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The zeta function of a birational Severi-Brauer scheme

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§1. Introduction.

Let k be a field and V be an algebraic k-scheme of dimension $n-1$. V is called a Severi-Brauer k-scheme if there exists a separable algebraic extension L/k such that $V \times_k L$ and $\mathbb{P}_{n-1}(L) = \mathrm{Proj}\ L\,[X_1, \ldots X_n]$ are isomorphic as L-schemes ([11], p. 168). V is said to be split by L/k. V is called a trivial Severi-Brauer k-scheme if V and $\mathbb{P}_{n-1}(k)$ are isomorphic as k-schemes.

Let K/k be a finite Galois extension and let $G = \mathrm{Gal}\ (K/k)$. The isomorphism classes (as k-schemes) of Severi-Brauer k-schemes of dimension $n-1$ which are split by K/k are in canonical one-one correspondence with the elements of the cohomology set $H^1(G, PGL(n, K))$ ([1, 1], loc. cit.). The exact sequence $1 \to K^* \to GL(n, K) \to PGL(n, K) \to 1$ induces a map $\Delta\colon H^1(G, PGL(n, K)) \to H^2(G, K^*)$. Δ is injective and Im Δ is described as follows: let $\gamma \in H^2(G, K^*)$ and let $D(\gamma)$ denote the central division algebra over k defined by γ, let $[D(\gamma)\colon k] = d^2$, then $\gamma \in \mathrm{Im}\ \Delta$ if and only if $d \mid n$ ([10]). Assume now that $\gamma \in \mathrm{Im}\ \Delta$ and let $\beta = \Delta^{-1}(\gamma)$, then the Severi-Brauer k-scheme $V(\beta)$ defined by β is isomorphic as a k-scheme to the Grassmann variety of left ideals of rank n in the matrix algebra $A = M_{n/d}(D(\gamma))$. Actually, A does not depend on K but only on n and the image of γ in $Br(k)$, the Brauer group of k. It was Chatelet ([6]) who first defined Severi-Brauer varieties and established their basic properties over fields of characteristic zero. In particular, he determined the function field, $K(V(\beta))$, of $V(\beta)$. Amitsur ([2]) interpreted Chatelet's work from another point of view, obtained new results, and extended Chatelet's theorems to fields of arbitrary characteristic. Today, his determination of $K(V(\beta))$ is more accessible than Chatelet's. The defining relations for $K(V(\beta))$ are simplest when A is a cyclic algebra ([1,7]) i.e. there exists

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a cyclic splitting field k'/k for A with $[k':k]=n$. In this case, let $z \in k'$ such that $k'=k(z)$. Chatelet and Amitsur proved that there exists a $\theta \in k^*$ such that $K(V(\beta))$ is k-isomorphic to the homogeneous quotient field of $M=k[X_0,\ldots,X_n]/I$, where I denotes the ideal generated by

$$g(X_0,\ldots,X_n)=\prod_{g\in Gal(k'/k)}(X_0+g(z)X_1+g(z^2)X_2+\ldots g(z^{n-1})X_{n-1})-\theta X_n^n.$$

Since the coefficient of each monomial of $g(X_0,X_1,\ldots,X_n)$ is a symmetric polynomial in $\{z,\sigma(z),\ldots,\sigma^{n-1}(z)\}$, it is clear that each coefficient is actually an element of k.

The element θ arises in a natural way. To see this let $\hat{G}$ be the group of characters of $G=\text{Gal}\ (k'/k)$. There is a canonical pairing

$$\eta\colon \hat{G}\times k^*\to H^2(G,k'^*)\subset \text{Br}(k)$$

([11], p. 211; [15], p. 181). Let χ be a generator for $\hat{G}$ and $\eta_\chi\colon k^*\to H^2(G,k'^*)$ be the homomorphism defined by $\eta_\chi(b)=\eta(\chi,b)$. Ker η_χ consists of the elements of k^* which are norms from k'^*. By construction $\gamma\in Im\,\eta_\chi$. For θ one can choose any element of k^* such that $\eta(\chi,\theta)=\gamma$. As there may exist many cyclic splitting fields for A of degree n over k, the representation given by Chatelet and Amitsur of $K(V(\beta))$ as the homogeneous quotient field of the grade k-algebra M depends on the choice of k'. It also depends on the choice of χ, on the choice of the primitive element z for k'/k, and on the choice of θ. For an arbitrary field k there is no natural way to make these choices. However, in case k is a locally compact field, distinct from $\mathbb{R}$ or $\mathbb{C}$ (a p-field in the terminology of [15]) or in case k is an algebraic number field it is possible to use the result of Chatelet and Amitsur to construct projective models for $K(V(\beta))$ which are defined over the ring of integers in k. The construction of these models is carried out in §2 and §5.

It will be convenient, particulary in §4, to avoid mention of the algebras by using the following.

Definition. Let $\gamma\in Br(k)$ and let $D(\gamma)$ denote the central division algebra over k defined by γ. γ will be called a cyclic Brauer class of degree n in case there exists a cyclic extension k'/k of degree n such that $D(\gamma)\otimes_k k'$ is a full matrix algebra over k'. k' will be called a splitting field for γ of degree n over k.

As it will be necessary to work in a "coordinate-free" context, we make the following.

Definition. Let $V(\beta)$ be a Severi-Brauer k-scheme of dimension $n-1$ such that $\Delta(\beta)$ is a cyclic Brauer class of degree n. Let k' be a splitting field for γ of degree n over k. Let $\hat{G}$ be the group of characters of $G=$ $=Gal(k'/k)$ and let χ be a generator for $\hat{G}$. Let $z\in k'$ such that $k'=k(z)$ and $\theta\in k^*$ such that $\eta(\chi,\theta)=\Delta(\beta)\in H^2(G,k'^*)$. Let $M=k[X_0,\ldots,X_n]/I$, where I is the ideal generated by

$$\prod_{g\in G}(X_0+g(z)X_1+g(z^2)X_2+\ldots+g(z^{n-1})X_{n-1})-\theta X_n^n.$$

Let $M^\sim$ denote the quasi-coherent grade $\mathcal{O}_{Spec\,k}$-algebra defined by M. A k-scheme k-isomorphic to Proj $(M^\sim)$ will be called a Chatelet model for $V(\beta)$.

It is important to observe that Proj $(M^\sim)$ is birationally, and not biregularly, isomorphic to $V(\beta)$. Indeed, there are singular points on the "hyperplane at infinity". Their existence is reflected in the results of §3 where the specializations of models for Severi-Brauer schemes defined over discrete valuation rings are determined.

§2 begins with the study of a non-trivial Severi-Brauer scheme $V(\beta)$ defined over a p-field. A projective model for $V(\beta)$, defined over the ring of integers of the field, is constructed. Then trivial Severi-Brauer schemes defined over an arbitrary field with a discrete valuation are studied. Two types of models for these schemes are constructed. In §3 the specializations of the models given in §2 are determined and in case the discrete valuation ring has finite residue field the zeta function of the specialization of each model is computed. Severi-Brauer schemes defined over an algebraic number field k are investigated in §4 and §5. §4 contains a detailed analysis of cyclic Brauer classes of degree n. The arguments are based on the Grunwald-Wang theorem and on strong approximation. As it contains results in class field theory which seem to be new and may be of interest apart from their application to Severi-Brauer schemes, we have written this § so that it can be read independently of the rest of the paper. §5 contains the construction of a projective model of a Severi-Brauer k-scheme that is defined over the ring of integers of k. The basic properties of this model are established. In §6 the zeta function of this model is calculated and its functional equation is established.

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§2. Models for Severi-Brauer schemes over local fields.

Let k be a p-field with ring of integers r and residue field $\mathbb{F}_q$. Let $V(\beta)$ be a non-trivial Severi-Brauer k-scheme of dimension $n-1$. By local class field theory $V(\beta)$ is split by k_n/k, where k_n denotes the unramified extension of k of degree n. k_n contains a primitive q^n-1 root of unity ζ and $k_n = k(\zeta)$. Let $G = Gal(k_n/k)$ and $\hat{G}$ denote the group of characters G. There is a canonical surjective pairing $\eta: \hat{G} \times k^* \to H^2(G, k_n^*)$. ([15]), proposition 9, p. 182 and theorem 1, p. 222). Let π be a prime element of k, then $\chi \to \eta(\chi, \pi)$ gives an isomorphism of $\hat{G}$ onto $H^2(G, k_n^*)$ which is independent of the choice of π. Hence $H^2(G, k_n^*)$ is cyclic of order n. Let $\sigma \in G$ be the Frobenius automorphism and $\chi_F \in \hat{G}$ such that $\chi_F(\sigma) = e^{2\pi i/n}$, then $\eta(\chi_F, \pi)$ is a canonical generator for $H^2(G, k_n^*)$ which does not depend on the choice of π. As in the introduction, let γ denote the image of β in $H^2(G, k_n^*)$. Then $\gamma = \eta(\chi_F, \pi^t)$ for a unique t, $1 \le t \le n-1$. Let $I(\beta)$ be the ideal of $r[X_0, X_1, \ldots, X_n]$ generated by

$$g(X_0, X_1, \ldots, X_n) = \prod_{0 \le i \le n-1} (X_0 + \sigma^i(\zeta) X_1 + \sigma^i(\zeta^2) X_2 + \ldots + \sigma^i(\zeta^{n-1}) X_{n-1}) - \pi^t X_n^n.$$

Let $(r, \zeta, \pi^t) = r[X_0, \ldots, X_n]/I(\beta)$.

Proposition 1. *(r, ζ, π^t) is an integral domain.*

Proof. Since $r[X_0, \ldots, X_n]$ is factorial, $I(\beta)$ is a prime ideal if and only if $g(X_0, X_1, \ldots, X_n)$ is irreducible in $k[X_0, X_1, \ldots, X_n]$ and the greatest common divisor of the coefficients of g is one. The latter condition is clearly satisfied. Suppose that $g_1, g_2 \in r[X_0, X_1, \ldots, X_n]$ are non-constant polynomials such that $g_1 g_2 = g$. g_1 and g_2 are necessarily homogeneous $g_1(X_0, \ldots, X_{n-1}, 0)$. $g_2(X_0, \ldots, X_{n-1}, 0) = g(X_0, \ldots, X_{n-1}, 0)$, but it is well known that $g(X_0, \ldots, X_{n-1}, 0)$ is irreducible (cf. [4], theorem 2, p. 80). So we may assume that $g_i(X_0, \ldots, X_{n-1}, 0) = a \in r$, $a \neq 0$. Since g_1 is non-constant and homogeneous, $g_1 = aX_n^j$ for some $j, j \ge 1$, but $X_n \nmid g$ so certainly $X_n \nmid g_1$.

Corollary 1. *(r, ζ, π^t) is a flat r-module.*

Proof. By the proposition (r, ζ, π^t) is a torsion free r-module. Since r is a principal ideal domain, (r, ζ, π^t) is a flat r – module ($AC, I, \S 2.4$, proposition 3).

Let $(r, \zeta, \pi^t)^{\sim}$ denote the quasi-coherent graded $\underline{o}_{Spec\, r}$-algebra defined by (r, ζ, π^t). Since r is noetherian, $(r, \zeta, \pi^t)^{\sim}$ is a coherent graded $\underline{o}_{Spec\, r}$-algebra (EGA I, 1.5.1). By the preceeding corollary $(r, \zeta, \pi^t)^{\sim}$ is a flat $\underline{o}_{Spec\, r}$-module (EGA, IV, 2.1.1).

Corollary 2. *Proj $((r, \zeta, \pi^t)^{\sim})$ is an integral (i.e. reduced and irreducible) r-scheme.*

Proof. Proj $((r, \zeta, \pi^t)^{\sim})$ and Proj $((r, \zeta, \pi^t))$ are isomorphic as r-schemes (EGA, II, 3.1.3), so by EGA II, 3.1.14 it suffices to prove that (r, ζ, π^t) is integral domain.

Definition. An r-scheme r-isomorphic to Proj $((r, \zeta, \pi^t)^{\sim})$ will be called a canonical model for $V(\beta)$.

The generic fiber F of Proj $((r, \zeta, \pi^t)^{\sim})$ can be regarded as a k-scheme. Since Proj $((r, \zeta, \pi^t)^{\sim})$ is r-isomorphic to Proj $((r, \zeta, \pi^t))$, F is k-isomorphic to Proj $((r, \zeta, \pi^t) \otimes_r k)$ (EGA II, 2.8.9). Since k is flat over r, $(r, \zeta, \pi^t) \otimes_r k$ is k-isomorphic to $k[X_0, X_1, \ldots, X_n]/(g)$. Thus we have.

Proposition 2. *The generic fiber of a canonical model for $V(\beta)$ is a Chatelet model for $V(\beta)$.*

The following proposition will be needed in §3.

Proposition 3. *Let k be a p-field with ring of integers r and residue field $\mathbb{F}_q$. Let k_n be the unramified extension of k of degree n, r_n be the ring of integers in k_n, and ζ be a primitive q^n-1 root of unity in k_n. Let $\sigma \in Gal(k_n/k)$ be the Frobenius automorphism and $\Delta(\zeta) = det\, (\sigma^i(\zeta^j))_{0 \le i, j \le n-1}$. Then $r_n = r[\zeta]$ and if n is odd, $\Delta(\zeta)$ is a unit in r.*

Proof. The first assertion is a well known fact ([11], proposition 16, p. 84). $\Delta(\zeta)$ is a Vandermonde determinant so

$$\Delta(\zeta) = \prod_{n-1 \ge i > j \ge 0} (\sigma^i(\zeta) - \sigma^j(\zeta)) \text{ and } \sigma(\Delta(\zeta)) = \prod_{i > j} (\sigma^{i+1}(\zeta) - \sigma^{j+1}(\zeta)).$$

For odd n multiplication by σ induces an even permutation on $\{\sigma^0, \sigma, \sigma^2, \ldots, \sigma^{n-1}\}$. So $\sigma(\Delta(\zeta)) = \Delta(\zeta)$ and $\Delta(\zeta) \in k$. Since $\Delta(\zeta) \in r_n$, $\Delta(\zeta) \in k \cap r_n = r$. $\{\zeta, \sigma(\zeta), \ldots, \sigma^{n-1}(\zeta)\}$ belong to distinct cosets of (π) in r_n([15], theorem 7, p. 16) so for $i > j$, $\sigma^i(\zeta) - \sigma^j(\zeta)$ is a unit in r_n. Hence $\Delta(\zeta)$ is a unit in r_n, therefore a unit in r.

We now define two types of models for Severi-Brauer schemes defined over arbitrary local fields, not necessarily complete. First we need some algebraic preliminaries.

Let A be a discrete valuation ring with group of units A^* and quotient field K. Let L/K be a cyclic extension of degree n, n odd, and let B be the integral closure of A in L. Let $G = Gal\, (L/K)$ and let τ be a generator for G. For $x \in L$ define $\Delta(x) = \det\, (\tau^i(x^j))_{0 \le i, j \le n}$. The argument used in the proof of proposition 3 shows that $\Delta(x) \in K$. Let $\delta_{B/A}$ denote the discriminant of B with respect to A.

Proposition 4. *Assumptions and notations being as above the following conditions are equivalent:*

i) *There exists* $z \in B$ *such that* $L = K(z)$ *and* $\Delta(z) \in A^*$.
ii) *There exists* $z \in B$ *such that* $B = A[z]$ *and* $\delta_{B/A} = A$.
iii) *There exists* $z \in B$ *such that* $B = A[z]$ *and* $\Delta(z) \in A^*$.

Proof. i) implies ii). By [16], theorem 30, p. 307, $\Delta(z)^2 \in \delta_{B/A}$. Since $\Delta(z)^2 \in A^*$, $\delta_{B/A} = A$. By the same theorem $\delta_{B/A} = \Delta(z)^2 A$ if and only if $\{1, z, z^2, \ldots, z^{n-1}\}$ is a basis for B as a free A-module. Hence $B = A[z]$.
ii) implies iii). Since $z \in B$, the minimal polynomial $f(X)$ for z over K yields an equation of integral dependence $f(z) = 0$ for z over A ([16], theorem 4, p. 260). $B = A(z)$ implies $L = K(z)$ so $f(X)$ has degree n and $\{1, z, \ldots, z^{n-1}\}$ are linearly independent over K. Therefore $\{1, z, \ldots, z^{n-1}\}$ is a basis for B as a free A-module. So $\delta_{B/A} = \Delta(z)^2 A$. By assumption $\delta_{B/A} = A$ so $\Delta(z)^2 \in A^*$. Since $\Delta(z) \in K$ and A is a discrete valuation ring, $\Delta(z) \in A$.
That iii) implies i) is obvious.

Remark. It follows from i) that any conjugate $\tau^i(z)$ of z also satisfies the conditions of the proposition.

Remark. The equivalent conditions of proposition 4 are satisfied in two important cases:

a) A has perfect residue field, B is a discrete valuation ring, and m_A, the maximal ideal of A, is unramified in B ([11], proposition 12, p. 66).
b) m_A splits completely in B, i.e. there exist n distinct prime ideals of B lying over m_A, and $n \le$ card (A/m_A) ([11], exercise 3, p. 67).

Let $\hat{K}$ be the completion of K and let V be a Severi-Brauer K-scheme of dimension $n-1$, n odd, such that $V \times_{Spec K} Spec\ \hat{K}$ is a trivial Severi-Brauer $\hat{K}$-scheme and such that V is split by L/K. Let $z \in B$ satisfy the conditions of proposition 4 and let $\phi \in A^* \cap N_{\hat{L}_i/\hat{K}}(\hat{L}_i)$ for each completion $\hat{L}_i$ of L such that $\hat{K} \subset \hat{L}_i$. Let $\hat{A}$ be the completion of A and I be the ideal of $\hat{A}[X_o, \ldots, X_n]$ generated by $\prod_{0 \le i \le n-1} (X_o + \tau^i(z)X_1 + \tau^i(z^2)X_2 + \ldots + \tau^i(z^{n-1})X_{n-1}) - \phi X_n^n$. Let $(\hat{A}, z, \phi) = \hat{A}[X_0, \ldots, X_n]/I$. The arguments used to prove proposition 1 and its corollaries show that $(\hat{A}, z, \phi)$ is an integral domain and a flat $\hat{A}$-module. They also show that Proj $((\hat{A}, z, \phi)\tilde{\ })$ is an integral $\hat{A}$-scheme.

Definition. An $\hat{A}$-scheme $\hat{A}$-isomorphic to Proj $((\hat{A}, z, \phi)\tilde{\ })$ will be called a completed model with respect to L/K for the Severi-Brauer K-scheme V.

Proposition 5. *If* m_A *splits completely in B, then the graded* $\hat{A}$*-algebra* $(\hat{A}, z, \phi)$ *is isomorphic to* $\hat{A}[Y_0, Y_1, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \phi Y_n^n)$.

Proof. Since the minimal polynomial $f(X)$ for z over K yields an equation of integral dependence $f(z) = 0$ for z over A, each of the conjugates $\tau^i(z)$, $i = 0, 1, \ldots, n-1$, of z in L is contained in B. On the other hand, $f(X)$ decomposes into linear factors in $\hat{K}$, the completion of K, because m_A splits completely in B. Let $z' \in \hat{K}$ be a root of $f(X) = 0$. There is a unique K-embedding λ of L in $\hat{K}$ such that $\lambda(z) = z'$. Let $x \in B$, since x is integral over A, $\lambda(x)$ is integral over $\hat{A}$, but $\hat{A}$ is integrally closed, so $\lambda(x) \in \hat{A}$; hence $\lambda(B) \subset \hat{A}$. Let $\tau^i(z')$ denote $\lambda(\tau^i(z))$, $i = 0, 1, \ldots, n-1$, $\tau^i(z') \in \hat{A}$.

$$\text{Let } D = \begin{pmatrix} 1 & z' & z'^2 & & z'^{n-1} & 0 \\ 1 & \tau(z') & \tau(z'^2) & \ldots & \tau(z'^{n-1}) & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \tau^{n-1}(z') & \tau^{n-1}(z'^2) & \ldots & \tau^{n-1}(z'^{n-1}) & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$$

Det $D = \Delta(z')$. By assumption $\Delta(z') \in A^* \subset \hat{A}^*$, so $D \in GL\ (n, \hat{A})$. Let $Y_i = (X_0 + \tau^i(z')X_1 + \ldots + \tau^i(z'^{n-1})X_{n-1})$, for $0 \le i \le n-1$

$Y_n = X_n$.

Then D defines a degree preserving $\hat{A}$-algebra isomorphism of $\hat{A}[X_0, \ldots, X_n]$ onto $\hat{A}[Y_0, \ldots, Y_n]$ which induces an isomorphism of $\hat{A}[X_0, \ldots, X_n]/I$ onto $\hat{A}[Y_0, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \phi Y_n^n)$, as required.

Corollary. $\hat{A}[Y_0, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \phi Y_n^n)$ *is a flat* $\hat{A}$*-module.*

We conclude §2 with the definition of another type of model for a Severi-Brauer scheme defined over a local field. Let $K, L, A, B, G\ \tau$ and V be as above. Let $x \in B$, such that $K(x) = L$ and x is contained in all maximal ideals of B. Let $\phi \in m_A \cap N_{\hat{L}_i/\hat{K}}(\hat{L}_i)$ for each completion $\hat{L}_i$ of L such that $\hat{K} \subset \hat{L}_i$. Let J be the ideal of $A[X_0, \ldots, X_n]$ generated by

$$\prod_{0 \le i \le n-1} (X_0 + \tau^i(x)X_1 + \tau^i(x^2)X_2 + \ldots + \tau^i(x^{n-1})X_{n-1}) - \phi X_n^n.$$

Let $(\hat{A}, x, \phi) = \hat{A}[X_0, \ldots, X_n]/J$. As before we see that $(\hat{A}, x, \phi)$ is an integral domain and a flat $\hat{A}$-module and that Proj $((\hat{A}, x, \phi)\tilde{\ })$ is an integral $\hat{A}$-scheme.

Definition. An $\hat{A}$-scheme $\hat{A}$-isomorphic to Proj $((\hat{A}, x, \phi)\tilde{\ })$ will be called a degenerate model for the Severi-Brauer K-scheme V.

§3. Specialization of models for Severi-Brauer schemes.

Let k be a p-field with ring of integers r and residue field $\mathbb{F}_q$. Let k_n be the unramified extension of k of degree n, r_n be the ring of integers of k_n, and p_n be the maximal ideal of r_n. Let $\rho\colon r_n \to r_n/p_n = \mathbb{F}_{q^n}$ be the projection homomorphism; for $x \in r_n$, let $\bar{x} = \rho(x)$. ρ extends naturally to a homomorphism $r_n[X_0, \dots, X_n] \to \mathbb{F}_{q^n}[X_0, \dots, X_n]$ which will also be denoted by ρ. Let $G = \mathrm{Gal}(k_n/k)$ and $\sigma \in G$ be the Frobenius automorphism. Let $\hat{G} = \mathrm{Gal}\,(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and $\bar{\sigma} \in \bar{G}$ be the Frobenius automorphism.

Lemma 1. *Let S be a symmetric polynomial in $\mathbb{Z}[X_0, \dots, X_{n-1}]$ with each coefficient equal to ± 1. For all $x \in r_n$, let $s(x) = S(x, \sigma(x), \sigma^2(x), \dots, \sigma^{n-1}(x))$. Then $\overline{s(x)} = S(\bar{x}, \bar{\sigma}(\bar{x}), \dots, \bar{\sigma}^{n-1}(\bar{x})) = s(\bar{x})$.*

Proof. Since each coefficient of S equals ± 1, $\overline{s(x)} = S(\bar{x}, \overline{\sigma(x)}, \overline{\sigma^2(x)}, \dots, \overline{\sigma^{n-1}(x)})$. Since σ is characterized by the property $\sigma(x) \equiv x^q \pmod{p_n}$, $\overline{\sigma^i(x)} = \overline{x^{q^i}} = \bar{x}^{q^i} = \bar{\sigma}^i(\bar{x})$, for all i, $0 \le i \le n-1$.

Corollary. *If $x \in r_n$ such that $\Delta(x) \in r^*$, then $\{1, \bar{\sigma}^i(\bar{x}), \bar{\sigma}^i(\bar{x}^2), \dots, \bar{\sigma}^i(\bar{x}^{n-1})\}$ is a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ for each i, $0 \le i \le n-1$.*

Proof. $\Delta(x)$ is a symmetric polynomial in $\{x, \sigma(x), \sigma^2(x), \dots, \sigma^{n-1}(x)\}$ with each coefficient equal to ± 1, so $\overline{\Delta(x)} = \Delta(\bar{x})$. $\Delta(x) \in r^*$ implies $\overline{\Delta(x)} \in \mathbb{F}_q^*$, so $\Delta(\bar{x}) = \det(\bar{\sigma}^i(\bar{x}^j))_{0 \le i, j, \le n-1} \ne 0$. Hence $\{1, \bar{\sigma}^i(\bar{x}), \dots \bar{\sigma}^i(\bar{x}^{n-1})\}$ is a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$.

Let $z \in r_n$ such that $r_n = r[z]$ and $\Delta(z) \in r^*$. Let $\phi \in r$ and $M = r[X_0, \dots X_n]/I$, where I is the ideal generated by $f(X_0, X_1, \dots, X_n) = \prod_{0 \le i \le n-1} (X_0 + \sigma^i(z)X_1 + \dots \sigma^i(z^{n-1})X_{n-1}) - \phi X_n^n$. Then Proj (M) is an r-scheme and Proj $(M) \times_{\mathrm{Spec}\, r} \mathbb{F}_q$ can be regarded as an $\mathbb{F}_q$-scheme. Let $\bar{M} = \mathbb{F}_q[X_0, \dots, X_n]/\bar{I}$ where $\bar{I}$ is the ideal generated by

$$\prod_{0 \le i \le n-1} (X_0 + \bar{\sigma}^i(\bar{z})X_1 + \dots \bar{\sigma}^i(\bar{z}^{n-1})X_{n-1}) - \bar{\phi} X_n^n.$$

Proposition 1. *Proj $(M) \times_{\mathrm{Spec}\, r} \mathbb{F}_q$ and Proj $(\bar{M})$ are isomorphic $\mathbb{F}_q$-schemes.*

Proof. The closed fiber of Proj (M) is canonically isomorphic to Proj $(M \otimes_r \mathbb{F}_q)$ (EGA II, 2.8.9) so it suffices to prove that $M \otimes_r \mathbb{F}_q$ and M are isomorphic graded $\mathbb{F}_q$-algebras. Since M is a flat r-module (cf. corollary 1 of proposition 1, §2), $M \otimes_r \mathbb{F}_q$ is isomorphic as an $\mathbb{F}_q$-vector space to $\mathbb{F}_q[X_0, \dots, X_n]/I'$, where I' is the ideal generated by $\rho(f(X_0, \dots X_n))$. Write $f(X_0, \dots, X_n) = \sum_{|\alpha| = n} S_\alpha(z, \sigma(z), \dots, \sigma^{n-1}(z))X^\alpha) - \phi X_n^n$, where α is a multi-index and where the S_α are symmetric polynomials as in the lemma.

By the lemma $\rho(f(X_0, \dots X_n)) = (\sum_{|\alpha| = n} \rho(S_\alpha(z, \sigma(z), \dots \sigma^{n-1}(z))X^\alpha) - \rho(\phi)X_n^n =$
$= (\Sigma\, S_\alpha(\bar{z}, \bar{\sigma}(\bar{z}), \dots \bar{\sigma}^{n-1}(\bar{z})) - \bar{\phi}\, X_n^n =$
$= \prod_{0 \le i \le n-1} (X_0 + \bar{\sigma}(\bar{z})X_1 + \dots \bar{\sigma}^i(\bar{z}^{n-1})X_{n-1}) - \bar{\phi}\, X_n^n.$

Corollary. *Proj $(M^{\sim}) \times_{\mathrm{Spec}\, r} \mathbb{F}_q$ and Proj $(M^{\sim})$ are isomorphic $\mathbb{F}_q$-schemes.*
Proof. EGA II, 3.1.3.

Remark. The corollary applies in two important cases: first, if Proj $(M^{\sim})$ is a completed model with respect to k_n/k for a trivial Severi-Brauer k-scheme and second, if Proj $(M^{\sim})$ is a canonical model for $V(\beta)$, a non-trivial Severi-Brauer k-scheme.

We now calculate the zeta function of Proj $(\bar{M})$ in case n is prime, $n > 2$. Recall that the zeta function $\zeta(T)$ of an $\mathbb{F}_q$-scheme X is the formal power series in T such that $\log \zeta(T) = \sum_{v=1}^{\infty} \frac{N_v}{v} T^v$, where N_v denotes the number of $\mathbb{F}_{q^v}$-valued points of X. There are two cases to consider: I) $\phi \in r^*$, so $\bar{\phi} \in \mathbb{F}_q^*$, and II) $\phi \notin r^*$, so $\bar{\phi} = 0$.

Case I) Let $\bar{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. The $\mathbb{F}_{q^v}$-valued points of Proj $(\bar{M})$ correspond to the $\mathbb{F}_{q^v}$-rational points on the variety W in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by

$$\prod_{0 \le i \le n-1} (X_0 + \bar{\sigma}^i(\bar{z})X_1 + \dots \bar{\sigma}^i(\bar{z}^{n-1})X_{n-1}) - \bar{\phi}\, X_n^n = 0.$$

Lemma 2. $N_1 = q^{n-1} + q^{n-2} + \dots + q + 1$.
Proof. Since $\{1, \bar{\sigma}^1(\bar{z}), \dots \bar{\sigma}^1(\bar{z}^{n-1})\}$ is a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, for each $i, 0 \le i \le n-1$, $W \cap \{X_n = 0\}$ contains no $\mathbb{F}_q$-rational points. The points of W belonging to the complement of the hyperplane $X_n = 0$ can be identified with the points of the variety W' in $\mathbb{A}_n(\bar{\mathbb{F}}_q)$ defined by

$$\prod_{0 \le i \le n-1} (Y_0 + \bar{\sigma}^i(\bar{z})Y_1 + \dots \bar{\sigma}^i(\bar{z}^{n-1})Y_{n-1}) - \bar{\phi} = 0.$$

Since $\{1, \bar{z}, \dots \bar{z}^{n-1}\}$ is a basis for $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$, $(y_0, y_1, \dots, y_{n-1}) \in W'$ for $y_i \in \mathbb{F}_q$ if and only if $N(y_0 + zy_1 + \dots z^{n-1}y_{n-1}) = \bar{\phi}$, where N denotes the norm, $N\colon \mathbb{F}_{q^n}^* \to \mathbb{F}_q^*$. Since N is surjective, ker N contains $q^{n-1} + q^{n-2} + \dots + 1$ elements. By assumption $\bar{\phi} \in \mathbb{F}_q^*$ so there are $q^{n-1} + \dots + 1$ points $(y_0, \dots, y_{n-1})$ in $\mathbb{A}_n(\mathbb{F}_q)$ satisfying $N(y_0 + zy_1 + \dots + z^{n-1}y_{n-1}) = \bar{\phi}$. Hence N_1 has the required value.

Lemma 3. *Let $v \in N$ such that $n \nmid v$, then $N_v = q^{v(n-1)} + q^{v(n-2)} + \ldots + q^v + 1$.*
Proof. Since n is prime, $\mathbb{F}_{q^v} \cap \mathbb{F}_{q^n} = \mathbb{F}_q$ and $\mathbb{F}_{q^v}$ and $\mathbb{F}_{q^n}$ are linearly disjoint over $\mathbb{F}_q$. Hence $\{1, \bar{\sigma}^i(\bar{z}), \ldots \bar{\sigma}^i(\bar{z}^{n-1})\}$ is again a basis for $\mathbb{F}_{q^{nv}}$ over $\mathbb{F}_{q^v}$ and the argument of lemma 2 shows that N_v has the required value.

Lemma 4. *Let $v \in N$ such that $n | v$, then $N_v = q^{n(n-1)} + q^{v(n-2)} + \ldots + q^v + 1$.*
Proof. Since $n | v$, $\mathbb{F}_{q^n} \subset \mathbb{F}_{q^v}$ and $\bar{\sigma}^i(\bar{z}^j) \in \mathbb{F}_{q^v}$ for all i, j, $0 \le i, j \le n-1$. By assumption $\Delta(z) \in r^*$, so $\bar{\Delta}(\bar{z}) \in \mathbb{F}_q^*$. Therefore $(\bar{\sigma}^i(\bar{z}^j))_{0 \le i, j \le n-1} \in GL(n, \mathbb{F}_{q^v})$. Let $Y_i = X_0 + \bar{\sigma}^i(\bar{z}) X_1 + \ldots + \bar{\sigma}^i(\bar{z}^{n-1}) X_{n-1}$, $0 \le i \le n-1$

$Y_n = X_n$.

The argument of proposition 5 of §2 shows that W, considered as a variety defined over $\mathbb{F}_{q^v}$, is $\mathbb{F}_{q^v}$-isomorphic to the projective variety W'' in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by $Y_0 Y_1 \ldots Y_{n-1} - \bar{\phi}\, Y_n^n = 0$. Hence to calculate N_v it suffices to count the $\mathbb{F}_{q^v}$-rational points on W''.

$W'' \cap \{Y_n = 0\}$ can be regarded as a variety in $\mathbb{P}_{n-1}(\bar{\mathbb{F}}_q)$. The affine cone over this variety is the variety in $\mathbb{A}_n(\bar{\mathbb{F}}_q)$ defined by $Y_0 Y_1 \ldots Y_{n-1} = 0$. This affine variety contains $(q^v)^n - (q^v - 1)^n$ $\mathbb{F}_{q^v}$-rational points, so the projective variety $W'' \cap \{Y_n = 0\}$ contains $\dfrac{q^{nv} - (q^v - 1)^n - 1}{q^v - 1}$ $\mathbb{F}_{q^v}$-rational points. $W'' \cap \{Y_n \neq 0\}$ can be identified with the points on the variety in $\mathbb{A}_n(\bar{\mathbb{F}}_q)$ defined by $Z_0 Z_1 \ldots Z_{n-1} - \bar{\phi} = 0$. This variety contains $(q^v - 1)^{n-1}$ $\mathbb{F}_{q^v}$-rational points. So $N_v = q^{v(n-1)} + q^{v(n-2)} + \ldots + q^v + 1$.

Combining lemmas 2, 3, and 4 with proposition 1 and the remark following the corollary of proposition 1 yields.

Proposition 2. *Let n be a prime, $n > 2$, and let V be a trivial Severi-Brauer k-scheme of dimension $n-1$. Then N_v, the number of $\mathbb{F}_{q^v}$-valued points on the closed fiber of a completed model for V with respect to k_n/k, is given by $N_v = q^{v(n-1)} + q^{v(n-2)} + \ldots q^v + 1$, for all $v \in N$.*

Corollary. *The zeta function of the closed fiber of a completed model for V is given by*

$$\zeta(T) = [(1 - q^{n-1}T)(1 - q^{n-2}T) \ldots (1 - q\,T)(1 - T)]^{-1}.$$

Case II). Since $\bar{\phi} = 0$, the $\mathbb{F}_{q^v}$-valued points of Proj $(\bar{M})$ correspond to the $\mathbb{F}_{q^v}$-rational points on the projective variety W_1 in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by $\prod_{0 \le i \le n-1} (X_0 + \bar{\sigma}^j(\bar{z}) X_1 + \ldots \bar{\sigma}^j(\bar{z}^{n-1}) X_{n-1}) = 0$. Observe that the point $(0, 0, \ldots, 0, x_n)$, $x_n \neq 0$ is an $\mathbb{F}_q$-rational point, hence $N_v \ge 1$ for all $v \in N$.

Lemma 5. *Let $v \in N$ such that $n \nmid v$, then $N_v = 1$.*
Proof. Since $n \nmid v$, $\{1, \bar{\sigma}^i(\bar{z}), \ldots, \bar{\sigma}^i(\bar{z}^{n-1})\}$ form a basis for $\mathbb{F}_{q^{nv}}$ over $\mathbb{F}_{q^v}$, for each i, $0 \le i \le n-1$. Hence the only $\mathbb{F}_{q^v}$-rational point on W_1 is $(0, 0, \ldots, x_n)$, $x_n \neq 0$.

Lemma 6. *Let $v \in N$ such than $n | v$, then $N_v = n(q^{v(n-1)} + q^{v(n-2)} + \ldots + q^v) + 1$.*
Proof. Since $n | v$, $\mathbb{F}_{q^n} \subset \mathbb{F}_{q^v}$ and $\bar{\sigma}^i(\bar{z}^j) \in \mathbb{F}_{q^n}$ for all $i, j, 0 \le i, j \le n-1$. By assumption $\Delta(z) \in r^*$, so $\Delta(\bar{z}) \in \mathbb{F}_q^*$. Therefore $(\bar{\sigma}^i(\bar{z}^j)) \in GL(n, \mathbb{F}_{q^v})$. Let $Y_i = X_0 + \bar{\sigma}^i(\bar{z}) X_1 + \ldots + \bar{\sigma}^i(\bar{z}^{n-1}) X_{n-1}$, $0 \le i \le n-1$. The argument of proposition 5, §2 shows that W_1, considered a variety defined over $\mathbb{F}_{q^v}$, is isomorphic to the projective variety W_1' in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by $Y_0 Y_1 \ldots Y_{n-1} = 0$. W_1' is a union of the hyperplanes $\{Y_i = 0\}$, $i = 0, \ldots n-1$. Their common intersection is the point $(0, 0, \ldots, 0, y_n)$, $y_n \neq 0$. The complement of this point in each hyperplane contains $q^{v(n-1)} + q^{v(n-2)} \ldots + q^v$ $\mathbb{F}_{q^v}$-rational points. Counting the points with their appropriate multiplicity gives $N_v - 1 = n(q^{v(n-1)} + q^{v(n-2)} + \ldots + q^v)$.
Combining lemmas 5 and 6 with proposition 1 and the remark following the corollary of proposition 1 yields.

Proposition 3. *Let n be a prime, $n > 2$, and let $V(\beta)$ be a non-trivial Severi-Brauer k-scheme of dimension $n-1$. Then N_v, the number of $\mathbb{F}_{q^v}$-valued points on the closed fiber of a canonical model for V, is given by $N_v = 1$, if $n \nmid v$, and $N_v = n(q^{v(n-1)} + q^{v(n-2)} + \ldots + q^v) + 1$, if $n | v$.*

Corollary. *The zeta function of the closed fiber of a canonical model for V is given by*

$$\zeta(T) = [(1 - (q^{n-1}T)^n)(1 - (q^{n-2}T)^n) \ldots (1 - (q\,T)^n)(1 - T)]^{-1}.$$

This concludes the discussion of Case II).

Let A be a discrete valuation ring with maximal ideal m_A and quotient field K. Let L/K be a cyclic extension of degree n, n odd, and let B be the integral closure of A in L. Assume that m_A splits completely in B and that $n \le$ card (A/m_A). Let $\hat{K}$ be the completion of K and let V be a Severi-Brauer K-scheme of dimension $n-1$, n odd, such that $V \times_{Spec\, K} Spec\, \hat{K}$ is a trivial Severi-Brauer $\hat{K}$-scheme and such that V is split by L/K. By proposition 5, §2 a completed model for V with respect to L/K is $\hat{A}$-isomorphic to Proj (N) where $N = \hat{A}[Y_0, Y_1, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \phi\, Y_n^n)$ with $\phi \in \hat{A}^*$. Let $F = \hat{A}/m_A\hat{A}$ and let $\bar{\phi}$ be the image of ϕ in F. The closed fiber of Proj (N) is canonically isomorphic to Proj $(N \otimes_{\hat{A}} F)$ (EGA II, 2.8.9). Since N is a flat $\hat{A}$-module (corollary of proposition 5, §2) $N \otimes_{\hat{A}} F$ is isomorphic as an F-vector space to

$$F[Y_0, Y_1, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \bar{\phi} Y_n^n).$$

Assume now that $F = \mathbb{F}_q$, then the $\mathbb{F}_{q^v}$-valued points on the closed fiber of Proj (N) correspond to the $\mathbb{F}_{q^v}$-rational points on the variety W'' in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by $Y_0 Y_1 \ldots Y_{n-1} - \bar{\phi} Y_n^n = 0$. The argument of lemma 4 shows that $N_v = q^{v(n-1)} + \ldots + q^v + 1$.

Proposition 4. *Let n be odd and let V be a Severi-Brauer K-scheme of dimension $n-1$, n odd, such that $V \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \hat{K}$ is a trivial Severi-Brauer $\hat{K}$-scheme and such that V is split by L/K. Then N_v, the number of $\mathbb{F}_{q^v}$-valued points on the closed fiber of a completed model for V with respect to L/K, is given by $N_v = q^{v(n-1)} + \ldots + q^v + 1$, for all $v \in N$.*

Corollary. *The zeta function of the closed fiber of a completed model for V with respect to L/K is given by*

$$\zeta(T) = [(1 - q^{n-1} T)(1 - q^{n-2} T)\ldots(1 - q\, T)(1 - T)]^{-1}.$$

Let K, L, A, B, F and V be as above. Let $G = \mathrm{Gal}\ (L/K)$ and let $\tau \in G$ be a generator for G. Let $x \in B$, such that $K(x) = L$ and x is contained in all maximal ideals of B. Recall that a degenerate model for V is $\hat{A}$-isomorphic to Proj (P) where $P = \hat{A}[X_0, \ldots X_n]/J$, J being the ideal generated by $\prod_{0 \le i \le n-1} (X_0 + \tau^i(x) X_1 + \ldots + \tau^i(x^{n-1}) X_{n-1}) - \phi X_n^n$, with $\phi \in m_A$. As before, the closed fiber of Proj (P) is canonically isomorphic to Proj $(P \otimes_{\hat{A}} F)$. Since P is a flat $\hat{A}$-module, x belongs to each maximal ideal of B, and $\phi \in m_A$, $P \otimes_{\hat{A}} F$ is isomorphic as an F-vector space to $F[X_0, \ldots, X_n]/X_0^n$. Assume now that $F = \mathbb{F}_q$. Then the $\mathbb{F}_{q^v}$-valued points on the closed fiber of Proj (P) correspond to the $\mathbb{F}_{q^v}$-rational points on the variety in $\mathbb{P}_n(\bar{\mathbb{F}}_q)$ defined by $X_0^n = 0$. This is a hyperplane, so we have.

Proposition 5. *Let n be odd and let V be a Severi-Brauer K-scheme of dimension $n-1$, n odd, such that $V \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \hat{K}$ is a trivial Severi-Brauer $\hat{K}$-scheme and such that V is split by L/K. Then N_v, the number of $\mathbb{F}_{q^v}$-valued points on the closed fiber of a degenerate model for V, is given by $N_v = q^{v(n-1)} + \ldots + q^v + 1$.*

Corollary. *The zeta function of the closed fiber of a degenerate model for V is given by*

$$\zeta(T) = [(1 - q^{n-1} T) \ldots (1 - q\, T)(1 - T)]^{-1}.$$

§4. Cyclic Brauer classes.

Throughout the rest of the paper k will denote an algebraic number field. Let Ω be the set of places of k; for $v \in \Omega$, let k_v be the completion of k at v. Let $Br(k_v)$ be the Brauer group of k_v. In case k_v is isomorphic to $\mathbb{R}$, $Br(k_v) \cong \{\pm 1\}$; in case k_v is isomorphic to $\mathbb{C}$, $Br(k_v) \cong \{1\}$; and in case k_v is a p-field, $Br(k_v)$ is canonically isomorphic to the group of roots of unity in $\mathbb{C}^*$. For each v there is a canonical map $Br(k) \to Br(k_v)$. These maps give rise to an injection $Br(k) \to \prod_{v \in \Omega} Br(k_v)$([3], theorem 2, p. 16; [15], p. 252). Let $\gamma \in Br(k)$ and let γ_v be the image of γ in $Br(k_v)$. Let n be the order of γ; n is clearly equal to the least common multiple of the orders of the γ_v.

Let $S = \{v \in \Omega | \gamma_v \neq 1\}$. S is finite ([15], theorem 1, p. 202). In case $v \in S$ is a finite place, let $k_{v,ab}$ be the maximal abelian extension of k_v (in a fixed algebraic closure of k_v). Let $\mathfrak{A}_v = \mathrm{Gal}\ (k_{v,ab}/k_v)$ and $\mathfrak{H}_n$ be the open subgroup of $\mathfrak{A}_v$ with fixed field $k_{v,n}$, the unramified extension of k of degree n. Let $\phi: \mathfrak{A}_v \to \mathfrak{A}_v/\mathfrak{K}_n$ be the projection and $\sigma_v \in \mathfrak{A}_n/\mathfrak{K}_n$ be the Frobenius automorphism. Let ψ_v be the character on $\mathfrak{A}_v/\mathfrak{K}_n$ such that $\psi_v(\sigma_v) = e^{2\pi i/n}$. Let $a_v: k_v^* \to \mathfrak{A}_v$ be the canonical morphism of local class field theory. Then $\chi_v = \psi_v \circ \phi \circ a_v$ is a character of order n on k_v^*. In case $v \in S$ is an infinite place, let ψ_v be the non-trivial character on $\mathfrak{A}_v = \mathrm{Gal}\ (\mathbb{C}/\mathbb{R})$. Let $a_v: \mathbb{R}^* \to \mathfrak{A}_v$ be the canonical morphism. $\chi_v = \psi_v \circ a_v$ is the character on $\mathbb{R}^*$ which maps $\mathbb{R}_+^*$ to 1 and $\mathbb{R}_-^*$ to -1.

Let $P(n, S)$ be the group of elements $z \in k^*$ such that $z \in (k_v^*)^n$ for all $v \notin S$. Assume that $P(n, S) = (k^*)^n$ in order to avoid the notorious "special case" discussed in [3], pp. 93-6. Let J_k denote the idele group of k. There exists a chacter χ of J_k of order n, trivial on k^*, such that the restriction of χ to k_v^* is equal to χ_v for all $v \in S$ ([3], theorem 5, p. 103). Let k_{ab} be the maximal abelian extension of k and $\mathfrak{A} = \mathrm{Gal}(k_{ab}/k)$ and $a: J_k \to \mathfrak{A}$ be the canonical morphism of global class field theory. a is surjective. The map induced by a from $\mathfrak{A}\,\hat{}$, the group of characters of $\mathfrak{A}$, to $J_k\,\hat{}$, the group of characters of J_k, gives an isomorphism of $\mathfrak{A}\,\hat{}$ onto the group of characters of J_k of finite order, trivial on k^*([15], theorem 5, p. 271). So χ can be regarded as a character on $\mathfrak{A}$. Let k' be the fixed field of ker χ. By construction k'/k is a cyclic extension of degree n such that for all $v \in S$, $k' \otimes_k k_v$ is a field, k'_w; in case v is finite k'_w is k_v-isomorphic to $k_{v,n}$; in case v is infinite, k'_w is k_v-isomorphic to $\mathbb{C}$. For every place $v \in \Omega$ the degree of k'_w over k_v is a multiple of the order of γ_v in $Br(k_v)$, so γ is a cyclic Brauer class of degree n with splitting field k' of degree n over k ([15], proposition 5, p. 253).

χ can be regarded as a character on $\mathfrak{J} = \mathrm{Gal}(k_{sep}/k)$ constant on cosets of $\mathfrak{A}$ in $\mathfrak{J}$ and χ_v can be regarded as a character on $\mathfrak{J}_v = \mathrm{Gal}(k_{v,sep}/k_v)$

constant on cosets of $\mathfrak{A}_v$ in $\mathfrak{J}_v$. Let $\eta: \mathfrak{J} \times k^* \to Br(k)$ be the canonical pairing for k ([15], p. 181) and $\eta_v: \hat{\mathfrak{J}}_v \times k_v^* \to Br(k_v)$ be the canonical pairing for k_v. There exists $\theta \in k^*$ such that $\gamma = \eta(\chi, \theta)$. θ is defined up to an element of $N_{k'/k}(k')$ and $\gamma_v = \eta_v(\chi_v, \theta)$. Let $v \in S$, v finite, and π be a prime element of k_v. From §2 recall that there exists a unique integer t_v, $1 \le t_v \le n-1$, such that $\gamma_v = \eta_v(\chi, \pi^{t_v})$.

Proposition 1. *Let A denote the ring of integers of k and let T be a finite set of places of k such that* $S \cap T = \phi$. *Then there exists* $\theta' \in A$ *satisfying the conditions*:

i) $\gamma = \eta(\chi, \theta')$,

ii) *for all* $v \in S$, v *finite*, $ord_v(\theta') = t_v$,

iii) *for all* $v \in T$, v *finite*, $ord_v(\theta') \ge 1$.

Proof. Let θ be as above and let r_v^* be the units in the ring of integers of k_v. For $v \in S$, v finite, $\gamma_v = \eta_v(\chi_v, \theta) = \eta_v(\chi_v, \pi^{t_v})$ so $\theta^{-1}\pi^{t_v} \in N_{k'_w/k_v}(k'_w)$. Since k'_w/k_v is unramified of degree n, $N_{k'_w/k_v}(k'_w) = \{\pi^{na} r_v^*\}_{a \in \mathbb{Z}}$ ([15], p. 139 and corollary p. 226), so $t_v - ord_v(\theta) = b_v n$ for some $b_v \in \mathbb{Z}$.

For $x \in k'$, $N_{k'/k}(x) = N_{k'_w/k_v}(x)$ ([15], corollary 3, p. 58) and $ord_v(N_{k'_w/k_v}(x)) = n\ ord_w(x)$. Hence if $ord_w(x) = b_v$, then $ord_v(\theta N_{k'/k}(x)) = t_v$. For any $v \in \Omega$, v finite, if $x \in k'$ has sufficiently high order at a place w of k', $w|v$, then $ord_v(\theta N_{k'/k}(x)) \ge 1$.

Let $U = \{v \in \Omega | v \notin S \cup T$ and $ord_v(\theta) < 0\}$. U is a finite set. Let $\Omega_{k'}$ be the set of places of k'. Using the strong approximation theorem ([5], p. 67) in k' select $y \in k'$ such that

a) for $v \in S$, v finite, and for $w \in \Omega_{k'}$, $w|v$, $ord_w(y) = b_v$,

b) for $v \in T$, v finite, and for some $w \in \Omega_{k'}$, $w|v$, $ord_w(y)$ is sufficiently large that $ord_v(\theta N_{k'/k}(y)) \ge 1$,

c) for $v \in U$ and for some $w \in \Omega_{k'}$, $w|v$, $ord_w(y)$ is sufficiently large that $ord_v(\theta N_{k'/k}(y)) \ge 0$,

d) for all finite v, $v \notin S \cup T \cup U$ and for all $w \in \Omega_{k'}$, $w|v$, $ord_w(y) \ge 0$.

Then for $v \in S$, v finite, $ord_v(\theta N_{k'/k}(y)) = t_v$; for $v \in T$, v finite, $ord_v(\theta N_{k'/k}(y)) \ge 1$; and for all other $v \in \Omega$, v finite, $ord_v(\theta N_{k'/k}(y)) \ge 0$. So $\theta' = \theta N_{k'/k}(y) \in A$. θ' satisfies i) – iii).

Remark. Proposition 1 was proven under the assumption that k, S and n do not give rise to the "special case". Proposition 1 is sufficient for applications later in this paper. However, a stronger result can be established using a similar argument. We describe the result as it may be of independent interest. Again, let k be an algebraic number field and let $\gamma \in Br(k)$. γ defines a central division algebra over k, $D(\gamma)$. The index $i = i(\gamma)$

of $D(\gamma)$ is defined by $[D(\gamma): k] = i(\gamma)^2$. The Albert-Brauer-Hasse-Noether theorem ([7], cf. [3] corollary, p. 105) implies that γ is a cyclic Brauer class of degree $i(\gamma)$ – even under the conditions of the special case. Let $S' = \{v \in \Omega | \gamma_v \ne 1\}$ and let S_0 be the subset of Ω defined on p. 95 of [3]. For all $v \in S'$, $v \notin S_0$, let $\mathfrak{K}_{v,i}$ denote the subgroup of $\mathfrak{A}_v = \mathrm{Gal}\,(k_{v,ab}/k_v)$ with fixed field $k_{v,i}$. Let $\sigma_v \in \mathfrak{A}_v/\mathfrak{K}_{v,i}$ be the Frobenius automorphism and ψ_v be the character on $\mathfrak{A}_v/\mathfrak{K}_{v,i}$ such that $\psi_v(\sigma_v) = e^{2\pi i/i(\gamma)}$. Let χ_v be the character of order i on k_v^* defined by ψ_v as above. Then there exists a character χ of J_k of order i, trivial on k^*, such that the restriction of χ to k_v^* equals χ_v([3], theorem 5, p. 105). For $v \in S'$, $v \notin S_0$, let t_v be defined as above. An argument similar to the one used in the proof of proposition 1 gives.

Proposition 1′. *Let A denote the ring of integers of k and let T be a finite set of places such that* $S' \cap T = \phi$. *Then there exists* $\theta' \in A$ *satisfying the following conditions*:

i) $\gamma = \eta(\chi, \theta')$

ii) *for all* $v \in S'$, v *finite*, $v \notin S_0$, $ord_v(\theta') = t_v$

iii) *for all* $v \in T$, v *finite*, $ord_v(\theta') \ge 1$.

Before giving an important corollary to proposition 1 we recall a result of Hensel concerning the discriminant of a finite extension of an algebraic number field. As before, let k be an algebraic number field and A be the ring of integers of k. For $v \in \Omega$, let p_v denote the prime ideal of A defined by v. Let L/k be a finite Galois extension of k of degree m with $G = \mathrm{Gal}(L/k)$. Denote the elements of G by τ_i, $0 \le i \le m-1$. Let B be the integral closure of A in L and let $\delta_{B/A}$ be the discriminant of B with respect to A. For $z \in L$, let $\Delta(z) = \det(\tau_i(z^j))_{0 \le i, \le m-1}$; $\Delta(z) \ne 0$ if and only if $L = k(z)$. $\{\Delta(z)^2 | z \in B,\ k(z) = L\}$ generate an ideal $I_{B/A}$ of A. $I_{B/A} \subset \delta_{B/A}$; if $p_v | I_{B/A}$ and $p_v \nmid \delta_{B/A}$, then p_v is called a common inessential factor of the discriminant of L/k ([8], p. 439). From Hensel's criterion ([8], p. 442) it follows that if $p_v B$ is a prime ideal of B, then p_v is not a common inessential factor of the discriminant of L/k. A necessary condition for p_v to be a common inessential factor of the discriminant is that $card\,(A/p_v) < m$. This condition is also sufficient if p_v splits completely in B. We apply these results in case $L = k'$, the extension of k described above, under the additional assumption that n is and odd prime.

Corollary. *Notations being as in Proposition* 1, *assume that* $[k': k] = n$ *is and odd prime. Let* $T_1 = \{v \in \Omega | p_v | I_{B/A}\}$. *Then there exists* $\theta' \in A$ *satisfying the following conditions*:

i) $\gamma = \eta(\chi, \theta')$,

ii) *for all* $v \in S$, *v finite*, $ord_v(\theta') = t_v$,
iii) *for all* $v \in T$, *v finite*, $ord_v(\theta') \geq 1$.

Proof. As n is odd, k, n, and S do not give rise to the "special case". As n is prime, $p_v | I_{B/A}$ implies that p_v is totally ramified in B or that p_v splits completely in B and card $(A/p_v) < m$. Hence $S \cap T_1 = \emptyset$. The corollary now follows from the proposition.

Throughout the rest of §4 the assumptions will be those of the corollary. We let θ(instead of θ') denote an element of A satisfying conditons i) – iii) of the corollary. Define

$$T_2 = \{v \in \Omega | v \notin S \cup T_1, ord_v(\theta) \geq 1\}, \text{so } S \cup T_1 \cup T_2 = \{v \in \Omega | ord_v(\theta) \geq 1\}.$$

For any v such that k'_w/k_v is unramified of degree n, the restriction map Gal $(k'_w/k_v) \to$ Gal (k'/k) is an isomorphism. Identify these two groups using this isomorphism. Then $\Delta(x)$ is well-defined for any $x \in k'_w$. $\Delta: k'_w \to k_v$ is a polynomial map, therefore continuous. Let q_v be the cardinality of the residue field of k_v. k'_w is generated over k_v by ζ_w, a primitive $q_{v^n} - 1$ root of unity. Recall that $\Delta(\zeta_w) \in A_v^*$, the group of units of A_v (proposition 3, §2).

Lemma. *Let B_w be the ring of integers of k'_w. There exists $\varepsilon > 0$ such that for all $z \in k'$, $|z - \zeta_w|_w < \varepsilon$, the following conditions are satisfied:*

i) $k_v(z) = k_v(\zeta_w) = k_w$,
ii) $z \in B_w$,
iii) $\Delta(z) \in A_v^*$.

Proof. Let $\varepsilon_1 = \min_{\sigma \in Gal(k_w/k_v), \sigma \neq id} |\sigma\zeta_w - \zeta_w|_w$. Let $z \in k'$ such that $|z - \zeta_w|_w < \varepsilon_1$. By Krasner's lemma ([9], proposition 3, p. 43) $k_v(\zeta_w) \subset k_v(z)$, so $k'_w = k_v(z)$. Since $\zeta_w \in B_w$, for some $\varepsilon_2 > 0$, $|z - \zeta_w|_w < \varepsilon_2$ implies $z \in B_w$. Since $\Delta: k'_w \to \to k_v$ is continuous and $\Delta(\zeta_w) \in A_v^*$, for some $\varepsilon_3 > 0$, $|z - \zeta_w|_w < \varepsilon_3$ implies $\Delta(z) \in A_v^*$. Set $\varepsilon = \min \varepsilon_i$, $i = 1, 2, 3$.

Proposition 2. *Let B be the ring of integers of k'. There exists $z_1 \in B$ such that*

i) $k(z_1) = k'$,
ii) *for all* $v \in S$, *v finite*, $\Delta(z_1) \in A_v^*$
iii) *for all* $w | v$, $v \in T_1 \cup T_2$, $ord_w(z_1) \geq 1$.

Proof. If there is no finite $v \in S$ and $T_1 \cup T_2 = \emptyset$, then ii) and iii) are vacuous and i) is obviously satisfied for some $z_1 \in B$. If there is no finite $v \in S$, but $T_1 \cup T_2 \neq \emptyset$, let $z \in B$ be a primitive element for k' and let $a \in A$ be such that $ord_w(az) \geq 1$ for all $w | v$, $v \in T_1 \cup T_2$. Then $z_1 = az$ satisfies i) and iii), while ii) is vacuous. Finally, assume that there is a finite $v \in S$. For each such v select $\varepsilon_v > 0$ as in the lemma; let $\varepsilon = \min_{v \in S} \varepsilon_v$. Using the strong approximation theorem in k' select $z_1 \in k'$ such that $|z_1 - \zeta_w|_w < \varepsilon$ for all $w | v$, $v \in S$, v finite, such that $ord_w(z_1) \geq 1$ for all $w | v$, $v \in T_1 \cup T_2$, and such that $ord_w(z_1) \geq 0$ for all other finite places w of k'. $z_1 \in B$ and ii) and iii) are satisfied. To verify i) observe that $[k(z_1): k] \geq [k(z_1)_w: k_v] \geq n$ for all finite $v \in S$, so $k(z_1) = k'$.

By definition of T_1, for each finite $v \in \Omega$, $v \notin T_1$, there exists an $x_v \in B$ such that $k' = k(x_v)$ and $\Delta(x_v)^2 \in A_v^*$ (or, equivalently, $\Delta(x_v) \in A_v^*$). Let $\Sigma_1 = \{v \in \Omega | v \text{ finite}, v \notin S \cup T_1 \cup T_2, \Delta(z_1) \notin A_v^*\}$. Σ_1 is finite set. Let v_0 be a finite place of k, $v_0 \notin S \cup T_1 \cup T_2 \cup \Sigma_1$, such that there is only one place w_0 of k' lying over v_0, so k'_{w_0}/k_{v_0} is unramified of degree n. Let q_{v_0} be the cardinality of the residue field of k_{v_0} and ζ_{w_0} be a primitive $q_{v_0}^n - 1$ root of unity in k'_{w_0}.

Proposition 3. *There exists $z_2 \in B$ such that*

i) $k(z_2) = k'$,
ii) *for all* $v \in \Sigma_1$, $\Delta(z_2) \in A_v^*$.

Proof. If $\Sigma_1 = \emptyset$, ii) is vacuous and i) is obviously satisfied for some $z_2 \in B$. The rest of the proof being essentially the same as the proof of proposition 2 it suffices to remark that one choose $z_2 \in B$ close to ζ_{w_0} in the topology of k'_{w_0} and close to each x_v in the topology of each k'_w, $w | v$, $v \in \Sigma_1$.

§5. A model for Severi-Brauer schemes over algebraic number fields.

Let $Y = Spec\ A$ and for $v \in \Omega$, v, finite, let A_v denote the valuation ring in k_v. For an A-scheme X, let X_v denote the A_v-scheme $X \times_Y Spec\ A_v$. Let X_0 denote $X \times_Y Spec\ k$. Let V be a Severi-Brauer k-scheme of dimension $n-1$, n an odd prime, defined by $\gamma \in Br(k)$. Let γ_v be the image of γ in $Br(k_v)$. For each $v \in \Omega$, let ${}_vV = V \times_{Spec\ k} Spec\ k_v$. The purpose of this § is to prove:

Theorem 1. *There exists a quasi-coherent graded $\underline{o}_Y$-algebra $\mathscr{S}$, flat over $\underline{o}_Y$, such that Proj ($\mathscr{S}$) is a projective Y-scheme and such that*

i) *Proj* $(\mathscr{S})_0$ *is a Chatelet model for V.*
ii) *For all* $v \in S$, *v finite, Proj* $(\mathscr{S})_v$ *is canonical model for* V (γ_v).
iii) *For all* $v \in T_1 \cup T_2$, *Proj* $(\mathscr{S})_v$ *is a degenerate model for* ${}_vV$.
iv) *For all other* $v \in \Omega$, *v finite, Proj* $(\mathscr{S})_v$ *is a completed model for* ${}_vV$.

The construction of $\mathscr{S}$.

We make the following notational convention: if D is an integral domain with quotient field E, F is a finite Galois extension of E with $H = \mathrm{Gal}(F/E)$, $x \in F$ such that $F = E(x)$, $\phi \in E^*$, then the graded D-algebra

$$D[X_0, X_1, \ldots, X_n]/(\prod_{\sigma\in H}(X_0 + \sigma(x)X_1 + \ldots \sigma(x^{n-1})X_{n-1} - \phi X_n^n))$$

will be denoted by (D, x, ϕ).

The quasi-coherent $\underline{o}_Y$-algebra $\mathscr{S}$ is constructed by first assigning to each member of a cover for Y, consisting of subsets of the form $D(f)$, $f \in A$, a quasi-coherent graded $\underline{o}_Y$-algebra and then joining these sheaves using *recollement de faisceaux* (EGA 0, 3.3).

Let Ω_{fin} be the set of finite places of k and let $U_1 = \Omega_{fin} - \Sigma_1$. $(S \cup T_1 \cup T_2) \cap \Sigma_1 = \phi$, so $S \cup T_1 \cup T_2 \subset U_1$. Identify U_1 with an open subset of Y and let $\{W_i | W_i \subset U_1\}_{i\in I}$ be a covering of U_1 by distinguished affine open subsets of Y. Each $W_i = D(f_i)$ for some $f_i \in A$. The affine scheme $(W_i, \underline{o}_Y|_{W_i})$ is canonically isomorphic to Spec A_{f_i}. (A_{f_i}, z_1, θ) is a graded A_{f_i}-algebra of finite type. (A_{f_i}, z_1, θ) can also be regarded as a graded A-algebra. Let $(A_{f_i}, z_1, \theta)^\sim$ be the associated quasi-coherent graded $\underline{o}_Y$-algebra. Then $(A_{f_i}, z_1, \theta)^\sim \rceil W_1$ is canonically isomorphic to the quasi-coherent graded $o_{\mathrm{Spec}\, A f_i}$-algebra associated to the A_{f_i}-algebra (A_{f_i}, z_1, θ) (EGA I, 1.3.6). For $h, i \in I$, $(A_{f_h}, z_1, \theta)|_{W_h \cap W_i}$ and $(A_{f_i}, z_1, \theta)^\sim|_{W_h \cap W_i}$ are isomorphic as quasi-coherent graded $\underline{o}_Y$-algebras because each is isomorphic to $(A_{f_h f_i}, z_1, \theta)^\sim|_{W_h \cap W_i}$.

Let $\Sigma_2 = \{v \in \Omega | v$, finite, $v \in S \cup T_1 \cup T_2$ or $\Delta(z_2) \notin A_v^*\}$. By construction $\Sigma_1 \cap \Sigma_2 = \phi$. Let $U_2 = \Omega_{fin} - \Sigma_2$. Identify U_2 with an open subset of Y. Let $\{X_j | X_j \subset U_2\}_{j\in J}$ be a covering of U_2 by distinguished affine open subsets of Y. If $X_j = D(g_j)$ and $X_k = D(g_k)$, $g_j, g_k \in A$, then as before $(A_{g_j}, z_2, \theta)^\sim|_{X_j \cap X_k}$ and $(A_{g_k}, z_2, \theta)^\sim|_{X_j \cap X_k}$ are isomorphic as quasi-coherent graded $\underline{o}_Y$-algebras.

$\{W_i, X_j\}_{i\in I, j\in J}$ is an open cover for Y. In order to apply *recollement de faisceaux* to establish the existence and uniqueness of a quasi-coherent graded $\underline{o}_Y$-algebra $\mathscr{S}$ such that $\mathscr{S}|_{W_i}$ is isomorphic to $(A_{f_i}, z_1, \theta)^\sim|_{W_i}$ for all $i \in I$ and $\mathscr{S}|_{X_j}$ is isomorphic to $(A_{g_j}, z_2, \theta)^\sim|_{X_j}$ for all $j \in J$ it suffices to verify that $(A_{f_i}, z_1, \theta)^\sim|_{W_i \cap X_j}$ and $(A_{g_j}, z_2, \theta)^\sim|_{W_i \cap X_j}$ are isomorphic. $(A_{f_i}, z_1, \theta)^\sim|_{W_i \cap X_j}$ is canonically isomorphic to the quasi-coherent graded $o_{\mathrm{Spec}\, A f_i g_j}$-algebra associated to $(A_{f_i g_j}, z_1, \theta)$ and $(A_{g_j}, z_2, \theta)^\sim|_{W_i \cap X_j}$ is canonically isomorphic to the $o_{\mathrm{Spec}\, A f_i g_j}$-algebra associated to $(A_{f_i g_j}, z_2, \theta)$. Since the functor $\sim$ from the category of graded $A_{f_i g_j}$-algebras to

the category of quasi-coherent graded $o_{\mathrm{Spec}\, A_{f_i g_j}}$-algebras is faithful, it suffices to prove

Proposition 1. *$(A_{f_i g_j}, z_1, \theta)$ and $(A_{f_i g_j}, z_2, \theta)$ are isomorphic graded $A_{f_i g_j}$-algebras.*

Proof. Let C be the $A_{f_i g_j}$-algebra $A_{f_i g_j}[Y_0, \ldots, Y_n]/(Y_0 Y_1 \ldots Y_{n-1} - \theta Y_n^n)$. There is a $A_{f_i g_j}$-algebra homomorphism of $(A_{f_i g_j}, z_1, \theta)$ onto C which maps $X_0 + \sigma^i(z_1)X_1 + \ldots + \sigma^i(z_1^{n-1})X_{n-1}$ to Y_i for $i = 0, \ldots n-1$ and maps X_n to Y_n. This homomorphism is defined by the matrix

$$Z_1 = \begin{pmatrix} 1 & z_1 & z_1^2 & \ldots & z_1^{n-1} & 0 \\ 1 & \sigma(z_1) & \sigma(z_1^2) & \ldots & \sigma(z_1^{n-1}) & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \sigma^{n-1}(z_1) & \sigma^{n-1}(z_1^2) & \ldots & \sigma^{n-1}(z_1^{n-1}) & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Similarly, there is an $A_{f_i g_j}$-algebra homomorphism of $(A_{f_i g_j}, z_2, \theta)$ onto C defined by the matrix

$$Z_2 = \begin{pmatrix} 1 & z_2 & z_2^2 & \ldots & z_2^{n-1} & 0 \\ 1 & \sigma(z_2) & \sigma(z_2^2) & \ldots & \sigma(z_2^{n-1}) & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & \sigma^{n-1}(z_2) & \sigma^{n-1}(z_2^2) & \ldots & \sigma^{n-1}(z_2^{n-1}) & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

An easy calculation (recall that n is odd) shows that $\sigma(Z_2^{-1} Z_1) = Z_2^{-1} Z_1$, so $Z_2^{-1} Z_1$ has entries in k. By construction $\Delta(z_h) = \det Z_h \in A_v^*$ for all $v \in W_i \cap X_j$, $h = 1,2$. Hence $Z_2^{-1} Z_1$ has entries in $A_v \cap k$ and $\det Z_2^{-1} Z_1 \in A_v^* \cap k$ for all $v \in W_i \cap X_j$, so $Z_2^{-1} Z_1 \in \mathrm{GL}(n, A_v \cap k)$. Since $A_{f_i g_j} = \bigcap_{v \in W_i \cap X_j} (A_v \cap k)$, $Z_2^{-1} Z_1$ has entries in $A_{f_i g_j}$. Since $\det Z_2^{-1} Z_1$ is a unit in each $A_v \cap k$, it is a unit in $A_{f_i g_j}$. Hence $Z_2^{-1} Z_1 \in \mathrm{GL}(n, A_{f_i g_j})$. So $Z_2^{-1} Z_1$ defines an isomorphism of $(A_{f_i g_j}, z_1, \theta)$ and $(A_{f_i g_j}, z_2, \theta)$ as graded $A_{f_i g_j}$-algebras.

The properties of $\mathscr{S}$

The assertions of theorem 1 are proven in proposition 2-10.

Proposition 2. *$\mathscr{S}$ is a coherent graded $\underline{o}_Y$-algebra.*

Proof. By construction $\mathscr{S}$ is a quasi-coherent graded $\underline{o}_Y$-algebra of finite type, so since A is noetherian, $\mathscr{S}$ is a coherent $\underline{o}_Y$-module (EGA I, 1.5.1).

Proposition 3. *$\mathscr{S}$ is a flat $\underline{o}_Y$-module.*

Proof. Since $\mathscr{S}|_{W_i}$ is isomorphic to $(A_{f_i}, z_1, \theta)^{\sim}|_{W_i}$ and (A_{f_i}, z_1, θ) is an integral domain (cf. proposition 1,§2) $\mathscr{S}_x$ is an integral domain for all $p \in W_i$. Applying the same argument to $\mathscr{S}|_{X_j}$ we conclude that $\mathscr{S}_p$ is an integral domain for all $p \in Y$. But for each $p \in Y$, $\underline{o}_{p,Y}$ is either a discrete valuation ring or a field. So $\mathscr{S}_p$ is a flat $\underline{o}_{p,Y}$-module for all $p \in Y$.

Proposition 4. *Proj $(\mathscr{S})$ is a scheme of finite type over Y and the morphism f: Proj $(\mathscr{S}) \to Y$ is projective.*

Proof. Let $\mathscr{S}_i$ denote the homogeneous component of $\mathscr{S}$ of degree i. By construction $\mathscr{S}_0$ is isomorphic to $\underline{o}_Y$, $\mathscr{S}_1$ is an $\mathscr{S}_0$-module of finite type, and $\mathscr{S}$ is generated by $\mathscr{S}_1$ (EGA II, 3.1.9). The two assertions now follow from EGA II, 3.4.1 and 5.5.1-2 respectively.

Proposition 5. *Proj $(\mathscr{S})$ is integral i.e. reduced and irreducible.*

Proof. In the proof of proposition 3 we observed that $\mathscr{S}$ is integral (EGA II, 3.1.12) and that Y is an integral scheme. Since $\mathscr{S}_0$ is isomorphic to $\underline{o}_Y$, the conclusion follows from EGA II, 3.1.14.

Proposition 6. *Let ψ_0: Spec $k \to Y$ denote the canonical map of Spec k to the generic point of Y. Then Proj $(\mathscr{S})$ x_Y Spec k is a Chatelet model for the Severi-Brauer k-scheme V.*

Proof. As before, let $D(f)$ be a *basic* open set of Y contained in U_1. The schemes $(D(f), \underline{o}_Y|_{D(f)})$ and *Spec A_f* are canonically isomorphic. Identify them using this isomorphism. The inclusion homomorphisms $A \to A_f \to k$ induce maps *Spec $k \to$ Spec $A_f \to$ Spec A*; ψ_0 is the composite of these maps. Proj $(\mathscr{S})$ x_Y *Spec k* and *(Proj $(\mathscr{S})x_Y$ Spec $A_f)x_{\text{Spec}\,A_f}$ Spec k* are canonically isomorphic as k-schemes (EGA 0, 1.3.2), so it suffices to show that latter is a Chatelet model for V.

By the definition of *Proj $(\mathscr{S})$* (EGA II, 3.1.2-3) *Proj $(\mathscr{S})x_Y$ Spec A_f* is canonically isomorphic as an A_f-scheme to *Proj* $(\Gamma(D(f), \mathscr{S}))$. By construction $\mathscr{S}|_{D(f)}$ is isomorphic to the quasi-coherent graded $\underline{o}_{\text{Spec}\,A_f}$-algebra $(A_f, z_1, \theta)^{\sim}|_{D(f)}$, so *Proj $(\mathscr{S})x_Y$ Spec A_f* and *Proj* $((A_f, z_1, \theta)^{\sim}|_{D(f)})$ are canonically isomorphic as A_f-schemes. From the proof of EGA II, 3.5.3 one sees that $Proj((A_f, z_1, \theta)^{\sim}|_{D(f)})x_{\text{Spec}\,A_f}$*Spec k* and $Proj((A_f, z_1, \theta) \otimes_{A_f} k)^{\sim}$ are canonically isomorphic as k-schemes. Since k is a flat A_f-module, $(A_f, z_1, \theta) \otimes_{A_f} k$ and (k, z_1, θ) are canonically isomorphic graded k-algebras. So $Proj((A_f, z_1, \theta)^{\sim}|_{D(f)})x_{\text{Spec}\,A_f}$*Spec k* and $Proj(k, z_1, \theta)^{\sim}$ are canonically isomorphic k-schemes. The latter is a Chatelet model for V.

In order to state and prove the next proposition without introducing an awkwardly complicated notation we make the following convention:

$(A_v, z_i, \theta)^{\sim}$ (resp. $(A_f, z_i, \theta)^{\sim}$) will denote the graded $\underline{o}_{\text{Spec}\,A_v}$-(resp. $\underline{o}_{\text{Spec}\,A_f}$−) algebra (instead of the graded $\underline{o}_{\text{Spec}\,A}$ − algebra) associated to (A_v, z_i, θ) (resp. (A_f, z_i, θ)).

Proposition 7. *Let v be a finite place of k, k_v be the completion of k at v, A_v the valuation ring in k_v, and ψ_v: Spec $A_v \to Y$ be the map induced by the inclusion $A \to A_v$. If $v \in U_i$, $i = 1, 2$, then Proj$(\mathscr{S})x_Y$ Spec A_v and Proj$(A_v, z_i, \theta)^{\sim}$ are canonically isomorphic A_v-schemes.*

Proof. Assume $v \in U_1$ and let $D(f)$ be a basic set of Y contained in U_1 such that $v \in D(f)$. As in the proof of proposition 6 one sees that *Proj$(\mathscr{S})$ x_Y Spec A_v* and *(Proj$(\mathscr{S})$ x_Y Spec A_f) $x_{\text{Spec}\,A_f}$* are canonically isomorphic as A_v-schemes; and also that *Proj$(\mathscr{S})x_Y$ Spec A_f* and *Proj* $(A_f, z_1, \theta)^{\sim}$ are canonically isomorphic as A_f-schemes. From the proof of EGA II 3.5.3 one sees that *Proj(A_f, z_1, θ) $x_{\text{Spec}\,A_f}$ Spec A_v* and *Proj* $((A_f, z_1, \theta) \otimes_{A_f} A_v)^{\sim}$ are canonically isomorphic as A_v-schemes. Since A_v is obtained from A_f by localization and completion, A_v is a flat A_f-module, hence $(A_f, z_1, \theta) \otimes_{A_f} A_v$ and (A_v, z_1, θ) are isomorphic as graded A_v-algebras. So *Proj(A_f, z_1, θ) $x_{\text{Spec}\,A_f}$ Spec A_v* and *Proj*$(A_v, z_1, \theta)^{\sim}$ are canonically isomorphic A_v-schemes.

Proposition 8. *For all $v \in S$, v finite, Proj$(\mathscr{S})x_Y$ Spec A_v is a canonical model for $V(\gamma_v)$.*

Proof. Let q_v be the cardinality of the residue field of A_v and ζ be a primitive $q_v^n - 1$ root of unity in $k_{v,n}$, the unramified extension of k_v of degree n. Let σ be the Frobenius automorphism of $k_{v,n}$ over k_v. Since $v \in S$, $v \in U_1$. By proposition 7 it suffices to show that (A_v, z_1, θ) is isomorphic as a graded A_v-algebra to an (A_v, ζ, ϕ) where $\phi \in A_v$ and $ord_v(\phi) = t$, t being the positive integer defined in the first paragraph of §2. But by the corollary to proposition 1, §4, $ord_v(\theta) = t$. An isomorphism between (A_v, z_1, θ) and (A_v, ζ, θ) can be constructed by noting that

$$\begin{pmatrix} 1 & \zeta & \zeta^2 & \dots & \zeta^{n-1} & 0 \\ 1 & \sigma(\zeta) & \sigma(\zeta^2) & \dots & \sigma(\zeta^{n-1}) & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \sigma^{n-1}(\zeta) & \sigma^{n-1}(\zeta^2) & \dots & \sigma^{n-1}(\zeta^{n-1}) & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & z_1 & \dots & z_1^{n-1} & 0 \\ 1 & \sigma(z_1) & \dots & \sigma(z_1^{n-1}) & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \sigma^{n-1}(z_1) & \dots & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$\in GL(\dot{n}, A_v)$ (because $\Delta(\zeta) \in A_v^*$ by proposition 3, §2, and $\Delta(z_1) \in A_v^*$ by proposition 2, §4) and employing the argument used in the proof of proposition 1.

Proposition 9. *For all* $v \in T_1 \cup T_2$, $Proj(\mathscr{S}) x_Y \; Spec \; A_v$ *is a degenerate model for* ${}_vV$.

Proof. Since $v \in T_1 \cup T_2, v \in U_1$. Let $D(f)$ be a basic open set of Y contained in U_1 such that $v \in D(f)$. The inclusion homomorphisms $A \to A_f \to \to A_v \cap k \to A_v$ induce maps $Spec \; A_v \to Spec \; A_v \cap k \to Spec \; A_f \to Spec \; A$. Let Z_1 denote the $A_v \cap k$-scheme $(Proj \; (\mathscr{S}) \; x_Y \; Spec \; A_f) \; x_{\,Spec \; A_f} \; Spec \; (A_v \cap k)$. Then $Z_1 \, x_{\,Spec(A_v \cap k)} \; Spec \; A_v$ is A_v-isomorphic to $Proj(\mathscr{S}) x_Y \; Spec \; A_v$. As in the proofs of proposition 6 and 7 one sees that

$$Z_1 \, x_{\,Spec(A_v \cap k)} \, x \; Spec \; A_v \text{ and } Proj((A_v \cap k, z_1, \theta) \otimes_{A_v \cap k} A_v)^{\sim}$$

are isomorphic A_v-schemes. Since A_v is a flat $A_v \cap k$-module, $(A_v \cap \cap k, z_1, \theta) \otimes_{A_v \cap k} A_v$ and (A_v, z_1, θ) are isomorphic graded A_v-algebras. It suffices to show that $k' = k(z_1)$, $ord_w(z_1) \geq 1$ and that $\theta \in m_{A_v \cap k} \cap N_{k'_w/k_v} (k'_w)$ for all places w of k', $w | v$. That the first two conditions are satisfied follows from proposition 2, §4; that the last condition is satisfied follows from the corollary to proposition 1, §4, and the definition of T_2.

Proposition 10. *For all* $v \in \Omega$, *v finite*, $v \notin S \cup T_1 \cup T_2$, $Proj(\mathscr{S}) \, x_Y \; Spec \; A_v$ *is a completed model for* ${}_vV$.

Proof. In case $v \in U_1$ the proof follows that of proposition 9 up to the final two sentences. Beyond this point to conclude the proof of proposition 10 for $v \in U_1$ it suffices to show that $k' = k(z_1)$, $\Delta(z_1) \in (A_v \cap k)^*$ and that $\theta \in (A_v \cap k)^*$, $\theta \in N_{k'_w/k_v} (k'_w)$ for all places w of k', $w | v$. By proposition 2, §4, and the definition of $U_1, k' = k(z_1)$ and $\Delta(z_1) \in A_v^*$; since $\Delta(z_1) \in k$, $\Delta(z_1) \in (A_v \cap k)^*$. By the corollary to proposition 1, §4, and the definition of S, T_1, and $T_2, \theta \in A_v^*$ for all v, v finite, $v \notin S \cup T_1 \cup T_2$; since $\theta \in k^*$, $\theta \in (A_v \cap k)^*$ for these v. Also by the corollary to proposition 1, §4, $\theta \in N_{k'_w/k_v} (k'_w)$ for all $w | v$, $v \notin S$.

In case $v \notin U_1$, let $D(g)$ be a basic open subset of Y contained in U_2 such that $v \in D(g)$. The inclusion homomorphisms $A \to A_g \to A_v \cap k \to A_v$ induce maps $Spec \; A_v \to Spec \; A_v \cap k \to Spec \; A_g \to Spec \; A$. Let Z_2 denote the $A_v \cap k$-scheme $(Proj(\mathscr{S}) \, x_Y \; Spec \; A_g) \, x_{\,Spec \; A_g} \, x \; Spec \; (A_v \cap k)$.
$Z_2 \, x_{\,Spec(A_v \cap k)} \; Spec \; A_v$ is A_v-isomorphic to $Proj(\mathscr{S}) x_Y \; Spec \; A_v$. As in the proof of proposition 9 one sees that $Z_2 \, x_{Spec(A_v \cap k)} Spec \; A_v$ and $Proj(A_v \, z_2, \theta)^{\sim}$ are isomorphic as A_v-schemes. So it suffices to show that $k' = k(z_2)$, $\Delta(z_2) \in (A_v \cap k)^*$ and that $\theta \in (A_v \cap k)^*, \theta \in N_{k'_w/k_v} (k'_w)$ for all $w | v$.
By proposition 3, §4, $k' = k(z_2)$ and $\Delta(z_2) \in A_v^*$; since $\Delta(z_2) \in k$, $\Delta(z_2) \in \in (A_v \cap k)^*$. Since $v \notin U_1$, by the corollary to proposition 1, §4, $\theta \in A_v^*$; since $\theta \in k^*$, $\theta \in (A_v \cap k)^*$. Also by the same corollary $\theta \in N_{k'_w/k_v} (k'_w)$ for all $w | v$.

§6. The zeta function of proj ($\mathscr{S}$).

Let V be a Severi-Brauer k-scheme of dimension $n-1$, n an odd prime, defined by $\gamma \in Br(k)$. Let $X = Proj \, (\mathscr{S})$ be the Y-scheme constructed in §5. As X is of finite type over Y (proposition 4, §5) and Y is of finite type over $Spec \, \mathbb{Z}$, X is of finite type over $Spec \; \mathbb{Z}$ (EGA I, 6.3.4). For each $x \in X$, let o_x/m_x be the residue field of x and let $N(x) = \text{card} \; (o_x/m_x)$. Let $\bar{X}$ denote the set of closed points of X. Then $\bar{X} = \{x \in X \,|\, N(x) \text{ is finite}\}$. ζ_X is defined by $\zeta_X(s) = \prod_{x \in \bar{X}} \frac{1}{1 - N(x)^{-s}}$ ([13]). This product converges absolutely for $Re\,(s) > n-1$. Let $f: X \to Y$ be the projection. For each $y \in Y$, let o_y/m_y be the residue field of y, $N(y) = \text{card} \; (o_y/m_y)$, and $X_y = f^{-1}(y)$. Let $\bar{Y}$ denote the set of closed points of Y. Then $\zeta_X(s) = = \prod_{y \in \bar{Y}} \zeta_{X_y}(s)$. Each $\zeta_{X_y} = \zeta(N(y)^{-s})$, where ζ denotes the zeta function of X_y regarded as a scheme over o_y/m_y. Hence ζ_X contains factors corresponding to each of the finite places of k.

Recall that for $v \in S$, v finite, $X \, x_Y \; Spec \; A_v$, is a canonical model for $V(\gamma_v)$ (theorem 1, §5) and that the zeta function of the closed fiber of a canonical model is given by

$$\zeta_{X_v}(s) = [(1 - N(v)^{(n-1-s)n})(1 - N(v)^{(n-2-s)n}) \ldots (1 - N(v)^{(1-s)n})(1 - N(v)^{-s})]^{-1}$$

(corollary of proposition 3, §3). For all other finite v, $X \, x_Y \; Spec \; A_v$ is either a completed or a degenerate model for a trivial Severi-Brauer k_v-scheme and the zeta function of the closed fiber of $X \, x_Y \; Spec \; A_v$ is given by

$$\zeta_{X_v}(s) = [(1 - N(v)^{n-1-s})(1 - N(v)^{n-2-s}) \ldots (1 - N(v)^{1-s})(1 - N(v)^{-s})]^{-1}$$

(corollaries of propositions 2, 4, and 5, §3).

It is generally agreed that the full zeta function of X should also contain factors corresponding to each of the infinite places of k, but is not clear just how these factors should be defined. There are two possibilities. For an infinite place v one could use the factor corresponding to v which occurs in the zeta function $Z_{D(\gamma)}$ of the central division algebra $D(\gamma)$ defined by γ. The other possibility is to use a definition which has been proposed by J. P. Serre ([12]) and is based on the complex cohomology of ${}_vX = (X \; x_Y \; Spec \; k) \; x_{\,Spec \, k} \; Spec \; k_v$. Pursuing either alternative leads to the same definition for the factor corresponding to v, as we now show.

For each infinite v such that k_v is isomorphic to $\mathbb{R}$, $D(\gamma) \otimes_k k_v$ is a central simple $\mathbb{R}$-algebra of dimension n^2 over $\mathbb{R}$. Since n is odd, $D(\gamma) \otimes_k k_v$ is k_v-isomorphic to $M_n(\mathbb{R})$. Hence the factor in $Z_{D(\gamma)}$ corresponding to v is $\prod_{i=0}^{n-1} G_1(s-i)$, where $G_1(s) = \pi^{-s/2} \, \Gamma(s/2)$ ([15], prop. 1, p. 203). For each $v \in \Omega$ such that k_v is isomorphic to $\mathbb{C}$ the factor in $Z_{D(\gamma)}$ corresponding to v is $\prod_{i=0}^{n-1} G_2(s-i)$, where $G_2(s) = (2\pi)^{1-s}\,\Gamma(s)$.

For each infinite v, ${}_vX$ can be regarded as an $\mathbb{R}$-scheme in case k_v is isomorphic to $\mathbb{R}$, and as a $\mathbb{C}$-scheme in case k_v is isomorphic to $\mathbb{C}$ and an isomorphism between k_v and $\mathbb{C}$ has been chosen. In either case, let ${}_vX(\mathbb{C})$ denote the $\mathbb{C}$-valued points of ${}_vX$. ${}_vX(\mathbb{C})$ is a singular $\mathbb{C}$-variety birationally isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$. Since ${}_vX(\mathbb{C})$ is not a complex analytic manifold, the proceedure proposed by Serre for determining the factor corresponding to v in the zeta function of X is not applicable. However, this procedure can be applied to $\mathbb{P}^{n-1}(\mathbb{C})$, for which it gives a plausible result.

For each $i, 0 \le i \le 2(n-1)$, $H^i(\mathbb{P}^{n-1}(\mathbb{C}), \mathbb{C})$ is a $\mathbb{C}$-vector space of dimension one or zero, according as i is even or odd. In case i is even, the Hodge decomposition of $H^i(\mathbb{P}^{n-1}(\mathbb{C}), \mathbb{C})$ is simply $\Omega^{i/2,\, i/2}$. If k_v is isomorphic to $\mathbb{C}$, the factor corresponding to v for the i^{th} cohomology group is $(2\pi)^{-1}\, G_2(s - i/2)$ or 1, according as i is even or odd; the complete factor corresponding to v is $G_{\mathbb{C}}(s) = \prod_{\substack{0 \le i \le 2(n-1) \\ i,\, even}} (2\pi)^{-1}\, G_2(s - i/2)$. This is the same factor as the one corresponding to v in $Z_{D(\gamma)}$, except for the innocuous $(2\pi)^{-n}$. If k_v is isomorphic to $\mathbb{R}$, $G = \mathrm{Gal}\,(\mathbb{C}/\mathbb{R})$ acts on $\mathbb{P}^{n-1}(\mathbb{C})$ by conjugation, an anti-analytic automorphism. So G acts on each $H^i(\mathbb{P}^{n-1}(\mathbb{C}), \mathbb{C})$. Let g be the non-trivial element of G and $x \in H^i(P^{n-1}(\mathbb{C}), \mathbb{C})$, $x \neq 0$, then $g(x) = (-1)^{i/2}\, x$. Thus, the factor corresponding to v for the i^{th} cohomology group is $G_1(s - i/2)$ or 1, according as i is even or odd. The complete factor corresponding to v is $G_{\mathbb{R}}(s) = \prod_{\substack{0 \le i \le 2(n-1) \\ i,\, even}} G_1(s - i/2)$, the same factor as the one corresponding to v in $Z_{D(\gamma)}$.

Definition. Let r_1 be the number of real places of k and r_2 be the number of complex places of k. The function $Z_X(s) = G_{\mathbb{R}}(s)^{r_1} G_{\mathbb{C}}(s)^{r_2}\, \zeta_X(s)$, defined in the half-plane $Re(s) > n$ and homomorphic there, will be called the full zeta function of X.

Proposition 1. *Let $s \in \mathbb{C}$, $Re(s) > n$. Then*

$$Z_X(s) = $$
$$= Z_{D(\gamma)}(s) \cdot \prod_{\substack{v \in S \\ v,\, finite}} [(1 - N(v)^{(n-1-s)n})(1 - N(v)^{(n-2-s)n}) \ldots (1 - N(v)^{(1-s)n}]^{-1}.$$

Proof. Since n is prime, for each $v \in S$, v finite, $D(\gamma) \otimes_k k_v$ is a central division algebra over k_v and the factor corresponding to v in $Z_{D(\gamma)}$ is $(1 - N(v)^{-s})^{-1}$

([15], proposition 7, p. 197). For all other finite v the factor corresponding to v in $Z_{D(\gamma)}$ is $[(1 - N(v)^{n-1-s})(1 - N(v)^{n-2-s}) \ldots (1 - N(v)^{-s})]^{-1}$. So Z_X and $Z_{D(\gamma)}$ are formed from the same local factors for all infinite v and for all finite $v, v \notin S$. The conclusion follows by comparing the factors in Z_X corresponding to each $v \in S$, v finite, with their counterparts in $Z_{D(\gamma)}$.

Theorem 1. *Z_X can be continued analytically as a meromorphic function in the s-plane, holomorphic except for simples poles at $s = 0$ and $s = n$. Z_X satifies the functional equation*

$$Z_X(s) = (N\delta)^{(s/n - 1/2)(n-1)}\, |\, D\, |^{n^{(n/2 - s)}}\, Z_X(n - s),$$

where $N\delta$ is the norm of the different of $D(\gamma)$ over k and D is the discriminant of k.

Proof. $Z_{D(\gamma)}$ is a meromorphic function in the s-plane, holomorphic except for simple poles at $s = 0$ and $s = n$ ([14], theorem 2, II). The finite product which appears in the statement of proposition 1 defines a non-vanishing holomorphic function in the s-plane, so the first two assertions follow from proposition 1. $Z_{D(\gamma)}$ satisfies the functional equation

$$Z_{D(\gamma)}(s) = (N\delta)^{1/2 - s/n}\, |\, D\, |^{n(n/2 - s)}\, Z_{D(\gamma)}(n - s)$$

([14], loc. cit.). So by proposition 1

$$Z_X(s) = \prod_{\substack{v \in S \\ v,\, finite}} (N(v)^{s - n/2})^{n(n-1)} \cdot (N\delta)^{1/2 - s/n}\, |\, D\, |^{n\,(n/2 - s)}\, Z_X(n - s).$$

For $v \in S$, v finite, $N\delta_v$, the norm of the different of $D(\gamma) \otimes_k k_v$ over k_v, is given by $N\delta_v = N(v)^{n(n-1)}$([14]), proof of proposition 2, II). As $N\delta = \prod_{\substack{v \in S \\ v,\, finite}} N\delta_v$, the functional equation has the form asserted.

References

[1] A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publications, XXIV, 1939.
[2] S. Amitsur, *Generic splitting fields of central simple algebras*, Ann. of Math. 62 (1955), 8-43.
[3] E. Artin and J. Tate, *Class Field Theory*, Benjamin, 1967.
[4] Z. Borevich and I. Shafarevich, *Number Theory*, Academic Press, 1966.
[5] J. W. S. Cassels and A. Fröhlich, eds., *Algebraic Number Theory*, Thompson Book Company, 1967.
[6] F. Chatelet, *Variations sur un thème de H. Poincaré*, Ann. ENS, 59 (1944), 249-300.
[7] M. Deuring, *Algebren*, Springer-Verlag, 1968.
[8] H. Hasse, *Zahlentheorie*, Akademie-Verlag, 1969.
[9] S. Lang, *Algebraic Number Theory*, Addison-Wesley, 1970.
[10] P. Roquette, *On the Galois cohomology of the projective linear group...*, Math. Ann. 150 (1963), 411-439.
[11] J.-P. Serre, *Corps locaux*, Hermann, 1968.
[12] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algebriques* (définitions et conjectures), Séminaire Delange-Pisot-Poitou, 1969/70, exp. 19.
[13] J.-P. Serre, *Zeta and L-functions*, in *Arithmetical Algebraic Geometry*, Harper and Row, 1965, 82-92.
[14] S. Turner, *Zeta-functions of central simple algebras over global fields*, An. Acad. Brasil. Ci. 48 (2), (1976), 171-186.
[15] A. Weil, *Basic Number Theory*, Springer-Verlag, 1967.
[16] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I, D. Van Nostrand, 1958.
A.C. N. Bourbaki, *Eléments de Mathematique, Algèbre Commutative*, Hermann, 1961.
EGA. A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique*, I. Springer-Verlag, 1971; II, IV, Publications Mathématiques, IHES, 1961-65.

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