Записки научных
семинаров ПОМИ
Том 300, 2003 г.

H. R. Dullin, A. V. Ivanov

TWISTLESS TORI NEAR LOW ORDER RESONANCES

Abstract. In this paper we investigate the behaviour of the twist near low order resonances of a periodic orbit or an equilibrium of a hamiltonian system with two degrees of freedom. Namely, we analyse the case when a Hamiltonian has multiple eigenvalues (the hamiltonian Hopf bifurcation) or a zero eigenvalue near the equilibrium and the case when the system possesses a periodic orbit, which multipliers equal to 1 (the saddle-centre bifurcation) or -1 (the period-doubling bifurcation). We show that the twist does not vanish at least in a small neighborhood of the period- doubling bifurcation. For the saddle-center bifurcation and the resonances of an equilibrium under consideration we prove the existence of the "twistless" torus for sufficiently small values of the bifurcation parameter. The explicit dependence of the energy corresponding to the twistless torus on the bifurcation parameter is derived.

It is well known that a hamiltonian system with two degrees of freedom can essentially be described near an equilibrium or a periodic orbit in terms of its normal form, which represent an integrable hamiltonian system in that case. So in a small neighborhood of the equilibrium (periodic orbit) the original system can be thought of as a small perturbation of some integrable one. The motion of such an integrable system is easily described. Indeed, the phase space is divided by invariant Lagrangian tori $I = \text{const}$ (I stands for the action variables) and each of the tori possesses conditionally-periodic motion. The KAM theory states that most of these quasiperiodic tori persist under small perturbations being only slightly deformed. A particular torus survives a perturbation if its frequency vector $\omega(I)$ satisfies some Diophantine-type condition and the frequency map $I \to \omega(I)$ from actions to frequencies is nondegenerate. The latter is a transversality condition which has many different forms. In the isoenergetic case we are considering we will refer to it as the twist condition. It is connected to the geometry of the phase space foliated by invariant tori, which correspond to the intergable normal form. Break down of the twist condition can lead to a very interseting behaviour of a system. For example, "reconnection bifurcations" occur near an extremal point of the twist map [4] and the "meandering curves" may appear when the twistless torus passes through a rational rotational number [5].

In this paper we investigate the behaviour of the twist near low order resonances of a periodic orbit or an equilibrium of a hamiltonian system with two degrees of freedom, i.e. those resonances which play an essential role already in the quadratic terms of a Hamiltonian. Namely, we analyse the case when a Hamiltonian has multiple eigenvalues (the hamiltonian Hopf bifurcation) or a zero eigenvalue near the equilibrium and the case when the system possesses a periodic orbit, which multipliers equal to 1 (the saddle-center bifurcation) or -1 (the period-doubling bifurcation). One should note that the $1:3$ resonance of a periodic orbit has been treated in [3].

We begin with the case of a periodic orbit. In this situation the behaviour of a system near the orbit can be described by a one-dimensional integrable hamiltonian system with the Hamiltonian

$$H(\varphi, I) = \omega I + \frac{1}{2}\tau_0 I^2 + \frac{1}{6}\tau_1 I^3 + ..., \tag{1}$$

where (φ, I) denote the action-angle coordinates. Equivalently in a small vicinity of the periodic orbit the system can be rewritten in terms of the Poincaré map, whose normal form is

$$(\varphi, I) \to (\varphi + 2\pi\Omega(I), I), \tag{2}$$

where $\Omega(I) = \omega + \tau_0 I + \frac{1}{2}\tau_1 I^2 + ...$ is the rotation number.

If we define the "twist" to be the derivative of the rotation number with respect to the action

$$\tau(I) \equiv \frac{\mathrm{d}\Omega}{\mathrm{d}I}(I) = \tau_0 + \tau_1 I + ..., \tag{3}$$

the break down of the twist condition is equivalent simply to the vanishing of $\tau(I)$.

First let us consider the period-doubling bifurcation, which is more simple for technical reasons. The normal form of a hamiltonian system near this bifurcation is the following [1]

$$H(u, v) = \frac{v^2}{2} + D\frac{u^4}{4} + \varepsilon u^2, \tag{4}$$

where ε is a bifurcation parameter and D is non-zero constant. We can always assume $D = \pm 1$. If not one may rescale the parameters by means of the following change $\tilde{u} = D^{1/6}u$, $\tilde{v} = D^{-1/6}v$, $\tilde{\varepsilon} = D^{-2/3}\varepsilon$, $\tilde{H} = D^{-1/3}H$ to represent the system in the form (4).

There are two scenarios of the system behaviour which correspond to different values of D. They are represented in the Fig. 1.

It turns out that in the case of a periodic orbit the condition for vanishing of the twist can be reformulated in terms of the partial derivative of the period $T(h,\varepsilon)$ with respect to the energy value h. Namely the twist vanishes whenever $\frac{\partial T}{\partial h}(h,\varepsilon) = 0$. It is not difficul to show that the partial derivative $\frac{\partial T}{\partial h}(h,\varepsilon)$ can be expressed in terms of the elliptic integrals and the equation $\frac{\partial T}{\partial h}(h,\varepsilon) = 0$ is equivalent in this case to the following

$$\begin{cases} (1-2k^2)E(k) = (1-k^2)K(k), \quad 0<k<1, & h \geqslant 0 \\ \left(1-\frac{1}{2}k^2\right)E(k) = (1-k^2)K(k), \quad 0<k<1, & h<0,\ \varepsilon \leqslant -(-h)^{1/2} \end{cases} \tag{5}$$

for the case $D = 1$ and

$$\begin{cases} (1-2k^2)E(k) = (1-k^2)K(k), \quad 0<k<1, & h<0 \text{ or } h \geqslant \varepsilon^2 \\ (1+k^2)E(k) = (1-k^2)K(k), \quad 0<k<1, & \varepsilon \leqslant 0,\ 0 \leqslant h \leqslant \varepsilon^2 \\ \left(1-\frac{1}{2}k^2\right)E(k) = (1-k^2)K(k), \quad 0<k<1, & \varepsilon \geqslant 0,\ 0 \leqslant h \leqslant \varepsilon^2 \end{cases} \tag{6}$$

for the case $D = -1$.
Here $K(k)$ and $E(k)$ stand for the elliptic integrals of module k of the first and the second kind, respectively.

One can prove that all these equations have no solutions on the specified range of the parameter k. It means that in a small neighborhood of the period-doubling bifurcation the twist does not vanish and the nondegeneracy condition is fullfilled. In Fig. 2 we demonstrate the behaviour of constant lines of the rotation number on the bifurcation plane (ε, h). The numerical calculations are in a perfect agreement with the stated result.

Now let us consider the saddle-center bifurcation. The normal form of a hamiltonian system near this bifurcation is written in the following way [1]

$$H(u,v) = \frac{v^2}{2} + \frac{u^3}{3} + \varepsilon u, \tag{7}$$

where ε is the bifurcation parameter as in the previous case. When ε passes zero the phase space of the system (7) is subjected to qualitative changes which are shown in fig.3.

As in the case of period-doubling bifurcation it is possible to express the period $T(h,\varepsilon)$ and the partial derivative $\frac{T}{h}(h,\varepsilon)$ in terms of the elliptic

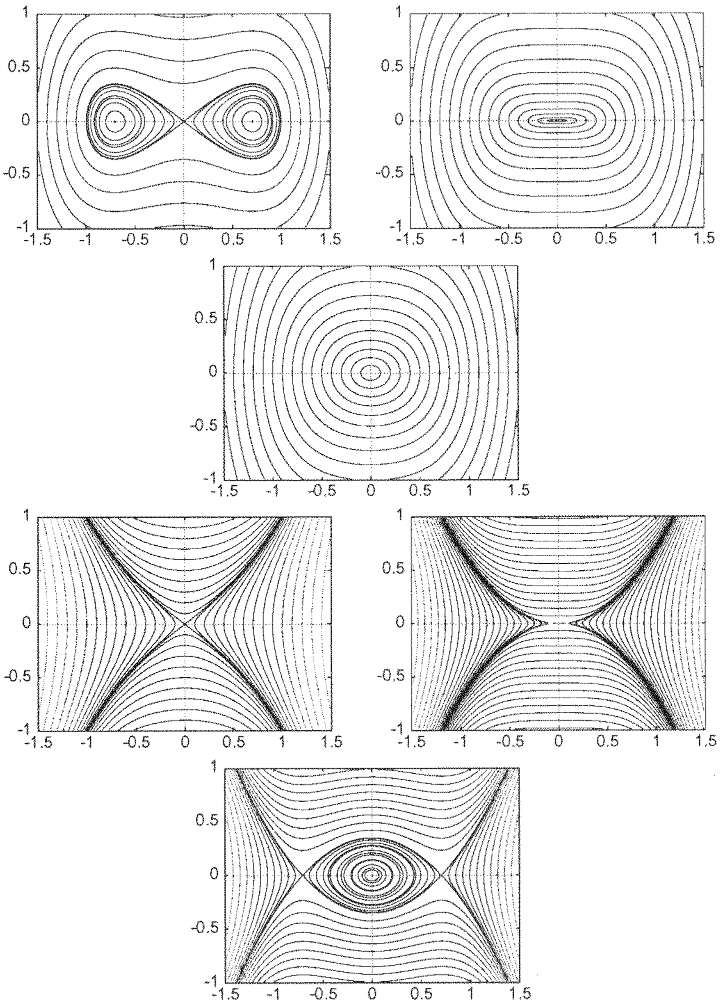

Fig. 1. Two series of three phase portraits, describing metamorphoses of the system (4) when the bifurcation parameter takes negative, zero and positive values, respectively. The first series corresponds to $(D = 1)$ and the second one to $(D = -1)$.

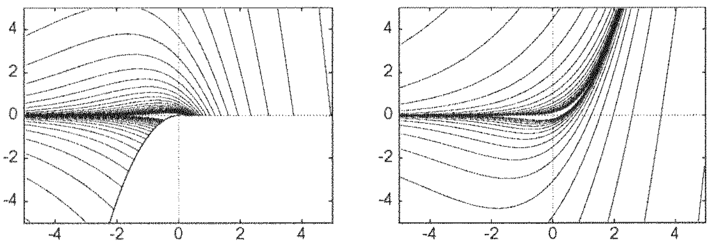

Fig. 2. Lines of the constant rotation number equidistant with $\Delta\Omega = 0.3$ on the bifurcation plane (ε, h) for the period-doubling bifurcation. Left picture corresponds to the case $(D = 1)$ and the right one to $(D = -1)$.

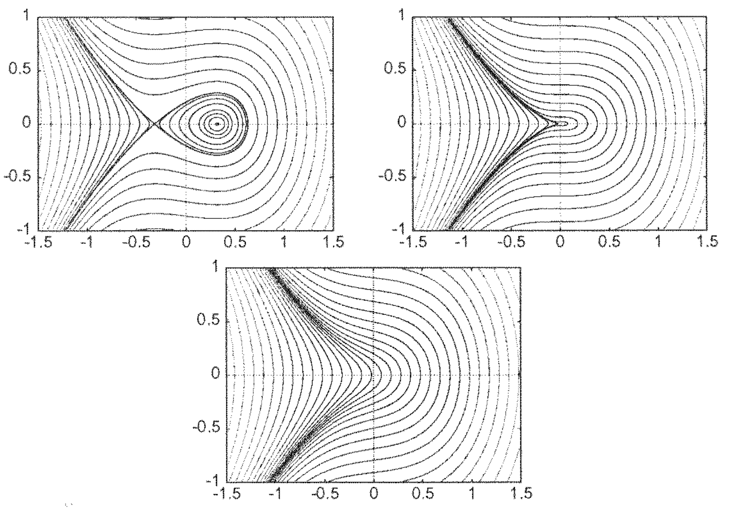

Fig. 3. A series of three phase portraits, describing metamorphoses of the system (7) when the bifurcation parameter takes negative, zero and positive values respectively..

integrals such that the equation $\frac{T}{h}(h,\varepsilon)=0$ is of the form

$$\begin{cases} (1-16k^2+16k^4)E(k)=(1-9k^2+8k^4)K(k), \\ 0<k<1,\, \varepsilon>0 \ \text{ or } \ \varepsilon<0, 9h^2>-4\varepsilon^3 \\ (1-k^2-k^4)E(k)=\left(1-\frac{3}{2}k^2+\frac{1}{2}k^4\right)K(k), \\ 0<k<1,\, \varepsilon<0, 9h^2<-4\varepsilon^3 \end{cases} \tag{8}$$

It can be proven that while the second equation has no solutions on the specified domain of the variable k, the first one has exactly one solution k_0, which is approximately equal to $k_0=0.84245....$ Note that k is a function of ε and h, so the equation $k(h,\varepsilon)=k_0$ defines a curve on the bifurcation plane (ε,h). To derive an expression for this curve we rewrite the first equation (8) for k such that $0.5<k<1$ in the following way:

$$\begin{gathered}\left(3\alpha-1+(1+\alpha)^{1/2}\right)K(k(\alpha))=2(3\alpha-1)E(k(\alpha)),\\ k(\alpha)=\left(\frac{1}{2}\left(1+\left(\frac{1}{1+\alpha}\right)^{1/2}\right)\right)^{1/2},\ \alpha>0\end{gathered} \tag{9}$$

Obviously, this equation has the unique solution α_0 such that $k_0=k(\alpha_0)$, which is equal to $\alpha_0=4.68340....$ But it turns out that

$$\alpha=\frac{1}{3}\left(\frac{1+(\gamma+\sqrt{1+\gamma^2})^{2/3}}{-1+(\gamma+\sqrt{1+\gamma^2})^{2/3}}\right)^2,\quad \gamma=\frac{3h}{2\varepsilon^{3/2}} \tag{10}$$

We can invert the formula (10) to resolve the equation $\gamma_0=\alpha^{-1}(\alpha_0)$ and find that $\gamma_0=0.91522...$ approximately. Thus the expression for the twist curve is

$$3h=2\gamma_0\varepsilon^{3/2} \tag{11}$$

It means that the "twistless" torus is created, when the bifurcation parameter passes through zero, it exists after creation for all values of the bifurcation parameter and moves outward the periodic orbit as can be seen in Fig. 4.

Now let us consider resonances near an equilibrium. First one may show [1] that for the case when a Hamiltonian has a zero eigenvalue near the equilibrium the normal form coincides with (7). So we can apply here results obtained for the saddle-center bifurcation and prove the existence of the twistless torus near this resonance and hence the break down of the transversality condition.

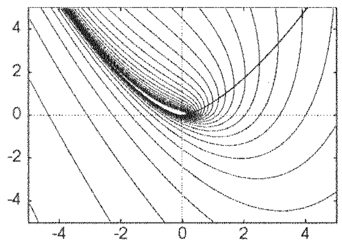

Fig. 4. Lines of the constant rotation number equidistant with $\Delta\Omega = 0.3$ on the bifurcation plane (ε, h) for the saddle-center bifurcation. The solid line corresponds to the twistless torus..

The most interesting case occurs near the hamiltonian Hopf bifurcation. Its normal form can be rewritten as [6], [2]

$$H(u, v) = \frac{1}{2}\left(P^2 + \frac{J^2}{r^2}\right) + \omega J + \frac{1}{2}(\varepsilon + 2BJ)r^2 + CJ^2 + \frac{1}{2}Dr^4, \quad D \neq 0, \tag{12}$$

where (r, ψ) are coordinates, (P, J) – the conjugate momenta, ω, B, C, D are some real constants and ε is the bifurcation parameter. In this paper we study the case of positive D only.

The Hamiltonian does not depend on the coordinate ψ, so J is an integral of motion, moreover it coincide with one of the actions of the system (12) $J = J_1$. It is to be noted that the nondegeneracy condition cannot be reformulated in terms of the period as in the previous cases, so we represent it in the standard way

$$\frac{\partial W}{\partial J_1} \neq 0, \quad W = \frac{\omega_1}{\omega_2} = -\frac{\partial J_2}{\partial J_1}, \tag{13}$$

where W is the winding number and (J_1, J_2) are the actions of the system (12).

We introduce the following notations:

$$Q(\xi) = -(\xi^3 + q_2\xi^2 + q_1\xi + q_0), \quad q_2 = \frac{\delta + 2BJ}{D},$$
$$q_1 = 2\frac{\omega J + CJ^2 - h}{D}, \quad q_0 = \frac{J^2}{D}. \tag{14}$$

Then the action J_2 and the winding number W can be expressed as

$$J_2 = \frac{D^{1/2}}{4\pi} \oint \frac{Q^{1/2}(\xi)}{\xi} \mathrm{d}\xi, \tag{15}$$

$$W = \frac{1}{4\pi D^{1/2}} \oint \frac{B\xi^2 + (\omega + 2CJ)\xi + J}{\xi Q^{1/2}(\xi)} \mathrm{d}\xi. \tag{16}$$

When $D > 0$ the polynomial Q has three real roots, which we denote by ξ_0, ξ_{min}, ξ_{max}, such that $\xi_0 \leqslant 0 \leqslant \xi_{min} \leqslant \xi_{max}$. Taking this into account we rewrite the winding number in terms of elliptic integrals such that

$$W = \frac{1}{\pi D^{1/2}\sqrt{\xi_{max} - \xi_0}}$$
$$\left((B\xi_0 + \omega + 2CJ)K(k) + B(\xi_{max} - \xi_0)E(k) + \frac{J}{\xi_{max}}\Pi(n,k)\right), \tag{17}$$

$$k^2 = \frac{\xi_{max} - \xi_{min}}{\xi_{max} - \xi_0}, \quad n = \frac{\xi_{max} - \xi_{min}}{\xi_{max}}, \tag{18}$$

where $K(k)$, $E(k)$, $\Pi(n,k)$ stand for the elliptic integrals of the first, second and third kind, respectively.

We can substitute this formulae into (13) and get an expression for the partial derivative $\frac{\partial W}{\partial J_1}$. Note that we have to consider the winding number as a function of three variables h, J_1 and ε (assuming ω, B, C, D to be fixed). It turns out that the analysis of the equation $\frac{\partial W}{\partial J_1} = 0$ is rather complicated problem in this case. To overcome this let us study the behaviour of the winding number near the origin of the plane (J_1, h), which is a singular point of W.

First let us introduce new variables

$$x = \frac{D}{\varepsilon^2}(\omega J - h), \quad y = \frac{D}{|\varepsilon|^{3/2}} J \tag{19}$$

and consider the following functions

$$\rho(x,y) = x^2 + y^2, \tag{20}$$

$$\Delta(x,y) = \arcsin\left(\left(1 + \frac{1}{2}\left(\mathrm{sign}(x)\sqrt{\frac{x^2}{x^2+y^2}} - 1\right)\right)^{1/2}\right). \tag{21}$$

One may prove the following

Theorem. *Provided ε to be negative the winding number $W = W(x,y)$ as a function of variables (x,y) has the following asymptotics in a small punctured neighborhood of the origin:*

$$W(x,y) = -\frac{\omega}{4\pi|\varepsilon|^{1/2}} \log(x^2+y^2)$$
$$+\frac{1}{\pi}\operatorname{sign}(y)\Delta(x,y) + O_1(\rho\log(\rho(x,y))). \tag{22}$$

We define the model function W_{mod} to be the leading term in the formula (22). In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ it takes the following simple form:

$$W_{\mathrm{mod}}(r,\theta) = -\frac{\omega}{4\pi|\varepsilon|^{1/2}}\log r + \frac{1}{2}\left\{\frac{\theta}{\pi}\right\}, \tag{23}$$

where $\{\}$ denotes the fractional part.

Now it is easy to show that $\frac{\partial W_{\mathrm{mod}}}{\partial J_1}$ vanishes on the line

$$h = h_{\mathrm{mod}}(J) = \left(\omega - \frac{2\varepsilon\omega}{\omega^2+\varepsilon}\right)J. \tag{24}$$

Taking into account the asymptotic (22) and using the implicit function theorem we prove that provided ε to be negative in a small punctured neighborhood of the origin on the (J,h)-plane there exists a unique twistless curve which has the following expansion

$$h = h_{\mathrm{twist}}(J) = \left(\omega - \frac{2\varepsilon\omega}{\omega^2+\varepsilon}\right)J + O_1(\rho(J,h)\log\rho(J,h)). \tag{25}$$

It is possible to show that when $\varepsilon > 0$ the partial derivative $\frac{\partial W}{\partial J_1}$ does not vanish in a small neighborhood of the plane (J_1,h). Thus in the case of the hamiltonian Hopf bifurcation we prove that when the bifurcation parameter passes through the origin the twistless tori appear near the resonant torus $(J,h) = (0,0)$. Fig. 5 illustrates the behaviour of the winding number near the origin of the plane (J,h).

Acknowledgements. This research is supported by the EPSRC grant GR/R44911/01; partial support by the European Research Training Network 'MASIE' (HPRN-CT-2000-00113) is also gratefully acknowledged. AVI was supported in part by INTAS grant 00-221, RFBI grant 01-01-00335 and RFME grant E00-1-120.

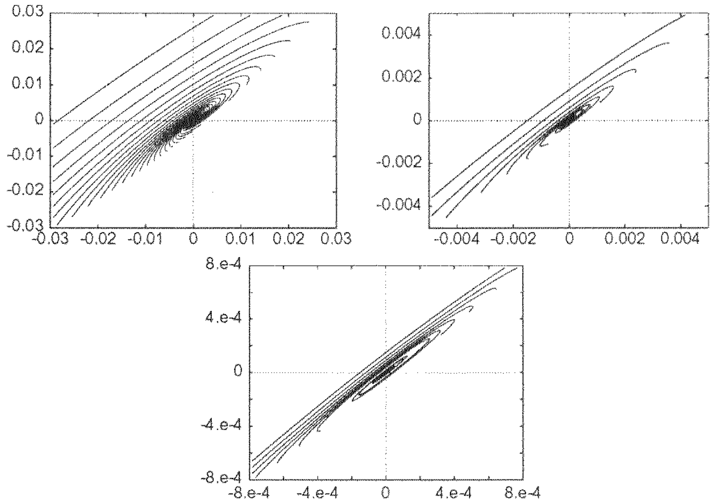

Fig. 5. A series of figures, representing lines of constant rotation number in a small neighborhood of the origin of the (J, h)-plane equidistant with $\Delta W = 1.5$ for $B = C = D = \omega = 1$; $\varepsilon = -0.15, -0.05, -0.02$ respectively..

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Department of Mathematical Sciences,
University of Loughborough, LE11 3TU, UK

E-mail: H.R.Dullin@lboro.ac.uk,
A.V.Ivanov@lboro.ac.uk

Поступило 30 ноября 2002 г.