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# The pile driver problem via a fixed point argument

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**Abstract.**

In this paper we study, via fixed point and subdifferential arguments, the existence of solutions for a variational inequality which models the dynamics of a pile penetrating into the ground through the action of a pile hammer. This line of reasoning reduces the variational inequality we considered to a nonlinear evolution equation involving a monotone operator.

## 1. Introduction.

On a recent work [2], Raupp, Feijoo and Moura proposed a mathematical model to simulate the dynamics of a pile driver into the ground through the action of a pile hammer. The postulates assumed for the friction force led to the formulation of this model as an evolutionary variational inequality involving a non-differentiable functional. Theorems concerning existence, uniqueness and stability of solution of that variational inequality were proved in [2] with arguments based on the regularization of the non-differentiable term.

The same authors wrote yet two other papers [3,4]. In [3], some numerical results were obtained with an algorithm that uses the regularization technique coupled with Galerkin and predictor-corrector discretizations. In [4], another discretization of the variational inequality, this one resting on an optimization algorithm, is analized.

The aim of the presente article is to treat with a different approach the problem of the existence of solutions for the same variational inequality. We use a fixed point argument and subdifferential techniques to reduce this problem to a question on the existence of solutions for a certain non-linear evolution equation, with a monotone term. The theorem we obtained generalizes slightly the one in [2]. In order to formulate that variational inequality we need to introduce some notation.

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Let $\Omega = [0, L]$ be an interval of the real line, $H = L^2(\Omega)$ the space of square integrable functions on $\Omega$, and $V = H^1(\Omega) \subset H$ the first order Sobolev spaces on $\Omega$. For $1 \leq p \leq \infty$, $D$ a Banach space and $T$ a positive constant, we shall denote by $L^p(D)$ the space of all strongly measurable functions

$$u : [0, T] \to D$$

such that $|u(t)|_D$ belongs to $L^p[0, T]$. These spaces are normed by

$$|u|^p_{L^p(D)} = \int_0^T |u(t)|^p_D \, dt,$$

if $1 \leq p < \infty$, and by

$$|u|_{L^\infty(D)} = \operatorname*{ess\,sup}_{t \in [0,T]} |u(t)|_D,$$

if $p = \infty$. The subspace of $L^\infty(D)$ consisting of all continuous functions on $[0, T]$ will be denoted by $C(D)$.

If $f$ and $g$ belong to $H$, then

$$(f, g) = \int_\Omega f(x) g(x) dx$$

denotes the $H$ usual inner product, and

$$|f|_2 = \sqrt{(f, f)}$$

its norm. The dual of $V$ will be denoted by $V'$. And if $f \in V'$ and $v \in V$,

$$\langle f, v \rangle = f(v)$$

will denote the dual pairing of $(V, V')$. Furthermore, for all functions $\alpha$ belonging to $C^1(\bar{\Omega})$-the space of continously differentiable functions on the closure of $\Omega$ - a functional $\alpha f \in V'$ can be defined by

$$\langle \alpha f, v \rangle = \langle f, \alpha v \rangle, \qquad \forall v \in V,$$

since $\alpha v$ belongs to $V$ whenever $\alpha$ belongs to $C^1(\bar{\Omega})$.

Now, let $\ell, A, \rho$ be positive functions of $C^1(\bar{\Omega})$ such that $\ell$ and $A$ are greater than a strictly positive constant; and $a, b, k_1, k_2$ and $R$ be arbitrary positive constants. Then, the variational inequality studied in this work has the following form:

$$\begin{aligned} &\langle \rho A \ddot{u}(t), v - \dot{u}(t) \rangle + a(A u_x(t), v_x - \dot{u}_x(t)) + \\ &+ b(A \dot{u}_x(t), v_x - \dot{u}_x(t)) + k_1 \langle \delta_1, u(t) \rangle \langle \delta_1, v - \dot{u}(t) \rangle + \\ &+ k_2 \langle \delta_1, \dot{u}(t) \rangle \langle \delta_1, v - u(t) \rangle + J(u(t), v) - J(u(t), \dot{u}(t)) \geq \\ &\geq \langle A f(t) + F(t) \delta_0, \ v - \dot{u}(t) \rangle, \qquad \forall v \in V, \ t \in [0, T], \end{aligned} \tag{1.1}$$



where $f : [0, T] \to V'$ and $F : [0, T] \to R$ are given data. In (1.1) we have used the usual notation for time derivatives:

$$\dot{u} = \frac{du}{dt}$$

and

$$\ddot{u} = \frac{d^2 u}{dt^2},$$

while $\delta_1$ and $\delta_0$ are elements of $V'$ defined by

$$\langle \delta_1, v \rangle = v(1),$$

and

$$\langle \delta_0, v \rangle = v(0),$$

for all $v$ that are in $V$. Finally, $J$ is a functional whose domain is the product $H \times H$ defined by the formula

$$J(u, v) = R \int_\Omega \ell(x) \, \Re(x + u(x) - L) \, (x + u(x) - L) \, |v(x)| \, dx, \tag{1.2}$$

where $|\cdot|$ is the norm of the real line and $\Re : R \to R$ is the Heaviside function

$$\Re(s) = \begin{cases} 1, & \text{if} \quad s \geq 0, \\ 0, & \text{if} \quad s < 0. \end{cases}$$

We point out the following properties of the functional $J$:

(1.3)
i) $J(u, v) \geq 0, \quad \forall u, v \in H,$
ii) $J(u, v) \leq C(\ell, R) |u|_2 |v|_2, \quad \forall u, v \in H,$
iii) $J(u, \cdot)$ is a proper convex functional for all fixed $u$ in $H$.

These properties are straightforward consequences of the definition (1.2).

Henceforth, and without any loss of generality, we shall take $\ell = A = \rho \equiv 1$ and $a = b = k_1 = k_2 = L = R = 1$. Under this normalization, the constant $C(\rho, R)$ in (1.3) equals unity and $J$ appear as

$$J(u, v) = \int_\Omega \Re(x + u(x) - 1) \, (x + u(x) - 1) \, |v(x)| \, dx. \tag{1.2$'$}$$

To clarify certain points in the argument, we shall also consider the bilinear form $a : V \times V \to R$ defined by

$$a(u, v) = (u_x, v_x) + \langle \delta_1, u \rangle \langle \delta_1, v \rangle.$$

This bilinear form is continuous and coercive. In other words, there are positive constants $B_a$ and $C_a$ such that

$$a(u, v) \leq B_a |u|_1 \, |v|_1, \qquad \forall u, v \in V,$$

and

$$a(v) = a(v, v) \geq C_a |v|_1, \quad \forall v \in V, \tag{1.4}$$

where $|\cdot|_1$ is the norm of $V$. For the proof of (1.4) see [2, sec. 3].

In this context, the existence problem we shall examine can be stated as.

To find a function $u \in L^\infty(V)$, whose derivative $\dot{u} \in L^1(V)$, and which satisfies for almost every $t \in [0, T)$

$$\langle \ddot{u}(t), v - \dot{u}(t) \rangle + a(\dot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + J(u(t), v) - J(u(t), \dot{u}(t)) \geq \langle \mathfrak{F}(t), v - \dot{u}(t) \rangle, \quad \forall v \in V, \tag{1.5}$$

as well as the initial conditions

$$u(0) = u_0, \tag{1.6}$$

and

$$\dot{u}(0) = u_1. \tag{1.7}$$

In (1.5) we have set $\mathfrak{F}(t) = f(t) + F(t)\delta_0$.

In the next section the following theorem will be proved.

**Theorem 1.1.** *Assume that* $u_0 \in V$, $u_1 \in H$ *and* $\mathfrak{F} \in L^2(V)$, *that is*, $f \in L^2(V)$ *and* $F(t) \in L^2(0, T)$. *Then there exists a unique function* $u$ *on* $\Omega \times [0, T]$ *satisfying:*

i) $u \in C(V)$,

ii) $\dot{u} \in C(H) \cap L^2(V)$,

iii) *the variational inequality* (1.5),

*and*

iv) *the initial conditions* (1.6) *and* (1.7).

## 2. The proof.

It was shown in [2] that (1.5) – (1.7) has at most one solution. Our existence proof shall be pursued in three steps:

i) First we reduce the problem to a fixed point question, by treating the functional $J$ as if its first variable was a given function $w$. This gives rise to a map that associates $w$ with the solution of a simpler variational inequality. Then the Schauder Fixed Point Theorem is applied to this map.

ii) Next, some standard results about subdifferentials are used to show that the weak solutions of a certain non-linear evolution equation with a monotone term are also solutions of the simpler variational inequality.



Moreover, the solutions of this variational inequality are unique, so that the mapping of step (i) is well defined.

iii) Last, we use a result from a book of Strauss [5] and the regularization technique to show that this evolution equation has weak solutions.

*Step* (i). Let $w$ belong to $L^2(H)$ and consider the problem below, which will be called *associated variational inequality*.

Find a function $u \in L^\infty(V)$ such that $u \in L^2(\mathrm{V})$ and satisfies, for almost every $t \in [0, T]$,

$$\langle \ddot{u}(t), v - \dot{u}(t) \rangle + a(\dot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + J(w(t), v) - J(w(t), \dot{u}(t)) \geq \langle \mathfrak{F}(t), v - \dot{u}(t) \rangle, \quad \forall v \in V, \tag{2.1}$$

and the initial conditions (1.6) and (1.7).

In steps (ii) and (iii) it will be shown that the variational inequality (2.1) has a unique solution for each $w$ in $L^2(H)$, whenever the given data satisfy $u_0 \in V$, $u_1 \in H$ and $\mathfrak{F} \in L^2(V')$. Call this solution $u_w$, reflecting with such notation its dependence on $w$. Then we have that, for each $w$ in $L^2(H)$, $u_w$ belongs to $L^\infty(V)$ and $\dot{u}_w$ belongs to $L^2(V)$.

Now choose $v = 0$ in (2.1) to conclude that

$$-\langle \ddot{u}_w(t), \dot{u}_w(t) \rangle - a(\dot{u}_w(t)) - a(u_w(t), \dot{u}_w(t)) - J(w(t), \dot{u}_w(t)) \geq -\langle \mathfrak{F}(t), \dot{u}_w(t) \rangle. \tag{2.2}$$

Since $J(w(t), \dot{u}_w(t)) \geq 0$, (2.2) implies that

$$\frac{1}{2} \frac{d}{dt} \left( |\dot{u}_w(t)|_2^2 + a(u_w(t)) \right) + a(\dot{u}_w(t)) \leq |\langle \mathfrak{F}(t), \dot{u}_w(t) \rangle|,$$

for all $t$. Integrating this last inequality and using (1.4), it follows that $u_w$ satisfies

$$|\dot{u}_w(t)|_2^2 + |u_w(t)|_1^2 + \int_0^t |\dot{u}_w(\tau)|_1^2 \, d\tau \leq K(a(u_0) + |u_1|_2^2) + K \int_0^t |\langle \mathfrak{F}(\tau), \dot{u}_w(\tau) \rangle| \, d\tau,$$

where $K = \max(2, 2/C_a)$. Thus,

$$|\dot{u}_w(t)|_2^2 + |u_w(t)|_1^2 + \int_0^t |\dot{u}_w(\tau)|_1^2 \, d\tau \leq K(a(u_0) + |u_1|_2^2) + \varepsilon K \int_0^t |u_w(\tau)|_1^2 \, d\tau + \frac{K}{4\varepsilon} \int_0^t |\mathfrak{F}(\tau)|_{V'}^2 \, d\tau,$$

so that, for any $\varepsilon$ small enough, we have

$$\text{(2.3)} \qquad |\dot{u}_w(t)|_2^2 + |u_w(t)|_1^2 + (1 - \varepsilon K)\int_0^T |\dot{u}_w(\tau)|_1^2\, d\tau \leq$$

$$K(a(u_0) + |u_1|_2^2 + \int_0^T |\mathfrak{F}(\tau)|_{V'}^2\, d\tau = K_0,$$

for almost every $t \in [0, T]$.

We now define the mapping

$$\mathbb{F} : L^\infty(H) \to L^\infty(H), \quad w \to \mathbb{F}(w) = u_w,$$

and let

$$W_0 = \{u \in L^\infty(V) \mid \dot{u} \in L^\infty(H) \text{ and } |u|_{L^\infty(V)} + |\dot{u}|_{L^\infty(H)} \leq K_0\}.$$

It follows that $\mathbb{F}$ maps the closed convex set $L^\infty(H)$ into it self in such a way that its range is contained in $W_0$. Moreover, as a consequence of the compactness criterion of Lions and Aubin [5, sec. 1.5, Theor. 2], the set $W_0$ is a pre-compact set of $L^\infty(H)$. And so is the range of $\mathbb{F}$.

We still need to show that $\mathbb{F}$ is a continuous mapping. So let $(w_n)_n$ be a sequence that converges to $w$ in $L^\infty(H)$ and set $u_n = \mathbb{F}(w_n)$ and $u = \mathbb{F}(w)$. Then

$$\text{(2.4)} \qquad \langle \ddot{u}_n, v - \dot{u}_n \rangle + a(\dot{u}_n, v - \dot{u}_n) + a(u_n, v - \dot{u}_n) + J(w_n, v) - J(w_n, \dot{u}_n) \geq \langle \mathfrak{F}, v - \dot{u}_n \rangle, \ \forall v \in V,$$

and

$$\text{(2.5)} \qquad \langle \ddot{u}, v - \dot{u} \rangle + a(\dot{u}, v - \dot{u}) + a(u, v - \dot{u}) + J(w, v) - J(w, \dot{u}) \geq \langle \mathfrak{F}, v - \dot{u} \rangle, \quad \forall v \in V,$$

for almost every $t$. Next choose, for each $t$, $v = u(t)$ in (2.4) and $v = u_n(t)$ in (2.5). Then add to obtain

$$\text{(2.6)} \qquad -\langle \ddot{u} - \ddot{u}_n, \dot{u} - \dot{u}_n \rangle - a(\dot{u} - \dot{u}_n) - a(u - u_n, \dot{u} - \dot{u}_n) \geq J(w, u) + J(w_n, \dot{u}_n) - J(w_n, \dot{u}) - J(w, \dot{u}_n),$$

which implies that

$$\text{(2.7)} \qquad \frac{1}{2}\frac{d}{dt}(|\dot{u} - \dot{u}_n|_2^2 + a(u - u_n)) + a(\dot{u} - \dot{u}_n) \leq |J(w, \dot{u}) - J(w_n, u)| + |J(w, \dot{u}_n) - J(w_n, \dot{u}_n)|.$$



But, from (1.3 ii), we have that

$$|J(w, \dot{u}) - J(w_n, \dot{u})| \leq |w - w_n|_2\, |\dot{u}|_2 \leq K_0 |w - w_n|_2$$

and

$$J(w, \dot{u}_n) - J(w_n, \dot{u}_n)| \leq |w - w_n|_2\, |\dot{u}_w|_2 \leq K_0 |w - w_n|.$$

Therefore, integrating (2.7) from 0 to $t$, it follows that

$$|\dot{u}(t) - \dot{u}_n(t)|_2^2 + C_a |u(t) - u_n(t)|_1^2 \leq 2K_0 \int_0^T |w(t) - w_n(t)|_2\, dt,$$

since the initial data do not depend on $u$. Consequently, as $w_n \to w$ in $L^\infty(H)$, we have that $u_n \to u$ in $L^\infty(V)$ and, a fortiori, in $L^\infty(H)$. That is, $\mathbb{F}$ is continuous.

Being $\mathbb{F}$ a continuous operator that maps the closed convex set $L^\infty(H)$ into a pre-compact set, the Schauder Fixed Point Theorem ensures the existence of at least one element $u$ of $L^\infty(H)$ such that $\mathbb{F}(u) = u$. This function $u$ is clearly a solution of (1.5) – (1.7). And, as $u$ belongs to the image of $\mathbb{F}$, it also has all the properties that are common to solutions of (2.1), the more relevant of which are included in the statement of Theorem 1.1.

*Step* (ii). We shall first show that (2.1) has at most one solution for each $w \in L^\infty(H)$. So, let $w_n = w$ in relation (2.6) and set $u_1 = u$ and $u_2 = u_n$. Then, $u = u_1 - u_2$ satisfies

$$-\langle \ddot{u}, \dot{u} \rangle - a(\dot{u}) - a(u, \dot{u}) \geq J(w, u_1) + J(w, u_2) - J(w, u_1) - J(w, u_2) = 0,$$

with $u(0) = \dot{u}(0) = 0$.

This inequality is equivalent to

$$\frac{d}{dt}(|\dot{u}(t)|_2^2 + a(u(t))) + 2a(\dot{u}(t)) \leq 0$$

which implies that

$$|\dot{u}(t)|_2^2 + a(u(t)) + 2\int_0^T a(\dot{u}(t))dt \leq |\dot{u}(0)|_2^2 + a(u(0)) = 0.$$

Since all the left hand side terms are non-negative, a fortiori,

$$C_a |u_1(t)|_1^2 \leq a(u(t)) \leq 0$$

for almost every $t$, and so $u = 0$.

Now, we shall prove that the variational inequality (2.1) is equivalent to a certain non-linear monotone equation. Due to the uniqueness of solutions of (2.1), it is enough to show that any solution of the equation is also a solution of (2.1).

Observe that $J(y,\cdot)$ is a continuous functional from $H$ into $R$, for any $y \in H$. This fact, together with (1.3 iii), implies that $J(y,\cdot)$ has a non empty subdifferential at all points of $H$ [1, Chap. 1], for all $y$ that belong to $H$.

Define the function $G : R \to R$ by

$$G(s) = \begin{cases} 1, & \text{if} \quad s > 0, \\ 0, & \text{if} \quad s = 0, \\ -1, & \text{if} \quad s < 0, \end{cases}$$

and the linear functional $\Phi(y, v_0)$ on $H$ by

$$(2.8) \qquad (\Phi(y,v_0), v) = \int_\Omega g(x, y(x))\, G(v_0(x))\, v(x)\, dx,$$

where $g(x, y) = \Re(x + y - 1)(x + y - 1)$, for all $y$ and $v_0$ that belong to $H$. This functional satisfies

i) $(\Phi(y, v_0), v_0) = J(y, v_0)$, for all $y \in H$,
ii) $(\Phi(y, v_0), v) \le J(y, v), \forall\, v \in H$, for all $y \in H$, and
iii) $\Phi(y, v_0)$ is a continuous functional for any $y$ and $v_0$ in $H$, since

$$|(\Phi(y,v_0), v)| \le |g(\cdot, y(\cdot))G(v_0(\cdot))|_2 \; |v|_2 \le |y|_2 \; |v|_2.$$

Properties (i) and (ii) imply that

$$J(y, v) - J(y, v_0) \ge (\Phi(y, v_0), v - v_0)$$

and thus $\Phi(y, v_0)$ is a subgradient of $J(y,\cdot)$ at the point $v_0$ [1, Chap. 1]. Consequently, as $\Phi(y, v_0)$ is also a continuous linear functional on $V$, it follows that if $\dot{u}$ belongs to $L^2(V)$ and $w$ belongs to $L^\infty(H)$ then

$$J(w(t), v) - J(w(t), \dot{u}(t)) \ge \langle \Phi(w(t), \dot{u}(t)), v - \dot{u}(t) \rangle,$$

for all $v$ in $V$ and almost every $t$ in $[0, T)$.

In this fashion, we can conclude that if $u$ is a function of $L^\infty(V)$ whose derivative $\dot{u}$ belongs to $L^2(V)$, and which satisfies for almost every $t \in [0, T)$

$$(2.9) \qquad \begin{aligned} &\langle \dot{u}(t), v - \dot{u}(t) \rangle + a(\dot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) + \\ &\langle \Phi(w(t), \dot{u}(t)), v - \dot{u}(t) \rangle \ge \langle \mathfrak{F}(t), v - \dot{u}(t) \rangle, \ \forall\, v \in V, \end{aligned}$$

as well as the initial conditions (1.6) and (1.7), then $u$ must necessarily be also a solution of the associated variational inequality (2.1). That is, (2.9) is equivalent to (2.1) and the solution of (2.9) does not depend on the chosen subgradient, given by (2.8).



After the equivalence between (2.1) and (2.9), the original problem is thus reduced to show the existence of solutions for (2.9). But note that one can choose, for each time $t$, $v = z \pm \dot{u}(t)$ in (2.9), so that it can be read as the variational equation

$$(2.10) \qquad \begin{aligned} &\langle \ddot{u}(t), z \rangle + a(\dot{u}(t), z) + a(u(t), z) + \\ &\langle \Phi(w(t), \dot{u}(t)), z \rangle = \langle \mathfrak{F}(t), z \rangle, \ \forall\, z \in V, \ t \in [0, T). \end{aligned}$$

Moreover, these exists a unique linear operator $A : V \to V'$, associated with the bilinear form $a(\cdot,\cdot)$, defined by $A(y) = a(y, \cdot)$ for all $y$ in $V$. Thus, if we define, for each element $w$ of $L^\infty(H)$, an operator $B : L^2(V) \to L^2(V')$ by

$$B(y)(t) = A(y(t)) + \Phi(w(t), y(t)),$$

it follows that (2.10) is the variational form of equation

$$(2.11) \qquad \ddot{u}(t) + A(u(t)) + B(\dot{u})(t) = \mathfrak{F}(t).$$

This is an evolution equation which contains a non-linear monotone operator $B$.

*Step* (iii). In [5, sec. 3.6] the following result is demonstrated.

Let $\mathfrak{A}$ be a linear operator from $V$ into $V'$ which is symmetric, coercive and continuous, and $\mathfrak{B}$ an operator from $L^2(V)$ such that

i) $\mathfrak{B}$ is bounded and semicontinuous,
ii) for some real $\lambda$, $e^{-\lambda t}(\lambda/2 + \mathfrak{B})$ is coercive in $L^2(V)$ and
iii) $e^{-\lambda t}(\lambda/2 + \mathfrak{B})$ is semimonotone on bounded subsets of a subspace of $L^2(V)$,

then, for any $u_0$ belonging to $V$, $u_1$ belonging to $H$ and $\mathfrak{F}$ in $L^2(V')$, there exists a function $u$ in $C(V)$ such that $\dot{u}$ belongs to $C(H) \cap L^2(V)$ and which is a solution of

$$(2.12) \qquad \begin{aligned} &\langle \ddot{u}(t), v \rangle + \langle \mathfrak{A}(u(t)), v \rangle + \langle \mathfrak{B}(\dot{u}(t)), v \rangle = \langle \mathfrak{F}(t), v \rangle, \\ &\forall\, v \in V, \ t \in [0, T), \end{aligned}$$

with the initial conditions (1.6) and (1.7).

Now, we shall use this result to show the existence of weak solutions for (2.11). So, for any positive $\varepsilon$, let

$$G_\varepsilon(s) = \begin{cases} 1, & \text{if} \quad s > \varepsilon, \\ \dfrac{s}{\varepsilon}, & \text{if} \quad |s| < \varepsilon, \\ -1, & \text{if} \quad s < -\varepsilon, \end{cases}$$

and consider the regularized operator $B_\varepsilon : L^2(V) \to L^2(V')$ defined by

(2.13) $$B_\varepsilon(v)(t) = A(v(t)) + \Phi_\varepsilon(w(t), v(t)),$$

where $w \in L^\infty(H)$, and $\Phi_\varepsilon : H \times V \to V'$ is such that

(2.14) $$\langle \Phi_\varepsilon(y, u), v \rangle = \int_\Omega g(x, y(x)) G_\varepsilon(u(x)) v(x) dx,$$

for any $y$ in $H$ and $u$ and $v$ in $V$.

The operator $A$, being defined through the bilinear form $a(\cdot, \cdot)$, is symmetric, coercive and continuous from $V$ into $V'$ and thus has the same characteristics as the operator $\mathfrak{A}$. Moreover, $A$ considered as an operator from $L^2(V)$ into $L^2(V')$ is also linear, coercive and continuous.

It is shown bellow that $B_\varepsilon$ has some properties that imply (i), (ii) and (iii) and so it has the same characteristics as the operator $\mathfrak{B}$. Indeed, it is an immediate consequence of (2.14) that

$$|\Phi_\varepsilon(w, v)|_{L^2(V')} \leq C\, |w|^2_{L^\infty(H)}.$$

Furthermore, from

$$|\Phi_\varepsilon(w, v_0) - \Phi_\varepsilon(w, v_1)|^2_{L^2(V')} \leq K\, |w|_{L^\infty(H)} \int_0^T |G_\varepsilon(v_0) - G_\varepsilon(v_1)|^2_2\, dt,$$

where $|\cdot|_\infty$ is the norm of $L^\infty(\Omega)$, it follows that $\Phi_\varepsilon(w, \cdot)$ is a bounded continuous operator from $L^2(V)$ into $L^2(V')$. Therefore, the operator $B_\varepsilon$ as defined in (2.13) satisfies condition (i) for any positive $\varepsilon$.

Next, note that

$$G_\varepsilon(s)s = |s|_\varepsilon = \begin{cases} |s|, & \text{if } |s| \geq \varepsilon \\ \dfrac{s^2}{2}, & \text{if } |s| \leq \varepsilon \end{cases}$$

and so

$$\langle \Phi_\varepsilon(w, v), v \rangle = \int_\Omega g(x, w(x))\, |v(x)|_\varepsilon\, dx \geq 0,$$

for any $v$ which belongs to $H$. This implies that $B_\varepsilon$ is coercive for

$$\int_0^T \langle B_\varepsilon(v(t)), v(t) \rangle\, dt \geq \int_0^T \langle A(v(t)), v(t) \rangle\, dt =$$

$$\int_0^T a(v(t)) dt \geq C_a \int_0^T |v(t)|^2_1\, dt = C_a\, |v|_{L^2(V)},$$

whenever $v$ belongs to $L^2(V)$. Therefore, we conclude that $B_\varepsilon$ satisfies condition (ii) with $\lambda = 0$.

Finally, if $w$ belongs to $H$ and $v$ belongs to $V$, $\Phi_\varepsilon(w, v)$ is a subgradient of the convex functional $J_\varepsilon : H \times V \to R$, defined by

(*) $$J_*(w, u) = \int_\Omega g(x, w(x))\, |v(x)|_*\, dx,$$

at the point $v$. Thus $B_\varepsilon$, as the sum of a linear coercive operator and a subgradient, is monotone and thus satisfies also condition (iii) for any positive $\varepsilon$.

Then, we conclude that for all positive $\varepsilon$ there exists a function $u^\varepsilon$ of $C(V)$ such that $\dot{u}^\varepsilon$ belongs to $C(H) \cap L^2(V)$, which satisfies the initial conditions (1.6) and (1.7) and the following equation, which is a regularization of (2.10):

(2.15) $$\langle \ddot{u}^\varepsilon(t), v \rangle + \langle A(u^\varepsilon(t)), v \rangle + \langle B_\varepsilon(\dot{u}^\varepsilon(t)), v \rangle = \langle \mathfrak{F}(t), v \rangle, \forall v \in V, \quad t \in [0, T).$$

Choosing $v = \dot{u}^\varepsilon(t)$ in (2.15) leads to

$$\frac{1}{2} \frac{d}{dt} (|\dot{u}^\varepsilon(t)|^2_2 + a(u^\varepsilon(t))) + a(\dot{u}^\varepsilon(t)) \leq |\langle \mathfrak{F}(t), \dot{u}^\varepsilon(t) \rangle|.$$

And, by the same reasoning used to obtain (2.3), we conclude that

(2.16) $$|\dot{u}^\varepsilon(t)|^2_2 + |u^\varepsilon(t)|^2_1 + (1 - \alpha K) \int_0^T |\dot{u}^\varepsilon(t)|^2_1\, dt \leq K_0$$

since the initial data and $\mathfrak{F}$ do not depend on $\varepsilon$. Moreover, as $A(u^\varepsilon)$ and $B_\varepsilon(\dot{u}^\varepsilon)$ belong to $L^2(V')$ for any positive $\varepsilon$, it follows that $\ddot{u}^\varepsilon \in L^2(V')$ and

$$|\ddot{u}^\varepsilon|_{L^2(V')} \leq |A(u^\varepsilon)|_{L^2(V')} + |B_\varepsilon(\dot{u}^\varepsilon)|_{L^2(V')} + |\mathfrak{F}|_{L^2(V')}.$$

However, since

(2.17) $$|A(u^\varepsilon)|_{L^2(V')} \leq C\, |u^\varepsilon|_{L^2(V)} \leq K_1,$$
$$|A(\dot{u}^\varepsilon)|_{L^2(V')} \leq C\, |\dot{u}^\varepsilon|_{L^2(V)} \leq K_2,$$

and

(2.18) $$|B_\varepsilon(\dot{u}^\varepsilon)|_{L^2(V')} \leq K_2 + C\, |w|_{L^\infty(H)} = K_3,$$

we conclude that

(2.19) $$|\ddot{u}^\varepsilon|_{L^2(V')} \leq K.$$

Thus, as a consequence of the bounds (2.16) – (2.19) and the weak and weak* compactness of bounded subsets of those spaces, by choosing successive subsequences, we can extract from $u^\varepsilon$ a subsequence such that

$$\begin{aligned} &\text{i) } u^\varepsilon \to u, \text{ weakly* in } C(V),\\ &\text{ii) } \dot{u}^\varepsilon \to \dot{u}, \text{ weakly* in } C(H) \cap L^2(V),\\ (2.20)\quad &\text{iii) } \ddot{u}^\varepsilon \to \ddot{u}, \text{ weakly in } L^2(V'),\\ &\text{iv) } A(u^\varepsilon) \to \psi, \text{ weakly in } L^2(V'),\\ \text{and}\quad &\text{v) } A(\dot{u}^\varepsilon) \to \varphi, \text{ weakly in } L^2(V'). \end{aligned}$$

For convenience, this last subsequence was also denoted by $u^\varepsilon$.

By the compactness criterion of Lions and Aubin [5, sec. 1.5, Theor. 2], we can also suppose that the subsequence $\dot{u}^\varepsilon$ converges to $\dot{u}$ strongly in $L^2(Q)$ and almost everywhere in $Q$, where $Q = \Omega \times [0, T)$. These last modes of convergence imply that $\Phi_\varepsilon(w, \dot{u}^\varepsilon) \to \Phi(w, \dot{u})$ in $L^2(V')$.

Next, it is easy to see that $\psi = A(u)$ and $\varphi = A(\dot{u})$ due to (2.20 i) and (2.20 ii) and the symmetry of operator $A$.

Therefore, we conclude that there exists an element $u$ of $C(V)$, whose derivative $\dot{u}$ belongs to $C(H) \cap L^2(V)$, and which satisfies

$$\int_0^T \langle \ddot{u}(t), v(t) \rangle \, dt + \int_0^T \langle A(u(t)), v(t) \rangle \, dt +$$

$$\int_0^T \langle B(\dot{u}(t)), v(t) \rangle \, dt = \int_0^T \langle \mathfrak{F}(t), v(t) \rangle \, dt,$$

whenever $v$ belong to $L^2(V)$. This equality, however, implies that $u$ is also solution of (2.10). Finally, as $u^\varepsilon(0) = u_0$ and $\dot{u}^\varepsilon(0) = u_1$ for all $\varepsilon$, it follows that $u$ satisfies the initial conditions (1.6) and (1.7).

**Remark 1.** Due to the use, in this proof, of the compactness criterion of Lions and Aubin, it is necessary to assume that $T < \infty$. However, we can take $T = \infty$ in Theorem 1.1 because the solution can always be extende to $T_1 > T$ if $\int_0^\infty |\mathfrak{F}(t)|^2_{V'} \, dt < \infty$.

**Remark 2.** The function $|\cdot|_*$, in equation (*), is defined by

$$|s|_* = \begin{cases} |s| - \dfrac{\varepsilon}{2}, & \text{if } |s| \geq \varepsilon \\ \dfrac{s^2}{2\varepsilon}, & \text{if } |s| \leq \varepsilon. \end{cases}$$



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