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**S. Yu. Slavyanov**

# INTEGRAL EQUATIONS OF FREDHOLM TYPE WITH RAPIDLY VARYING KERNELS AND THEIR RELATIONSHIP TO DYNAMIC SYSTEMS

ABSTRACT. The relationship between eigenfunctions of Fredholm type integral equation with rapidly oscillating kernel and dynamic mapping is analysed. The differential operators commuting with Fourier operator are constructed. These operators are closely connected with nontrivial solutions of unperturbed nonlinear functional equation related to the dynamic mapping.

In the early 70th both Vladimir Lazutkin and me had been working on the problem of the "bouncing ball" eigenfunctions. He attacked the problem starting with dynamics of rays and then enfolding the solution into the "wave meat" [1, 2]. I have been working with integral equations and with the related functional equations [3, 4]. In both approaches the problem of small denominators arose which caused major difficulties to construct the rigorous solution. Vladimir was more experienced in scrupulous mathematics and had overcome these difficulties [2]. I was lazy and mostly interested in physically sound conclusions and established only formal theory used further for various approximate formulae [5].

Here is the brief reconstruction of some of my ideas at that time added by new results. This text can be regarded not as a true paper but rather as an essay inspired by reminiscences and as an introduction to further studies.

## 1. Formulation of the problem

Consider the integral equation

$$y_n(q) = \mu_n \int_{-a}^{a} e^{icF(q,z)} A(q,z) y_n(z) dz \tag{1}$$

at $c \to \infty$. Here the functions $F(q,z)$ and $A(q,z)$ are supposed to satisfy the following symmetries with respect to variables $q$ and $z$

$$F(q,z) = F(z,q), \quad A(q,z) = A(z,q)$$
$$F(q,z) = F(-q,-z), \quad A(q,z) = A(-q,-z)$$

and are characterized by the following behavior at zero:

$$A = 1 + O(q^2 + z^2), \quad F(q,z) = qz - \frac{g}{2}(q^2 + z^2) + O(q^4 + z^4).$$

The parameter $g$ varies in the interval $[0,1)$.

The goal of the study in [3,4] was to find asymptotic expansions for eigenvalues $\mu_n$ and eigenfunctions $y_n(z)$. The formal procedure was proposed in these papers to construct formally at least the leading asymptotic term for the ground eigenfunction $y_0(z)$. Suppose we seek the asymptotics of $y_0(z)$ as

$$y_0(z) = h_0(z)\exp(-f(z)) \tag{2}$$

and use the saddle-point formula in order to evaluate the integral in (1). Let $z = \phi(q)$ be the adjacent saddle point (satisfying the assumption $\phi(q) \to 0$ at $q \to 0$). Then the condition that the left hand side expression in (1) is equal (up to smaller terms in $c$) to the right hand side expression in (1) implies that $\phi(q)$ must satisfy the following nonlinear functional equation

$$\frac{\partial F(q,s)}{\partial s} + \frac{\partial F(s,t)}{\partial s} = 0 \tag{3}$$

with $s = \phi(q)$, $t = \phi(\phi(q))$. Equation (3) is the analogue to nonlinear Hamilton–Jacobi equation used in semiclassical approach to differential equation. On the other hand it determines a dynamic mapping $\phi : q \to z$, $z \to q$. The particular case of pure Fourier transform in (1) corresponds to the equation

$$\phi(\phi(q)) + q = 0. \tag{4}$$

If terms beyond quadratic in $F(q,z)$ are omitted we arrive to the equation

$$\phi(\phi(q)) - 2g\phi(q) + q = 0. \tag{5}$$

As far as solution of (3) is known the function $f$ is restored with the help of integration

$$f(q) = -i\int_0^q \frac{\partial F(q,\phi)}{\partial q}dq. \tag{6}$$

For the amplitude function $h_0(z)$ we obtain a linear functional equation (an analogue for transport equation for semiclassical theory) of the form

$$h_0(q) = \mu_0 h_0(\phi(q))T(q, \phi(q)) \tag{7}$$

with $T(q, \phi(q))$ maintaining an explicit expression in terms of $A(q, \phi(q))$ and $F(q, \phi(q))$. The resolvability condition of (6) determines eigenvalues $\mu_0 \phi'(0)T(0, 0) = 1$.

Let

$$\rho_{1,2} = e^{\pm i\pi\alpha}, \quad \alpha \equiv (1/\pi)\arccos g$$

be solutions of the characteristic equation

$$\rho^2 - 2g\rho + 1 = 0.$$

Two cases can be distinguished: i) $\alpha$ is rational, ii) $\alpha$ is irrational. In the case ii) the above mentioned functional equations (3), (7) have solutions in the form of formal power series at $q = 0$ but the problem of small denominator arises. In the case i) even formal expansions fail and some other approaches are needed. This situation is typical for dynamic mappings.

The formulated conjecture is closely related to the problem whether "unperturbed" equations (4–5) have other solutions than linear ones

$$\phi_{1,2}(q) = \rho_{1,2} q. \tag{8}$$

In its turn the latter problem is related to the number of differential operators commuting with "unperturbed" integral operator (1) with $A(q, z) = 1$, $F(q, z) = qz - g(q^2 + z^2)/2$, $a = \infty$.

## 2. Differential operators commuting with the Fourier transform

We denote the Fourier transform as

$$\mathcal{F}[Y] : G(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqz} Y(z) dz$$

and use below its well known properties

$$\mathcal{F}[zY] = -iD_q \mathcal{F}[Y], \quad \mathcal{F}[D_z Y] = -iq\mathcal{F}[Y].$$

A simpler problem of a pure Fourier transform can be studied as a particular case of (1)

$$Y_n(q) = \frac{\lambda_n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqz} Y_n(z) dz. \tag{9}$$

There are four infinitely degenerated eigenvalues $\lambda_n = \exp(in\pi/2)$ of (9) laying on the unit circle. The eigenfunctions are conventionally supposed to be harmonic oscillator functions. This fact is due to the commutation relation

$$\mathcal{F}[L_z Y] = L_q \mathcal{F}[Y] \quad \text{with} \quad L_z = D_z^2 + (2n + 1 - z^2). \tag{10}$$

However, because of high degeneracy of eigenvalues other functions may be chosen as eigenfunctions corresponding to (9). If one tries to establish a perturbation techniques to operator (9), for instance, by turning to final limits of integration, he will meet crucial difficulties arising from the high degeneracy.

Is there another basis of eigenfunctions originated by a differential operator also commuting with Fourier operator? Yes, there is. The eigenfunctions on $[-1, 1]$ related to differential equation

$$D(1 - z^2)Dy + (\lambda - z^2)y = 0 \tag{11}$$

give another basis. These are prolate spheroidal functions $S_{0,n}(1, z)$ [6]. Another example is the equation

$$D(1 + z^2)Dy + (\lambda - z^2)y = 0. \tag{12}$$

Equations (10–12) give, so far as the author knows, the only examples of second-order differential operators commuting with Fourier transform. However, turning to higher order operators we arrive to the following list

$$\begin{aligned} L_z^{(1,k)} &= D_z^{2+4k} + (\Lambda - z^{2+4k}), \\ L_z^{(2,k)} &= D_z^{1+2k}(1 - z^{2+4k})D_z^{1+2k} + (\Lambda - z^{2+4k}), \\ L_z^{(3,k)} &= D_z^{1+2k}(1 + z^{2+4k})D_z^{1+2k} + (\Lambda - z^{2+4k}). \end{aligned} \tag{13}$$

It is important that the asymptotic study of eigenfunctions shows their different behavior of at $\pm\infty$.

Differential operator (10) corresponds to solution

$$\phi(q) = \pm iq \tag{14}$$

of (4). Differential operators (11–13) correspond to solution

$$\phi(q) = \frac{\pm iq}{(1 \pm q^{\beta})^{1/\beta}}, \quad \beta = 2 + 4k \tag{15}$$

of the same equation. Hence, infinite order of degeneracy of integral equation (9) is in relationship with infinite number of solutions of dynamic functional equation (4).

Beyond (9) the eigenfunctions of the Fourier transform in involution can be studied

$$Y_n(q) = \frac{\lambda_n}{2\pi} \mathcal{F}[\mathcal{F}[Y]]. \tag{16}$$

This extension permits to introduce additional parameterization. Namely, the following commutation identity holds

$$\mathcal{F}[L_z Y] = L_q^* \mathcal{F}[Y] \quad \text{with} \quad L_z = D_z^2 - 2igzD_z + (\nu - z^2) \tag{17}$$

with $L^*$ being the formally adjoint to $L$ differential operator. Eigenfunctions of this operator can be also explicitly expressed in terms of harmonic oscillator functions (see [3]). At rational $\alpha$ we arrive once again to infinite degeneracy. Differential operator (17) corresponds to solutions (8) of functional equation (5). An open question is: what are differential operators similar to (11–13) for this case and what are additional to (8) corresponding solutions of (5)?

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Department of Computational Physics,
St.Petersburg State University

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