to satisfy some vanishing condition. Therefore, the left hand side of the above equation, $\pi^! i^! \tau M$, always satisfies the same kind of vanishing condition. In the general situation, in order to apply the above equation to the determination of possibilities of orbit type, one needs to investigate the pull-back $\pi^! i^! \tau M$. The following simple observation is a useful first step for such an investigation.

Observation. Let G/H be the principal orbit type of a given connected G-manifold M. Then all imbeddings of G/H into M as principal orbits are obviously homotopic, because the principal orbit type theorem asserts that $G/H \to M_{(H)} \to M_{(H)}/G$ is a bundle over a *pathwise connected* base space. Moreover, it follows easily from the slice structure and the fact that principal orbits are everywhere dense that

$$G/H \subset M \quad \text{and} \quad G/H \to G/G_x = G(x) \subset M$$

are also homotopic for *any* $x \in M$.

As a consequence of the above observation, the image of $H^*(M) \to H^*(G/H)$ is included in the intersection of all images of $H^*(G/G_x) \to H^*(G/H)$.

Proposition 6.1. *Suppose the fundamental class of $H^*(G/H; \mathbb{Q})$ lies in the image of $H^*(M; \mathbb{Q}) \to H^*(G/H; \mathbb{Q})$ for the imbedding of the principal orbit G/H. Then all orbits in M are of the same dimension as that of G/H and $M \cong F(H, M) \times_W G/H \to G/N(H)$ is naturally a fibration associated to the W-bundle: $W = \dfrac{N(H)}{H} \to G/H \to G/N(H)$ with $F(H, M)$ as its fibre.*

Proof. Let $G(x)$ be an arbitrary orbit, Φ_x be the slice representation of G_x on the slice S_x at x. Then it follows from the slice structure that H is also the principal isotropy subgroup type of Φ_x and

$$\begin{array}{ccc} G/H & \hookrightarrow & M \\ \downarrow & & \uparrow \\ G/G_x & \cong & G(x) \end{array}$$

is homotopic commutative. Therefore

$$\begin{array}{ccc} H^*(G/H; \mathbb{Q}) & \longleftarrow & H^*(M; \mathbb{Q}) \\ & \nwarrow \quad \swarrow & \\ & H^*(G(x); \mathbb{Q}) & \end{array}$$

is also commutative and it follows from the assumption on fundamental class that $G(x)$ and G/H must be of the same dimension. Hence $G_x^0 = H^0$. Since $H^0 = G_x^0$ is a normal subgroup of G_x and (H) is also the principal isotropy subgroup type of the slice representation Φ_x, it is easy to see that $H^0 \subset \operatorname{Ker} \Phi_x$, namely, $\Phi_x : G_x \to G_x/G_x^0 \xrightarrow{\tilde{\Phi}_x} 0(k)$, $h = \dim S_x$. Hence (H/H^0) is the principal isotropy subgroup type of the representation, $\tilde{\Phi}_x$, of G_x/G_x^0. Then, it follows from Lemma (6.1) and the finiteness of G_x/G_x^0 that $\operatorname{Ker} \tilde{\Phi}_x = H/H^0$ and hence H is itself also normal in G_x, namely $H \subset G_x \subset N(H)$.

Let $Y = F(H, M)$. Then Y intersects every orbit in M and hence $G \times Y \subset G \times M \to M$ is onto. It is obvious that the above map factors through $G \times Y \to G/H \times_W Y$ where $W = \dfrac{N(H)}{H}$. Since $H \subset G_x \subset N(H)$ for all $x \in M$, it is not difficult to check that $G/H \times_W Y \cong M$ and hence $M \cong G/H \times_W Y \to G/N(H)$ is a fibration.

Proposition 6.2. *Let M be a given G-manifold with G/H as its principal orbit type. Let P (resp. W) be the subring of $H^*(G/H; \mathbb{Z})$ (resp. $H^*(G/H; \mathbb{Z}_2)$) generated by its Pontrjagin classes (resp. Stiefel-Whitney classes) together* *with cohomology operations. Let* I *(resp.* I_2*) be the image of* $H^*(M;\mathbb{Z}) \to \to H^*(G/H;\mathbb{Z})$ *(resp.* $H^*(M;\mathbb{Z}_2) \to H^*(G/H;\mathbb{Z}_2)$ *and*

$$J = \cap\{Im(H^*(G/G_x;\mathbb{Z}) \to H^*(G/H;\mathbb{Z}); x \in M\}$$
$$J_2 = \cap\{Im(H^*(G/G_x;\mathbb{Z}_2) \to H^*(G/H;\mathbb{Z}_2);\ x \in M\}.$$

Then $P \subset I \subset J$ *and* $W \subset I_2 \subset J_2$.

Proof. Since G/H imbeds in M with trivial normal bundle, it is clear that $P \subset I$, $W \subset I_2$.

On the other hand, it follows from the fact that $G/H \subset M$ and $G/H \to \to G/G_x \subset M$ are homotopic that $I \subset J$ and $I_2 \subset J_2$.

Remark. The above propositions indicate that non-vanishing characteristic classes of the principal orbit type G/H, in fact, impose strong restrictions on the orbit structure and the topology of the total space M.

Section 7. Orbit strucutures of SU(m) actions on manifolds with vanishing characteristic classes.

In this section, we shall study the possibilities of local orbit structures of $SU(m)$-manifolds M with $P_k(M) = 0$ for $k = 1, 2, 3$, and $(H_M^0) \neq \{Id\}$. In view of Theorems 1 and 4, (H_M^0) must be one of the type of linear subgroups of $SU(m)$ listed in Theorem 1. Hence, we shall divide our discussion of this section according to the type of (H_M^0).

(A) *The case* $H_M^0 = \mu_n(SU(n))$*; or* $2\mu_n(SU(n))$, $(n \geq 3)$.

Let us first consider the case $H_M^0 = \mu_n(SU(n))$. Suppose G_x^0 is an arbitrary connected isotropy subgroup. Then, up to conjugation, $G_x^0 \supset H_M^0 = = \mu_n(SU(n))$. It is easy to see that there exists a unique *simple* normal subgroup G_1 of G_x^0 containing H_M^0, and moreover, it is not difficult to determine the possibilities of G_1 and the representation $\psi: G_1 \subset SU(m)$ as follows:

Lemma 7.1. *Let* $(G_1, SU(n))$ *be a pair of compact connected simple Lie groups and* $\psi: G_1 \to SU(m)$ *is a representation* G_1 *with* $\psi\,|\,SU(n) = \mu_n$ *modulo trivial representations. Then the possibilities of* (G_1, ψ) *are given as follows:*

$$G_1 = \begin{cases} SU(\ell) \\ Sp(\ell) \text{ for the case } n = 2 \end{cases}, \quad \psi = \begin{cases} \mu_\ell \\ \nu_\ell \end{cases} \text{ modulo trivial reps.}$$

Proof. It is obvious that ψ must be irreducible modulo trivial representations, and the above assertion follows from a straightforward check of representation theory of simple compact connected Lie groups.

Theorem A1. *Let* $G = SU(m)$ *and* M *be a differentiable* G*-manifold with* $P_1(M) = 0$*. If* $H_M^0 = \mu_n(SU(n))$*, then all connected isotropy subgroups of* M*,* G_x^0*, are also of the type* $\mu_\ell(SU(\ell))$*,* $\ell \geq n$*, or in the special case* $n = 2$*, may also have the additional type* $\nu_\ell(Sp(\ell))$*.*

Proof. Suppose the contrary that there exist some $x \in M$ such that G_x^0 are not of the type $\mu_\ell(SU(\ell))$ or $\nu_\ell(Sp(\ell))$ for the case $n = 2$. Let x be a point such that G_x^0 is *minimal* among such isotropy subgroups, and $\bar{\Phi}_x$ be the reduced slice representation of G_x^0 excluding possible trivial representation. Then $\bar{\Phi}_x$ is a representation of G_x^0 whose proper isotropy subgroups are all of the type of $\mu_\ell(SU(\ell))$ or $\nu_\ell(Sp(\ell))$ (for the case $n = 2$ only), and the connected principal isotropy subgroup type of $\bar{\Phi}_x$ is $\mu_n(SU(n))$. On the other hand, it follows from Lemma 6.1 and representation theory that

$$\bar{\Phi}_x\,|\,G_1 = \begin{cases} (\ell - n)\,[\mu_\ell]_{\mathbb{R}} \\ (\ell - 1)\,[\nu_\ell]_{\mathbb{R}} \text{ (for } n = 2 \text{ only)} \end{cases} \text{(modulo trivial reps).}$$

Then, it is not difficult to see that $G_x^0 \sim SU(n) \times K$, $rk(K) = 1$ and $Ker\bar{\Phi}_x = = SU(n)$, for otherwise, there are proper isotropy subgroups in the slice not of the μ_ℓ or ν_ℓ types and hence G_x^0 is not minimal among such isotropy subgroups. Now, we shall divide the proof into two cases according to $K \sim SU(2)$ or $K \sim S^1$.

(i) $K \sim SU(2)$. Then the representation $\psi: SU(n) \times K \to G_x^0 \subset SU(m)$ is of the form $\psi = \mu_n + \psi_2$ where ψ_2 is a representation of K only, i.e. Ker $\psi_2 \supset SU(n)$. Let $\Omega'(\psi_2) = \{\pm w_i\lambda\}$ and $\Omega'(\Phi_x) = \{\pm h_j\lambda\}$ where λ is is the parameter of the Cartan subalgebra of $K = SU(2)$. Then

$$P\psi^2 = \sigma_2\ (\Sigma w_i^2) \cdot \lambda^2.$$

On the other hand, $\tau + \nu = -\alpha(Ad_{SU(n)} + Ad_K) + \alpha(\phi_x)$ and

$$\pi^* P_1(\tau + \nu) = i^*\{-2n\sigma_2 + 4\lambda^2 - (\Sigma h_j^2) \cdot \lambda^2\} =$$
$$= i^*\{(-2n\Sigma w_i^2 - \Sigma h_j^2 + 4) \cdot \lambda^2\} \neq 0$$

which is a contradiction to the assumption $P_1(M) = 0$.

(ii) $K \sim S^1$. Then the representation ψ: $SU(n) \times S^1 \to G_x^0 \subset SU(m)$ is of the form $\psi = \mu_n \otimes \xi + \psi_2$ where ξ and ψ_2 are representations of S^1 only. Let $\Omega(\xi) = \{a\lambda\}$, $\Omega'(\psi_2) = \{w_i\lambda\}$ and $\Omega'(\phi_x) = \{\pm h_j\lambda\}$. Then $S\psi^1 = (na + \Sigma w_i) \cdot \lambda = 0$; $\quad S\psi^2 = (na^2 + \Sigma w_i^2) \cdot \lambda^2 - 2\sigma_2$. Again, $\tau + \nu = = -\alpha(Ad_{SU(n)}) + \alpha(\Phi_x)$ and hence

$$\pi^* P_1(\tau + \nu) = i^*\{-2n\sigma_2 - \Sigma h_j^2\lambda^2\} =$$
$$= i^*\{-[n(na^2 + \Sigma w_i^2) + \Sigma h_j^2] \cdot \lambda^2\} \neq 0$$

which is again a contradiction to $P_1(M) = 0$.

All the above contradictions prove that G_x^0 is of the type $\mu_\ell(SU(\ell))$ or, in the case $n=2$, $\nu_\ell(Sp(\ell))$.

Next let us consider the case $H_M^0 = 2\mu_n(SU(n))$. Here, there are the following two possibilities:

(i) The *minimal normal* subgroup of G_x^0 containing H_M^0 is *non-simple.*

(ii) The *minimal normal* subgroup of G_x^0 containing H_M^0 is *simple*. In the first case and $n \geq 3$, it is easy to reduce to the previous case of μ_n to conclude that $SU(n) \subset SU(\ell_1) \times SU(\ell_2) \xrightarrow{\mu_{\ell_1}+\mu_{\ell_2}} SU(m)$, or in the case $n=2$, $SU(2) \subset Sp(\ell_1) \times Sp(\ell_2) \xrightarrow{\nu_{\ell_1}+\nu_{\ell_2}} SU(m)$, or $SU(2) \subset SO(4) \subset SU(m)$.

Lemma 7.2. *Let $(G_1, SU(n))$ be a pair of compact connected simple Lie groups and $\psi: G_1 \to SU(m)$ is a representation of G_1 with $\psi \mid SU(n) = 2\mu_n$ modulo trivial representations. Then the possibilities of such (G_1, ψ) are as follows modulo trivial representations:*

(i) *$G_1 = SU(\ell)$, $\psi = 2\mu_\ell$ or μ_ℓ, and in the special case $n=2$ one has the following additional possibilities:*

$$\text{(ii) } G_1 = \begin{cases} Sp(\ell) \\ SO(\ell), \\ G_2 \end{cases} \qquad \psi = \begin{cases} \nu_\ell \text{ or } 2\nu_\ell \\ \rho_\ell \\ \Phi_1, \ \dim \Phi_1 = 7 \end{cases}$$

Theorem A2. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$. If $H_M^0 = 2\mu_n(SU(n))$, $n \geq 3$, then all connected isotropy subgroups of M are of the type $2\mu_\ell(SU(\ell))$ with $\Phi_x = (\ell - n)[\mu_\ell]_{\mathbb{R}}$.*

Proof. Let G_x^0 be an arbitrary connected isotropy subgroup and $\tilde{G}$ be the minimal normal subgroup of G_x^0 containing H_M^0, Φ_x be the slice representation of G_x^0. Then the possibilities of $SU(n) \subset \tilde{G} \subset SU(m)$ are as follows:

(i) $SU(n) \xrightarrow{\mu_n} SU(\ell) \xrightarrow{2\mu_\ell} SU(m)$,

(ii) $SU(n) \xrightarrow{2\mu_n} SU(\ell) \xrightarrow{\mu_\ell} SU(m)$,

(iii) $SU(n) \xrightarrow{\mu_n \otimes \mu_n'} SU(\ell_1) \times SU(\ell_2) \xrightarrow{\mu_{\ell_1}+\mu_{\ell_2}} SU(m)$.

The latter two cases can *not* occur because it is then impossible to have a representation of $\tilde{G}$ with the prescribe subgroup $SU(n)$, $n \geq 3$, as its principal isotropy subgroup type. Therefore the only possibility left is $SU(n) \xrightarrow{\mu_n} SU(\ell) \xrightarrow{2\mu_\ell} SU(m)$.

(2) We claim $G_x^0 = \tilde{G} = 2\mu_\ell(SU(\ell))$. Suppose the contrary. Then, the same reason as that of the proof of Theorem A1 will show that there exist $G_x^0 \sim SU(n) \times K$, $rk\, K = 1$. Again, we can divide into two cases to deduce contradiction. If $K \sim SU(2)$, then $\psi: SU(n) \times SU(2) \to G_x^0 \subset SU(m)$ is of the form $\psi = 2\mu_n + \psi_2$ (or $\mu_n \otimes \mu_2' + \psi_2$), $\Omega'(\psi_2) = \{\pm w_i \lambda\}$ and $\Omega'(\Phi_x) = \{\pm h_j \lambda\}$. Then

$$P\psi^2 = 2\sigma_2 - (\Sigma w_i^2) \cdot \lambda^2 \text{ and } \pi^* P_1(\tau + \nu) = i^* \{(-n \cdot \Sigma w_i^2 - \Sigma h_j^2 + 4) \cdot \lambda^2\} \neq 0,$$

because Φ_x is a real representation of $SU(2)$, $\Sigma h_j^2 \geq 2$. If $K \sim S^1$ then $\psi: SU(n) \times S^1 \to G_x^0 \subset SU(m)$, $\psi = \mu_n \otimes \xi_1 + \mu_n \otimes \xi_2 + \psi_2$, $\Omega'(\psi_2) = \{w_i \lambda\}$, $\Omega'(\Phi_x) = \{\pm h_j \lambda\}$. Then $S\psi^1 = \{n(a+b) + \Sigma w_i\} \cdot \lambda = 0$, $S\psi^2 = (na^2 + nb^2 + \Sigma w_i^2) \cdot \lambda^2 - 4\sigma_2$ and again, it is easy to show that $\pi^* P_1(\tau + \nu) \neq 0$. All the above contradictions show that all G_x^0 must be of the type $2\mu_\ell(SU(\ell))$. Moreover, it is then quite easy to see that $\Phi_x^0 = [\mu_\ell]_{\mathbb{R}}$ modulo trivial representations This completes the proof of Theorem A2.

(B) *The case* $H^0 = \rho_n(SO(n))$, $n \geq 5$.

Theorem A3. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M = 0$. If $H_M^0 = \rho_n(SO(n))$, $n \geq 5$, then all connected isotropy subgroups of M, G_x^0, are also of the type $\rho_\ell(SO(\ell))$, $\ell \geqslant n$.*

Proof. Let x be an arbitrary point and $\tilde{G}$ be the minimal normal subgroup of G_x^0 containing $H_M^0 = \rho_\ell(SO(\ell))$, and Φ_x be the slice representation of G_x^0. Then it is not difficult to show that $\tilde{G} = \rho_\ell(SO(\ell))$ is the only possibility with $\rho_n(SO(n))$ as the principal isotropy subgroup type of $\Phi_x \mid \tilde{G}$. Suppose the contrary that there exists some point x with $G_x^0 \neq \tilde{G}$. Let x be such a point that G_x^0 is *minimal* among such isotropy subgroups. Then, again, it is not difficult to show that such G_x^0 must be of the type $SO(n) \times K$, $rk\, K = 1$. Let $\psi: SO(n) \times K \to G_x^0 \subset SU(m)$, $\psi = \rho_n + \psi_2$, $\Omega'(\psi_2) = \{w_i \lambda\}$. Similar computation as that of Theorem Al will show that $\pi^* P_1(\tau + \nu) \neq 0$, which is a contradiction to the assumption $P_1(M) = 0$. Therefore all connected isotropy subgroups, G_x^0, are of the type $\rho_\ell(SO(\ell))$.

(C) *The case* $H_M^0 = \nu_n\ (Sp(n))$, $n \geq 2$

Theorem A4. *Let $G = SU(m)$ and M be a G-manifold with $P_1(M) = 0$. If $H_M^0 = \nu_n(Sp(n))$, $n \geq 2$, then the connected isotropy subgroups of M, G_x^0, are also of the type $\nu_\ell(Sp(\ell))$, $\ell \geq n$.*

The proof of the above theorem is essentially the same as that of Theorem A1.

Theorem A5. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected principal isotropy subgroups (H_M) are of the type of exceptional compact Lie groups G_2, F_4, E_6 or E_7, and $\psi: H_M \subseteq SU(m)$ is given by the lowest basic representation modulo trivial one, then all connected isotropy subgroups are of the same type, namely $G_x^0 \sim H_M$ for all $x \in M$.*

Proof. Suppose the contrary that there exists some $x \in M$ with $G_x^0 \supsetneqq H_M^0$. Let x be such a point with G_x^0 of *minimal rank*. Then, there are the following two possibilities: (i) $G_x^0 = H_M^0 \times K$, $rk\, K = 1$ and $\mathrm{Ker}\, \Phi_x = H_M$, (ii) $G_x^0 = \mathrm{Spin}(7)$ and $\Phi_x = \Delta_7$, $H_M^0 = G_2$. In the first case, essentially the same proof as that of Theorem A1 will show it is impossible because it will contradict to $P_1(M) = 0$, $P_2(M) = 0$.

In the second case, $\tau + \nu = \tau + \alpha(\Delta_7) \equiv \tau(\mathrm{SU(m)}/\Delta_7 B_3)$ in $K\tilde{O}$ because $\alpha(\Delta_7) \equiv 0$ in $K\tilde{O}$. Therefore, the same computation as that of Proposition 3.2B will show that $P_2(\tau + \nu) = P_2(\tau) \neq 0$, which is again a contradiction.

(D) Next, let us consider the case that H_M = Adjoint of a semi-simple compact Lie group. The main result in this case is the following theorem:

Theorem A6. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$ and $P_2(M) = 0$. If the connected principal isotropy subgroups (H_M) are semi-simple without normal simple factors of A_1-type and $\psi: H_M \subseteq SU(m)$ is given by Ad_{H_M} modulo trivial ones, then all connected isotropy subgroups are of the same type, i.e., $G_x^0 \sim H_M$ for all $x \in M$.*

Proof. (1) We shall first consider the case that H_M is simple and of rank ≥ 2. Suppose the contrary that there exists $x \in M$ with $G_x^0 \supsetneqq H_M$. Let $G_x^0 \sim K_1 \times K_2 \times \ldots \times K_s$ be the local decomposition of such a G_x^0 into a product of simple normal factors. Since

$$\psi: H_M \subseteq G_x^0 \sim K_1 \times \ldots \times K_s \subseteq SU(m)$$

consists of *only one non-trivial irreducible* copy, i.e. Ad_{H_M}, it is clear that H_M must lie in one of the normal simple factors, say K_1. Therefore, the principal isotropy subgroup type of $\Phi_x | K_1$ is also equal to H_M and it is easy to see that this is, at all, possible only when $K_1 = H_M$. Now, again assume that x be such a point with G_x^0 of minimal rank among such isotropy subgroups. Then $G_x^0 = H_M \times K_2$, $rk\, K_2 = 1$ and $Ker(\Phi_x) = H_M$.

Case (i). $K_2 = S^1$. Then, the representation

$$\psi : H_M \times K_2 \to G_x^0 \subseteq SU(m)$$

is of the form $\psi = Ad_{H_M} \otimes \xi + \psi_2$. Similar computation will show that $\pi^* P_1(\tau + \nu) = i^*\{-(ka^2 + \Sigma w_i^2 + \Sigma h_j^2)\lambda^2\} \neq 0$, $k = \dim H_M$, which is a contradiction to the assumption $P_1(M) \neq 0$. Therefore, this case is impossible to occur.

Case (ii). $K_2 = SU(2)$. Then, the representation

$$\psi : H_M \times K_2 \to G_x^0 \subseteq SU(m)$$

is of the form $\psi = Ad_{H_M} \oplus \psi_2$. Therefore

$$\tau + \nu = -\alpha(Ad_{H_M}) - \alpha(Ad_{K_2}) + \alpha(\Phi_x) = \alpha(\psi_2) + \alpha(\Phi_x) - \alpha(Ad_{K_2}) \text{ in } KU(G(x)).$$

An easy computation will show that $\pi^* P_1(\tau + \nu) = 0$ implies that $\Phi_x = 2\mu_2$ and $\psi_2 = 2\mu_2$ (modulo trivial ones). Therefore

$$\begin{aligned} \pi^* P_*(\tau + \nu) &= i^*\{(1 - \lambda^2)^4 \cdot (1 - 4\lambda^2)^{-1}\} = \\ &= i^*\{(1 - 4\lambda^2 + 6\lambda^4 - 4\lambda^6 + \lambda^8) \cdot (1 + 4\lambda^2 + 16\lambda^4 + \ldots)\} = \\ &= i^*\{1 + 6\lambda^4 + \ldots\}, \text{ and } \pi^* P_*(\tau + \nu) = i^*(6\lambda^4). \end{aligned}$$

In case H_M is not of the following type, namely, $SU(3)$, G_2, D_4, F_4, E_6, E_7, E_8, it is rather easy to see that $j^*(6\lambda^4) \neq 0$. We shall compute $P\psi^2$ and $P\psi^4$ for the case that H_M is one of the above type. Observe that

$$1 + P\psi^2 + P\psi^4 + \ldots = (1 - PH^2 + PH^4 - \ldots) \cdot (1 - 2\lambda^2 + \lambda^4)$$

$$P\psi^2 = -PH^2 - 2\lambda^2, \; P\psi^4 = PH^4 + \lambda^4 + 2\lambda^2 PH^2 \equiv PH^4 - 3\lambda^4 \pmod{P\psi^2}$$

H	PH^2	PH^4	$P\psi^2$	$P\psi^4$
$SU(3)$	$-6\sigma_2$	$9\sigma_2^2$	$6\sigma_2 - 2\lambda^2$	$9\sigma_2^2 + \lambda^4 - 12\lambda^2\sigma_2$
G_2	$-8\sigma_2$	$22\sigma_2^2$	$8\sigma_2 - 2\lambda^2$	$22\sigma_2^2 + \lambda^4 - 16\lambda^2\sigma_2$
D_4	$6\bar{\sigma}_1$	$15\bar{\sigma}_1^2$	$-6\bar{\sigma}_1 - 2\lambda^2$	$15\bar{\sigma}_1^2 + \lambda^4 + 12\lambda^2\bar{\sigma}_1$
F_4	$9\bar{\sigma}_1$	$\frac{147\bar{\sigma}_1^2}{4}$	$-9\bar{\sigma}_1 - 2\lambda^2$	$\frac{147}{4}\bar{\sigma}_1^2 + \lambda^4 + 18\lambda^2\bar{\sigma}_1$
E_6	$24\omega_1$	$270\omega_1^2$	$-24\omega_1 - 2\lambda^2$	$270\omega_1^3 + \lambda^4 + 48\lambda^2\omega_1$
E_7	$-36\sigma_2$	$624\sigma_2^2$	$36\sigma_2 - 2\lambda^2$	$624\sigma_2^2 + \lambda^4 - 72\lambda^2\sigma_2$
E_8	$-60\sigma_2$	$1764\sigma_2^2$	$60\sigma_2 - 2\lambda^2$	$1764\sigma_2^2 + \lambda^4 - 120\lambda^2\sigma_2$

Based on the above results, it is then not difficult to compute the smallest positive integer d such that $d\lambda^4 \in Ker\, j^*$ for each of the above cases. For example, in case $H = A_2$, $d = 4$, $H = G_2$, $d = 16$; $H = D_4$, $d = 8$; etc. In all the above cases, fortunately $j^*(6\lambda^4) \neq 0$. That is $P_2(\tau + \nu) \neq 0$ which is a contradiction to the assumption $P_2(M) = 0$. Therefore, this case is again impossible and hence $G_x^0 \sim H_M$ for all $x \in M$.

(2) The general case can easily be reduced to the above special case that H_M is simple. Suppose

$$H_M = H_1 \times H_2 \times \ldots \times H_a$$
$$G_x^0 \sim K_1 \times K_2 \times \ldots \times K_s$$

are the decompositions of H_M and G_x^0 into (local) products of normal simple factors respectively. Again, the same reason will show that $G_x^0 \sim H_M \times (K_{a+1} \times \ldots \times K_s)$, and we may reduce to consider the case $G_x^0 \sim H_M \times K_{a+1}, rk(K_{a+1}) = 1$. If we pull back $(\tau + \nu)$ over $SU(m)/H_1 \times K_{a+1}$ via the projection

$$SU(m)/H_1 \times K_{a+1} \to SU(m)/H_M \times K_{a+1} = G(x),$$

then identical computation will show it is impossible to satisfy both $P_1(\tau+\nu) = 0$ and $P_2(\tau+\nu) = 0$. Hence $G_x^0 \sim H_M$ for all $x \in M$.

Proposition 7.1. $W_2(SU(m)/\rho B_k) \neq 0$

Corollary. *If $W_2(M) = 0$, then it is impossible to have $H_M = \rho B_k$.*
Proof. Let L be the usual $\mathbb{Z}_2$-maximal torus of $SO(2k+1)$ which consists of diagonal orthogonal matrices. Then

$$H^*(B_L; \mathbb{Z}_2) \cong \mathbb{Z}_2[t_1, t_2, \ldots, t_{2k+1}], \ \Sigma t_i = 0$$

and it is not difficult to whow that

$$\pi^* W_2(SU(m)/\rho_k) = i^*(\sigma_2), \quad \sigma_2 = \sum_{i<j} t_i t_j.$$

Since $Ker(j^*)$ is the ideal generated by $\{\sigma_2^2, \sigma_3^2, \ldots, \sigma_{2k+1}^2\}$, it is clear that $j^*(\sigma_2) \neq 0$.

Remarks. (i) If M also satisfies the condition $W_2(M) = 0$, then it follows from Proposition 7.1 that the *assumption H_M contains no factors of A_1-type* in Theorem *A*6 is *automatically satisfied.*

(ii) Observe that in the case of $SU(n)$ and $\psi = \mu_n \oplus \mu_n$,

$$P\psi^2 = PAd^2, \quad P\psi^4 = PAd^4.$$

Therefore, as far as the proof of Theorem *A*6 is concerned, it does not make any difference if we change the embedding of some factor of $SU(n)$-type in H_M from Ad to $\mu_n \otimes \mu_n$. We shall state such a variant version of Theorem *A*6 without proof.

Theorem A6′. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected principal isotropy subgroups (H_M) are of the following type, namely,*

$$\psi: H_M = SU(k_1) \times \ldots \times SU(k_a) \times \overline{H} \subseteq SU(m)$$
$$\psi = \mu_{k_1} \otimes \mu_{k_1} \oplus \ldots \oplus \mu_{k_a} \otimes \mu_{k_a} \oplus Ad_{\overline{H}},$$

$\overline{H}$ is semi-simple without factors of A_1-type and $k_j \geq 3$, then all connected isotropy subgroups G_x^0 are of the same type, i.e., $G_x^0 \sim H_M$ for all $x \in M$.

As a supplement to Theorem *A*3, similar computation of second Pontrjagin classes as in Theorem *A*6 will prove the following proposition.

Proposition 7.2. *Let $G = SU(m)$ and M be a differentiable G-manifold with $H_M = \rho_4(SO(4))$. If $P_1(M) = 0$ and $P_2(M) = 0$, then all connected isotropy subgroups G_x^0 are of the type $\rho(SO(\ell_x))$.*

(E) Next let us study the case that $H_M = (\nu_1 \oplus \nu_1 \oplus \ldots \oplus \nu_1) \cdot (Sp(1)^k)$, $k \geq 2$. First, let us look at the following example:

Example 1. Let $K = SU(k) \times Sp(\ell) \xrightarrow{\mu_k \oplus \nu_\ell} SU(m)$,

$$\Phi = \Lambda^2 \mu_k + \Lambda^2 \bar{\mu}_k + \Lambda^2 \nu_\ell,$$

and $\alpha(\Phi)$ be the associated equivariant bundle over $SU(m)/K$. Then, it is easy to see that the tangent bundle τ of $SU(m)/K$ is given by

$$\tau = \alpha(\mu_m \otimes \bar{\mu}_m \mid K - \mu_k \times \bar{\mu}_k - s^2 \nu_\ell)$$

and

$$\tau + \alpha(\Phi) = \alpha(\mu_m \otimes \bar{\mu}_m \mid K - \mu_k \times \bar{\mu}_k - s^2 \nu_\ell + \Lambda^2 \mu_k + \Lambda^2 \bar{\mu}_k + \Lambda^2 \nu_\ell) = \alpha((\Lambda^2 \mu_m + \Lambda^2 \bar{\mu}_m) \mid K)$$

which is a trivial bundle by reduction 1.

The above example shows that

$$G_x^0 = SU(k_1) \times \ldots \times SU(k_a) \times Sp(\ell_1) \times \ldots \times Sp(\ell_b)$$
$$\underset{SU(m)}{\cup} | \psi = \mu_{k_1} \oplus \ldots \oplus \mu_{k_a} \oplus \nu_{\ell_1} \oplus \ldots \oplus \nu_{\ell_b},$$

and with the slice representation

$$\Phi_x = (\Lambda^2 \mu_{k_1} + \Lambda^2 \bar{\mu}_{k_1}) \oplus \ldots \oplus (\Lambda^2 \mu_{k_a} + \Lambda^2 \bar{\mu}_{k_a}) \oplus \Lambda^2 \nu_{\ell_1} \oplus \ldots \oplus \Lambda^2 \nu_{\ell_b},$$

is certainly a possibility of local orbit structure for a stably parallelizable G-manifold M with $H_M = Sp(1)^k$. $k = \Sigma \left[\frac{k_i}{2}\right] + \Sigma \ell_j$.

The following theorem proves that there are *no other* possibilities.

Theorem A7. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected principal isotropy subgroups of M, (H_M), are of the type $Sp(1)^k$ with imbedding given by $\nu_1 \oplus \ldots \oplus \nu_1$, then the local orbit structure of M is as follows,*

$$G_x^0 = SU(k_1) \times \ldots \times SU(k_a) \times Sp(\ell_1) \times \ldots \times Sp(\ell_b)$$

$$\underset{SU(m)}{\cup} \psi = \mu_{k_1} \oplus \ldots \oplus \mu_{k_a} \oplus \nu_{\ell_1} \oplus \ldots \oplus \nu_{\ell_b} \quad (\textit{modulo trivial ones}),$$

$$\Sigma\left[\frac{k_i}{2}\right] + \Sigma \ell_j = k \text{ and with the slice representation}$$

$$\Phi_x = (\Lambda^2\mu_{k_1} + \Lambda^2\bar{\mu}_{k_1}) \oplus \ldots \oplus (\Lambda^2\mu_{k_a} + \Lambda^2\bar{\mu}_{k_a}) \oplus \Lambda^2\nu_{\ell_1} \oplus \ldots \oplus \Lambda^2\nu_{\ell_b}$$

modulo trivial ones.

Proof. (1) Let $x \in M$ be an arbitrary point and

$$G_x^0 \sim K_1 \times K_2 \times \ldots \times K_s$$

be the decomposition of G_x^0 into local product of simple normal factors. Then it is easy to see that each factor of $Sp(1)$ in H_M lies in only one of the above simple normal factors and the connected principal isotropy subgroup of $\Phi_x | K_i = (H_M \cup K_i)$. Therefore, it follows easily from Table I, II, III and IV that if $(H_M \cap K_i) = Sp(1)^{k_i}$, $k_i > 0$, then $K_i = \mu_\ell(SU(\ell))$ or $\nu_\ell(Sp(\ell))$ and $\Phi_x | K_i$ is given as follows:

(i) in case $k_i > 1$, $\Phi_x \,|\, K_i = (\Lambda^2\mu_\ell + \Lambda^2\bar{\mu}_\ell)$, or $\Lambda^2\nu_\ell$ respectively,
(ii) in case $k_i = 1$, $\Phi_x \,|\, K_i = (\ell - 2)(\mu_\ell + \bar{\mu}_\ell)$, or $2(\ell - 1)\nu_\ell$.

(2) Now, suppose to the contrary that there exists some point x such that the local orbit structures at x are different from the type given by the theorem. Assume x to be such a point and G_x^0 is minimal among them. Then, in view of the discussion of (1), there are only the following three possibilities, namely,

(i) $G_x^0 = H_M \times K$, $rk(K) = 1$ and $Ker(\Phi_x) = H_M$,
(ii) $G_x^0 = SU(4) \times Sp(1)^{(k-1)}$, $\Phi_x \,|\, SU(4) = 2(\mu_4 + \bar{\mu}_4)$,
(iii) $G_x^0 = Sp(2) \times Sp(1)^{(k-1)}$, $\Phi_x \,|\, Sp(2) = 2\nu_2$.

An almost identical proof as that of Theorem A6 will show that case (i) is impossible. Hence, the proof of Theorem A7 is reduced to showing that case (ii) and case (iii) are again impossible to occur.

Case (ii). Then the sum of the tangent and normal bundles of $G(x)$ is given as follows as an element of $\widetilde{KO}(G(x))$:

$$\begin{aligned}\tau + \nu &= \alpha(\mu_m \otimes \bar{\mu}_m \,|\, G_x^0 - Ad_{G_x^0}) + \alpha(2(\mu_4 + \bar{\mu}_4)) = \\ &= \alpha((\Lambda^2\mu_m + \Lambda^2\bar{\mu}_m) \,|\, G_x^0 - 2\Lambda^2\mu_4 + 2(\mu_4 + \bar{\mu}_4)) = \\ &= \alpha(2(\mu_4 + \bar{\mu}_4) - 2\Lambda^2\mu_4).\end{aligned}$$

Therefore, it follows from reduction 4 that

$$\begin{aligned}\pi^* P_*(\tau + \nu) &= i^*\left\{\frac{(1 + \sigma_2 - \sigma_3 + \sigma_4)^2(1 + \sigma_2 + \sigma_3 + \sigma_4)^2}{(1 + 2\sigma_2 + (\sigma_2^2 - 4\sigma_4) + \ldots)^2}\right\} = \\ &= i^*\{(1 + 12\sigma_4 + \ldots)\}.\end{aligned}$$

Hence $\pi^* P_2(\tau + \nu) = i^*(12\sigma_4) \neq 0$, which is a contradiction to the assumption $P_2(M) = 0$. That is case (ii) is impossible.

Case (iii). Then, the sum of the tangent and normal bundles of $G(x)$ is given as follows as an element of $\widetilde{KO}(G(x))$:

$$\tau + \nu = \alpha(2\nu_2 - \Lambda^2\nu_2).$$

Hence $\pi^* P_2(\tau + \nu) = i^*(12\sigma_4) \neq 0$, which is a contradiction to the assumption $P_2(M) = 0$. That is case (ii) is impossible.

Case (iii). Then, the sum of the tangent and normal bundles of $G(x)$ is given as follows as an element of $\widetilde{KO}(G(x))$:

$$\tau + \nu = \alpha(2\nu_2 - \Lambda^2\nu_2).$$

Therefore,

$$\begin{aligned}\pi^* P_*(\tau + \nu) &= i^*\{(1 - \bar{\sigma}_1 + \bar{\sigma}_2)^2 \cdot [1 - 2\bar{\sigma}_1 + (\bar{\sigma}_1^2 - 4\bar{\sigma}_2) + \ldots]^{-1}\} = \\ &= i^*\{1 + 6\bar{\sigma}_2 + \ldots\}.\end{aligned}$$

Hence, again $\pi^* P_2(\tau + \nu) = i^*(6\bar{\sigma}_2) \neq 0$, which is again a contradiction to the assumption $P_2(M) = 0$. This completes the proof of Theorem A7.

(E) Finally, we shall study the case that H_M is abelian, i.e., H_M are sub-tori of $SU(m)$. Again, let us begin with the following example:

Example 2. Let $K \subseteq SU(m)$ be an arbitrary compact connected subgroup of $SU(m)$ and $\alpha(Ad_K)$ be the associated equivariant bundle over $SU(m)/K$ of the adjoint representation of K. Then, it follows easily from reduction 1 that

$$\tau(SU(m)/K) + \alpha(Ad_L) = \alpha(Ad_{SU(m)} \,|\, K) \equiv 0 \text{ in } \widetilde{KU}(SU(m)/K).$$

Suppose M is a given differentiable $SU(m)$-manifold with $\tau(M) \equiv 0$ in $\widetilde{KU}(M)$ and $(H_M) = (T)$ are subtori. Then, it follows from the above computation that the following local orbit structure is certainly a possibility, namely,

$G_x^0 = K$ with the given T as its maximal tori, and the slice representation $\Phi_x = Ad_K$.

The following theorem proves that this is the only possibility.

Theorem A8. *Let $G = SU(m)$ and M be a differentiable G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected isotropy subgroups of M, $(H_M) = (T)$ are subtori of rank ≥ 2, then the local orbit structure of M is as follows, namely, G_x^0 are subgroups with T as their maximal tori and Φ_x are given by $Ad_{G_x^0}$ modulo trivial ones.*

Proof. The key step is to show that $rk(G_x^0) = rk(T)$ for all $x \in M$. Once this is proved, then it follows from the fact $H^0_{\Phi_x} = T$ that $\Phi_x = Ad_{G_x^0}$ modulo trivial ones. Suppose the contrary that there exist some $x \in M$ with $rk(G_x^0) > rk\, T$, and assume that x be such a point with G_x^0 to be minimal among such isotropy subgroups. Then, the slice representation Φ_x at x satisfies the following rather restrictive conditions: (i) the conected principal isotropy subgroups of the linear action Φ_x are subtori of G_x^0 of corank one, (ii) all isotropy subgroups of the linear action are of corank one (in G_x^0). The following lemma determines all such linear actions:

Lemma 7.3. *Let G be a compact connected Lie group and Φ be an almost faithful real representation of G on $\mathbb{R}^n$. Suppose that all proper isotropy subgroups of the linear Φ-action are of corank one, i.e., $rk(G_x^0) = (rk(G) - 1)$ or $G_x = G$ fol all $x \in \mathbb{R}^n$, and the principal connected isotropy subgroups are subtori of corank one. Then (G, Φ) must be one of the following list:*
(1) $G = SO(4)$, $\Phi = 2P_4$ *(modulo trivial ones),*
(2) $G = U(2)$, $\Phi = \mu_2 + \bar{\mu}_2$ *(modulo trivial ones),*
(3) $G = S^1$, $\Phi =$ *any non-trivial representation,*
(4) $G = SU(2)$, $\Omega(\Phi) \cap \Delta(G) = \phi$.

Proof of Lemma 7.3. Since adding and subracting trivial representations do not give the essential orbit structure of the given linear action, we may assume that Φ consists of no copy of trivial representation. Let $\Omega(\Phi)$ be the weight system of Φ. Because $rk(G_x^0) = (rk(G) - 1)$ for all $x \in \mathbb{R}^n - \{0\}$, $\Omega(\Phi)$ consists of no zero weight and hence $\Omega(\Phi) \cap \Delta(G) = \phi$. The case that $rk(G) = 1$ is obvious; we shall from now on assume that $rk(G) \geq 2$. Then it follows from the fact that $H_\Phi = T$, $rk(T) = rk(G) - 1$, that

(i) $\Phi \mid T = Ad_{\tilde{G}} \mid T$ modulo trivial ones, and

(ii) the directions of weight vectors in $\Omega(\Phi)$ are mutually conjugate with respect to $W(G)$, namely, the perpendicular directions of T's. Based on the above severe restriction on the weight system, it is then an easy exercise in representation theory to show that (1) and (2) are the only such possibilities.

Now, let us continue the proof of Theorem A8. It follows from Lemma 7.3 that the possibilities of local orbit structure at such an x are given as follows:

(i) $G_x^0 = H_M \times K$, $rk(K) = 1$ and $Ker(\Phi_x) = K$
(ii) $G_x^0 = SO(4) \times T_1$, $\Phi_x = 2\rho_4$ (modulo trivial ones)
(iii) $G_x^0 = U(2) \times T_1$, $\Phi_x = \mu_2 + \bar{\mu}_2$ (modulo trivial ones).

An almost identical proof as that of Theorems A6 and A7 will show that case (i) is impossible to occur. We shall show that cases (ii) and (iii) are also impossible.

Case (ii). Then

$$\tau + \nu = -\alpha(Ad_{SO(4)}) + \alpha(2\rho_4) \Rightarrow P_2(\tau + \nu) \neq 0$$

which is a contradiction to the assumption $P_2(M) = 0$.

Case (iii). Then

$$\tau + \nu = -\alpha(Ad_{U(2)}) + \alpha(\mu_2 + \bar{\mu}_2) \Rightarrow P_1(\tau + \nu) \neq 0$$

which is again a contradiction to the assumption $P_1(M) = 0$.

All the above contradictions prove that it is impossible to have G_x^0 with $rk\, G_x^0 > rk\, H_M$ and hence the proof of Theorem 8 is thus complete.

Concluding Remarks of §7. (i) There are still a few special cases such as (Spin (8), $\Delta^+ + \Delta^-$), ($Sp(2)$, $\nu_2 + \Lambda^2\nu_2$), ($G_2 \times G_2$, $\Phi_1 + \Phi'_1$), etc. which are not covered by the discussion of this section. In fact, those cases can also be treated by the same kind of procedure and computations, and in most cases, one will be able to show that all orbits are of equal dimension, namely, $G_x^0 = H_M^0$ for all $x \in M$.

(ii) In the above investigation of various cases, it is quite clear that the conditions: (i) $G \supset G_x^0 \supset H_M^0$ and (ii) $H_M^0 = H^0_{\Phi_x}$ are very restrictive both on the possibility of G_x^0 and that of Φ_x. This strong grip on the possibilities of G_x^0 and Φ_x makes the later computations of characteristic classes feasible and fruitful.

Section 8. Orbit structures of Sp(m) actions on manifolds with vanishing characteristic classes.

In this section, we study the orbit structures of $Sp(m)$ actions on manifolds satisfying suitable vanishing conditions on its characteristic classes such as $P_1(M) = 0$ and $P_2(M) = 0$. We shall only consider the case that the connected principal isotropy subgroups (H_M^0) are non-trivial, i.e. $\dim H_M^0 > 0$.

The main result of this section is the determination of local orbit structures for differentiable G-manifolds M with $P_\ell(M)=0$, $\ell=1,2,3$ and non-trivial connected principal isotropy subgroups of M (H_M). We shall divide the discussion according to the types of (H_M) and state the results separately as the following theorems:

Theorem C1. *Let $G=Sp(m)$ and M be a differentiable G-manifold with $P_1(M)=0$. If the principal connected isotropy subgroups of M, $(H_M)= = \nu_k(Sp(k))$, $k\geq 2$, then all connected isotropy subgroups G_x^0 are also of the type $\nu(Sp(\ell_x))$.*

Proof. Suppose to the contrary that there exists some $x\in M$ such that G_x^0 are *not* of the type $\nu(Sp(\ell_x))$. Let x be such a point with G_x^0 minimal among such isotropy subgroups. Then it follows from Lemmas 6.1 and 6.2 that

$$G_x^0=Sp(k)\times K,\quad rk(K)=1 \quad\text{and}\quad Ker(\Phi_x)=Sp(k).$$

(i) *Suppose* $K\sim SU(2)$.

$$\psi: Sp(k)\times K\to G_x^0\subseteq Sp(m)$$

is of the form $\psi=\nu_k\oplus\psi_2$, where ψ_2 is a representation of K only. Let $\Omega(\psi_2)=\{\pm w_i\lambda\}$ and $\Omega(\Phi_x)=\{\pm h_j\lambda\}$. Then

$$\begin{aligned}
P\psi^2 &= -\bar\sigma_1-\Sigma w_i^2\lambda^2,\\
\tau+\nu &= -\alpha(Ad_{Sp(k)})-\alpha(Ad_K)+\alpha(\Phi_x),\\
i^*P_1(\tau+\nu) &= j^*\{2(k+1)\bar\sigma_1+4\lambda^2-\Sigma h_j^2\lambda^2\}=\\
&= j^*\{-2(k+1)\cdot\Sigma w_i^2\lambda^2-\Sigma h_j^2\lambda^2+4\lambda^2\}\neq 0
\end{aligned}$$

which is a contradiction to the assumption $P_1(M)=0$.

(ii) *Suppose* $K=S^1$. Then the representation

$$\psi: Sp(k)\times S^1\to G_x^0\subseteq Sp(m)$$

is of the form $\psi=\nu_k\oplus\psi_2$, where ψ_2 is a representation of S^1 only. Let $\Omega(\psi_2)=\{\pm w_i\lambda\}$ and $\Omega(\Phi_x)=\{\pm h_j\lambda\}$. Then

$$\begin{aligned}
\tau+\nu &= -\alpha(Ad_{Sp(k)})+\alpha(\Phi_x),\ \text{and}\\
\pi^*P_1(\tau+\nu) &= i^*\{2(k+1)\bar\sigma_1-\Sigma h_j^2\lambda^2\}=\\
&= i^*\{(-2(k+1)\,\Sigma w_i^2-\Sigma h_j^2)\lambda^2\}\neq 0
\end{aligned}$$

which is again a contradiction to the assumption $P_1(M)=0$.

The above two contradictions show that it is impossible to have such G_x^0, and hence all G_x^0 must be of the type $\nu(Sp(\ell_x))$.

Theorem C2. *Let $G=Sp(m)$ and M be a differentiable G-manifold with $P_1(M)=0$, $P_2(M)=0$. If the principal connected isotropy subgroups of M $(H_M)=\nu_1\oplus\ldots\oplus\nu_1(Sp(1)^k)$, $k\geq 2$, then the local orbit structure of M is as follows:*

$$G_x^0=\nu_{k_1}\oplus\ldots\oplus\nu_{k_a}(Sp(k_1)\times\ldots\times Sp(k_a))$$

and

$$\Phi_x=\Lambda^2\nu_{k_1}\oplus\ldots\oplus\Lambda^2\nu_{k_a}\ \textit{modulo trivial ones.}$$

Proof. The proof of Theorem C2 is quite similar to the proof of Theorem A7 in §7.

Let $x\in M$ be an arbitrary point and

$$G_x^0\sim K_1\times\ldots\times K_s$$

be the decomposition of G_x^0 into local product of simple normal factors. Then it is easy to see that each factor of $Sp(1)$ in H_M lies in only one of the above simple normal factors and the connected principal isotropy subgroups of $\Phi_x|K_i=(H_M\cap K_i)$. Therefore, if $(H_M\cap K_i)=Sp(1)^{k_i}$, $k_i>0$, then $K_i=\nu_\ell(Sp(\ell))$ and $\Phi_x\,|\,K_i$ is given as follows (modulo trivial reps):

(i) in case $k_i>1$, $K_i=Sp(k_i)$ and $\Phi_x\,|\,K_i=\Lambda^2\nu_{k_i}$

(ii) in case $k_i=1$, $\Phi_x\,|\,K_i=2(\ell-1)\nu_\ell$.

Now, suppose to the contrary that there exists some point x such that the local orbit structures at x are different from the type stated in the above theorem. Assume x is such a point and G_x^0 is minimal among them. Then, there are only the following two possibilities, namely,

(i) $G_x^0=H_M\times K$, $rk(K)=1$ and $Ker(\Phi_x)=H_M$,

(ii) $G_x^0=Sp(2)\times Sp(1)^{k-1}$, $\Phi_x\,|\,Sp(2)=2\nu_2$.

However, the same proof as that of Theorem A3 will show that the above two cases are both impossible to be compatible with the assumption $P_1(M)=0$ and $P_2(M)=0$. Therefore, the local orbit structure at every point $x\in M$ must be of the type stated in Theorem *C*2.

Theorem C3. *Let $G=Sp(m)$ and M be a differentiable G-manifold with $P_1(M)=0$, $P_2(M)=0$. If the principal connected isotropy subgroups of M, (H_M), are subtori of rank at least 2, then $rk(G_x^0)=rk\,H_M$ for all $x\in M$ and $\Phi_x=Ad_{G_x}$ modulo trivial representations.*

Proof. The proof of Theorem C3 is essentially the same as that of Theorem A8 and hence is omitted.

Theorem C4. *Let $G=Sp(m)$ and M be a differentiable G-manifold with $P_1(M)=0$, $P_2(M)=0$. If the principal connected isotropy subgroups $(H_M)=$*

$= 2\Phi_1(G_2)$ (*resp.* $(\psi_1 + \overline{\psi}_1)(E_6)$, *dim* $\psi_1 = 27$, *or* $\xi_1(E_7)$, *dim* $\xi_1 = 56$), *then all connected isotropy subgroups are of the same type, and hence M is a fibration over* $G/N(H)$,

$$M \cong G/H \times {}_W F(H, M) \to G/N(H).$$

Proof. Suppose to the contrary that there are some $x \in M$ such that $G_x^0 \subsetneqq H_M$. Let x be such a point with minimal G_x^0 among that of Theorem C4. Again, assume G_x^0 is minimal among those isotropy subgroups $\neq H_M^0$. Then either (i) $G_x^0 = H_M^0 \times K$, $rk\,K = 1$, or (ii) $G_x^0 = (Spin(7), 2\Delta_7)$ and $\Phi_x = \Delta_7$, $H_M^0 = H_{\Phi_x} = (G_2, 2\Phi_1)$. The second case is impossible because straightforward computation will show that

$$\tau(G(x)) + \nu(G(x)) = \tau(Sp(m)/2\Delta_7 B_3) + \alpha(\Delta_7)$$

has non-zero second Pontrjagin class, which is a contradiction to $P_2(M) = 0$.

In the case $G_x^0 = H_M^0 \times K$, $rk\,K = 1$. The same proof as that of Theorem A5 will show that $P_1(\tau + \nu) \neq 0$ if $K = S^1$. Now, we shall prove that $K \sim SU(2)$ is again impossible as follows:

(1) $H_M^0 = 2\Phi_1(G_2)$, $G_x^0 \sim G_2 \times SU(2)$. Then the representation

$$\psi : G_2 \times SU(2) \to G_x^0 \subset SU(m)$$

is either $\Phi_1 \otimes \nu_1 \oplus \psi_2$ or $2\Phi_1 \oplus \psi_2$, where ψ_2 is a representation of K-part only. Straightforward computation will show that $P_1(\tau + \nu) = 0$ implies that

$$\psi = 2\Phi_1 + \nu_1 \text{ and } \Phi_x = 2\nu_1 \text{ (modulo trivial reps).}$$

Therefore, $P\psi^2 = 4\sigma_2 - \lambda^2$, $P\psi^4 = -4\lambda^2\sigma_2 + 6\sigma_2^2$ ond

$$\tau + \nu = -\alpha(Ad_{G_2}) - \alpha(Ad_K) + \alpha(2\nu_1).$$

Applying the splitting principle to the computation of $P_*(\tau + \nu)$, we have

$$\begin{aligned}\pi^* P_*(\tau + \nu) &= i^*\{(1 - \lambda^2)(1 - 4\lambda^2)^{-1}(1 + 8\sigma_2 + 22\sigma_2^2 + \ldots)^{-1}\} = \\ &= i^*\{(1 - 2\lambda^2 + \lambda^4)(1 + 4\lambda^2 + 16\lambda^4 + \ldots)(1 - 8\sigma_2 + 42\sigma_2^2 + \ldots)\} = \\ &= i^*\{1 + (2\lambda^2 - 8\sigma_2) + (9\lambda^4 - 16\lambda^2\sigma_2 + 42\sigma_2^2) + \ldots\}.\end{aligned}$$

Hence,

$$\begin{aligned}\pi^* P_2(\tau + \nu) &= i^*\{9\lambda^4 - 16\lambda^2\sigma_2 + 42\sigma_2^2\} = i^*\{5\lambda^4 - 42\sigma_2^2\} = \\ &= i^*\{-12\sigma_2^2\} = j^*\{2\sigma_2^2\} \neq 0\end{aligned}$$

which is a contradiction to the assumption $P_2(M) = 0$. Therefore, the G_2-case of this theorem is proved.

(2) $H_M = (\psi_1 + \overline{\psi}_1)E_6$. Then the representation

$$\psi : E_6 \times K \to G_x^0 \subseteq Sp(m)$$

is of the form $\psi = (\psi_1 + \overline{\psi}_1) \oplus \psi_2$, where ψ_2 is a representation of the K-part only. Again, it is not difficult to show that $P_1(\tau + \nu) = 0$ implies that $\psi_2 = \nu_1$, i.e., $\psi = (\psi_1 + \overline{\psi}_1) + \nu_1$ and $\Phi_x = 2\nu_1$ (modulo trivial reps). Therefore, $P\psi^2 = -12\omega_1 - \lambda^2$, $P\psi^4 = 12\omega_1\lambda^2 + 66\omega_1^2$, where ω_1 is the first Weyl invariant of E6 (cf. §2 Lemma E6). Again

$$\tau + \nu = -\alpha(Ad_{E_6}) - \alpha(Ad_K) - \alpha(2\nu_1)$$

and it follows from the splitting principle that

$$\begin{aligned}\pi^* P_*(\tau + \nu) &= i^*\{(1 - \lambda^2)^2(1 - 4\lambda^2)^{-1}(1 - 24\omega_1 + 270\omega_1^2 + \ldots)^{-1}\} = \\ &= i^*\{(1 + 2\lambda^2 + 9\lambda^4 + \ldots)(1 + 24\omega_1 + 306\omega_1^2 + \ldots)\} = \\ &= i^*\{1 + (2\lambda^2 + 24\omega_1) + (9\lambda^4 + 48\lambda^2\omega_1 + 306\omega_1^2) + \ldots\}.\end{aligned}$$

Therefore,

$$i^* P_2(\tau + \nu) = j^*\{9\lambda^4 + 48\lambda^2\omega_1 + 306\omega_1^2\} = j^*\{5\lambda^4 + 306\omega_1^2\} \neq 0$$

which is again a contradiction to the assumption that $P_2(M) = 0$. Hence the E_6-case of this theorem is also proved.

The case $H_M^0 = E_7$ is essentially the same as the above two cases and hence omitted.

Remark. Again, we shall leave out the discussion of a few special cases listed in (iv) of Theorem 2. Basically, the same kind of procedure and computation apply and there are no special difficulties in those cases.

Section 9. Orbit strucuture of SO(m) actions on manifolds with vanishing characteristic classes.

In this section, we investigate the local orbit structure of $SO(m)$-manifolds M with $P_k(M) = 0$ for $k = 1, 2, 3$ and $(H_M^0) \neq Id$. In view of Theorems 3 and 4, (H_M^0) must be one of the types of linear subgroups of $SO(m)$ listed in Theorem 3. For simplicity of presentation, we shall only state the results about the following major, general cases (and leave out a few special cases):

$$H_M^0 = \begin{cases} \rho_n(SO(n)),\ 2\rho_n(SO(n)),\ n \geq 3 \\ 2\nu_n(Sp(n)),\ n \geq 2 \\ Ad_H \text{ or } 2Ad_H \text{ for } H \text{ semi-simple} \\ \text{torus of } rk \geq 2 \\ Sp(1)^k \end{cases}$$

Theorem B1. *Let $G = SO(m)$ and M be a given G-manifold with $P_1(M) = 0$. If $H_M^0 = \rho_n(SO(n))$ (resp. $2\rho_n(SO(n))$), $n \geq 5$, then all connected isotropy subgroups of M, G_x^0, are also of the same type, namely,*

$$G_x^0 = \rho(SO(\ell_x)) \text{ (resp. } 2\rho(SO(\ell_x))), \ \ell_x \geq n.$$

The proof of Theorem $B1$ is essentially the same as that of Theorem A3 and hence is omitted.

Theorem B2. *Let $G = SO(m)$ and M be a given G-manifold with $P_1(M) = 0$. If $H_M^0 = 2\nu(Sp(n))$, $n \geq 2$, then all connected isotropy subgroups of M are also of the same type, namely, $G_x^0 = 2\nu(Sp(\ell_x))$.*

Again, the proof of Theorem B2 is essentially the same as that of Theorem A4; all involve similar computations as that of Theorem A1.

Theorem B3. *Let $G = SO(m)$ and M be a given G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected principal isotropy subgroup type (H_M^0) is semi--simple without normal simple factors of A_1-type and $\psi : H_M^0 \subset SO(m)$ is given by* Ad *or* 2Ad *modulo trivial representations, then all connected isotropy subgroups are of the same type, i.e., $G_x^0 \sim H_M^0$ for all $x \in M$ and M is a fibration over $G/N(H_M^0)$.*

Proof. In the case $\psi : H_M \subset SO(m)$ is equal to Ad_{H_M}, the proof is essentially the same as that of Theorem A6. Let us consider the case $\psi = 2Ad$. Suppose to the contrary that there exists some $G_x^0 \neq H_M^0$. It is not difficult to show that $G_x^0 = H_M^0 \times K, rk\,K = 1$. Again, we shall divide into two cases according to $K = S^1$ or $SU(2)$ to deduce contradiction.

(i) $G_x^0 = H_M^0 \times S^1, \psi : G_x^0 \subset SO(m), \psi = Ad \otimes_{\mathbb{R}} \xi + \psi_2$ where ξ may be trivial. Then, similar computation as that of Theorem A6 will show that $P_1(\tau + \nu) \neq 0$, which is a contradiction.

(ii) $G_x^0 = H_M^0 \times SU(2), \psi : G_x^0 \subset SO(m), \psi = 2 \cdot Ad + 2\psi_2$ (modulo trivial representations) where ψ_2 is a complex representation without zero weights. Then

$$\overline{P\psi^2} = \frac{1}{2}(P\psi^2) = PH^2 + P\psi_2^2,$$

hence, $\pi^* P_1(\tau + \nu) = i^*(-PH^2 + P\Phi_x^2) = i^*(P\psi_2^2 + P\Phi_x^2) = 0$ only when $\psi_2 = 2\mu_2$, $\Phi_x = 2\mu_2$. Therefore

$$P\psi^4 = 2PH^4 + (PH^2)\cdot(PH^2) + 8\lambda^2 \cdot PH^2 + 6\lambda^4,$$
$$\overline{P\psi^2} = PH^2 + 2\lambda^2$$
$$\pi^* P_2(\tau + \nu) = i^*\{-PH^4 + PH^2 \cdot PH^2 + PH^2 \cdot (2\lambda^2) + 9\lambda^4\} \not\equiv 0$$

(mod $\overline{P\psi^2}, \overline{P\psi^4}$), which is a contradiction to $P_2(M) = 0$.

Theorem B4. *Let $G = SO(m)$ and M be a given G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If the connected principal isotropy subgroup type of M, (H_M^0), is of the type $Sp(1)^k$ with the imbedding given by $2(\nu_1^{(1)} + \ldots + \nu_1^{(k)})$ $(k \geq 2)$, then the local orbit structure of M is as follows:*

$$G_x^0 = SU(k_1) \times \ldots \times SU(k_a) \times Sp(\ell_1) \times \ldots \times Sp(\ell_b)$$
$$\underset{SO(m)}{\cup} \ \psi = \mu_{k_1} + \bar{\mu}_{k_1} + \ldots + \mu_{k_a} + \bar{\mu}_{k_a} + 2\nu_{\ell_1} + \ldots + 2\nu_{\ell_b}$$

$\Sigma \left[\frac{k_i}{2}\right] + \Sigma \ell_j = k$ *and with the slice representation*

$$\Phi_x = (\Lambda^2\mu_{k_1} + \Lambda^2\bar{\mu}_{k_1}) + \ldots + (\Lambda^2\mu_{k_a} + \Lambda^2\bar{\mu}_{k_a}) + \Lambda^2\nu_{\ell_1} + \ldots + \Lambda^2\nu_{\ell_b}$$

modulo trivial ones.

Theorem B5. *Let $G = SO(m)$ and M be a given G-manifold with $P_1(M) = 0$, $P_2(M) = 0$. If H_M^0 are subtori of rank ≥ 2, then all isotropy subgroups are of the same rank as that of H_M^0 and $\Phi_x \mid G_x^0 = Ad$ modulo trivial representations.*

The proofs of the above two theorems are essentially the same as that of Theorems A7 and A8.

Concluding Remarks to §9. Similar computations can be applied to determine the local orbit structures of $SO(m)$-manifolds whose connected principal isotropy subgroup type (H_M^0) are of those special cases listed in (vi) and (v) of Theorem 3. For example, in the cases $H_M^0 = (\text{Spin}(8), \Delta^+ + \Delta^-)$ or $(SU(5), \mu_5 + \bar{\mu}_5)$ it is not difficult to prove that all orbits must be of equal dimension, and, in the case $H_M^0 = (SU(3), \mu_3 + \bar{\mu}_3)$ or $(SU(4), \mu_4 + \bar{\mu}_4)$, then $G_x^0 = (SU(n), \mu_n + \bar{\mu}_n)$ for $3 \leq n \leq 5$.

Section 10. Concluding Remarks.

(A) Throughout this paper, we only consider the connected components of isotropy subgroups of manifold M with a given action of a classical group G. Of course, after one determines the connected isotropy subgroup types (G_x^0), one may proceed to investigate the isotropy subgroup types (G_x) themselves. Technically, it is rather natural to study the possibility of the p-primary component of G_x/G_x^0 separately by computations of mod p Pontrjagin classes or Stiefel-Whitney classes for the case p = 2. Similar to the computations of Pontrjagin classes by means of symmetric products of weights and roots, one may reduce the computations of mod p Pontrjagin classes or Stiefel-Whitney classes by means of p-weights and p-roots with respect to chosen maximal p-tori of G_x. However, a more efficient way to accomplish the step or determining $\{G_x\}$ from that of $\{G_x^0\}$ is usually by menas of cohomology theory of transformation groups.

For example, in the case of actions of classical groups on spheres or acyclic manifolds studied in [*HH1*], one first determined the possibilities of (G_x^0) by computations of Pontrjagin classes and then, proved that in fact $G^x = G_x^0$ by suitable application of the Borel-formula of cohomology theory of transformation groups. This is a natural combination of the characteristic theory of local orbit structures and the cohomology theory of global orbit structures [cf. H3].

(B) There are many natural concrete problems to which one may apply the results of this paper. In fact, this paper was partially motivated by the following problems:

Problem 1. *Let G be a given simple, compact, connected Lie group and H be a non-trivial subgroup of G, $M = G/H$. Is it true that the only non--trivial differentiable G-action on the manifold M is the transitive one?*

Problem 2. *Let G be a given simple, compact, connected Lie group and Φ be a non-trivial, non-transitive differentiable action of the group G on the manifold G. Is it true that the principal orbit type must be G/T and its orbit structure models after that of the adjoint action of G?*

Some of the basic testing cases for Problem 1 are that of real, complex or quaternionic Stiefel manifolds. The results of this paper consist of an important step towards a successful solution of Problem 1 for most of Stiefel manifolds. We shall discuss such applications in a separate paper.

(C) In case the given manifold M satisfies some vanishing condition such as $P_k(M) = 0$, $k = 1, 2, 3$, then the results of this paper are directly applicable and constitute a convenient groundwork for further application of cohomology theory of transformation groups. In fact, even in the general case that M does not satisfy conditions of vanishing characteristic classes: those homogeneous apaces with vanishing characteristic classes listed in Theorems 1, 2 and 3 are still the majority of the possible candidates of principal orbit types for actions of classical groups. Because in the other possibility that the principal orbit type G/H has non-vanishing characteristic classes, it follows from Propositions 6.1 and 6.2 that those non--vanishing characteristic classes of G/H will impose strong restrictions on the cohomological structure of the total space M as well as possible orbit structures. In general, for a given manifold M and a given compact, connected simple Lie group, there are only rather limited possibilities for principal orbit types of G-actions on M. Moreover, once the principal isotropy subgroup type (H_M) is given and non-trivial, then the following three conditions become extremely restrictive and handy: (i) $G \supset G_x \supset H_M$,

(ii) the principal isotropy subgroups of the slice representation Φ_x belong to (H_M) and (iii) $i^!\tau M = \tau + \nu = \alpha(Ad_G | G_x - Ad_{G_x}) + \alpha(\Phi_x)$. Suitable combination of Lie group representation theory and characteristic class theory provide a powerful tool for the determination of local orbit structures. The result of this paper is only the beginning of such an approach which is equally useful in the study of orbit structures on manifolds with *non-vanishing* characteristic classes.

(D) From the viewpoint of geometry and topology of homogeneous spaces, the result of this paper indicates the effective computability of characteristic classes of homogeneous spaces on the one hand, and on the other hand, it also points out that characteristic classes provide a collection of highly resolutive invariants which can be very useful for topological classification of compact homogeneous spaces. Let us conclude this paper with the following conjecture:

Conjecture. *Two homeomorphic compact homogeneous spaces are necessarily diffeomorphic.*

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