Записки научных
семинаров ПОМИ
Том 300, 2003 г.

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GENERALIZED COHERENT STATES: A NOVEL APPROACH

Abstract. We define generalized coherent states for oscillator-like systems connected with orthogonal polynomials (classical, q-deformed etc.). In considered cases such polynomials play the same role, as the Hermite polynomials are defined in the case of usual boson oscillator.

1. Introductory remarks. The significance of coherent states (CS) [1] for physics [2] (especially for quantum mechanics [3], quantum optics [4, 7], quantum field theory [5] and for condensed matter physics [6, 7] as well as for mathematics (especially for the theory of group representations [8]) is difficult to overestimate. Introduced by E. Schrödinger [9] in the early days of quantum theory as non-spreading wave packets with quasi-classical behavior the coherent states became very popular at beginning of the sixties after the works of R. J. Glauber [10, 11], J. R. Klauder [12], and E. C. G.Sudarshan [13] in which the CS was applied to the problems of quantum optics. At the same time V. Bargmann developed the necessary mathematical background – the theory of Hilbert spaces of analytical functions [14, 15]. We refer the reader to the recent review of V. V. Dodonov [16] which contains the huge amount (more than 1000 items) of related bibliographical information about the CS, their numerous generalizations as well as several applications to quantum mechanical and mathematical physics problems.

Nowadays there are three different approaches to the construction of coherent states: 1) as eigenfunctions of the annihilation operator (they usually called Barut–Girardello CS) [17]; 2) as states generated by the given state (it is usually the vacuum state or the highest weight state) with the help of the unitary displacement operator. Such states are known as Perelomov-CS (PCS in the following) [8]; 3) as minimum uncertainty states. In the case of the usual boson oscillator all these definitions are equivalent in the sense that they give the same set of CS. But this is not true in the general case.

This research was supported by RFFI grant No 00-01-00500.

In recent publications [18–21] we suggested a novel approach to the construction of generalized coherent states for oscillator-like system connected with given Jacobi matrix J and a related system of the orthonormal polynomials.

Our construction is as follows: we associate the orthonormal polynomial system $\{\Psi_n(x)\}_{n=0}^{\infty}$ with J and consider a related generalized oscillator algebra with ladder operators $a^{\pm}$ and vacuum state $|\Psi_0\rangle$ (for the details of this construction we refer to [22, 23]). Note that the spectrum of the Hamiltonian (the energy operator) for the related oscillator is defined only by the coefficients of recurrent relation for considered system of orthonormal polynomials. Then CS of the Barut–Girardello type can be defined, as usual, as eigenstates of the annihilation operator. The main difficulties of our construction are in writing such defined CS in terms of known special functions and in obtaining the decomposition of the unity for these CS. For the solution of the latter task one must solve a classical moment problem [24] related with the considered Jacobi matrix J. Thus for a given energy spectrum of physical system one can define oscillator-like system with standard defined Hamiltonian, construct the system of coherent states and evaluate values of physically interesting observable quantities. In addition we avoid the difficult factorization problem of the given Hamiltonian operator.

In the following we briefly report some of our results connected with construction of the families of coherent states related with all, so called, classical orthonormal polynomials (Hermite, Laguerre, and Jacobi). Note that such construction of CS admits an extension to all families of the (orthonormal) Askey–Wilson polynomials as well as to their deformations (in particular deformed orthonormal polynomials related to quantum groups). The work in this direction is now in progress and the obtained results will be published in the future.

2. Coherent states for the Legendre oscillator. The Legendre polynomials can be defined by the relation ($n \in \mathbb{N}_0$)

$$P_n(x) = {}_2F_1\left(\begin{matrix}-n,\ n+1\\ 1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}, \tag{1}$$

where the symbol $[n]$ denotes the integer part of the number n. This polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{n\,m}. \tag{2}$$

The Legendre functions $\psi_n(x) = \sqrt{2n+1}\,P_n(x)$, $n \in \mathbb{N}_0$, make up the orthonormal basis $\{|n\rangle \equiv \psi_n(x)\}_{n=0}^{\infty}$ in the Hilbert space $\mathcal{H}\colon = L^2\left([-1,1], \frac{1}{2}\mathrm{d}\,x\right)$ and fulfill the recurrent relations

$$x\psi_n(x) = b_{n-1}\psi_{n-1}(x) + b_n\psi_{n+1}(x), \quad \psi_{-1}(x) = 0, \quad \psi_0(x) = 1, \tag{3}$$

with coefficients

$$b_n = \sqrt{\frac{(n+1)^2}{(2n+1)(2n+3)}}, \quad n \geq 0. \tag{4}$$

In the Hilbert space $\mathcal{H}$ we define the generalized position operator X, connected with the Legendre polynomials $P_n(x)$, as an operator of multiplication by argument:

$$X|n\rangle = x|n\rangle \quad \text{or} \quad X\psi_n(x) = x\psi_n(x) = b_n\psi_{n+1}(x) + b_{n-1}\psi_{n-1}(x), \tag{5}$$

where the coefficients b_n are defined by the relation (4). Because

$$\sum_{k=0}^{\infty}\frac{1}{b_k} = \infty,$$

the operator X is a selfadjoint operator in the space $\mathcal{H}$ (see [24, 25]).

Let us define the generalized momentum operator P by the method described in [22]. The action of the operator P on the basis elements in $\mathcal{H}$ is described by the following formula

$$P|n\rangle = i(b_n|n+1\rangle - b_{n-1}|n-1\rangle). \tag{6}$$

Calculating the commutator of operators X and P on the elements of the basis, we obtain

$$[X,P]|n\rangle = 2i({b_n}^2 - {b_{n-1}}^2)|n\rangle = \frac{2i}{(2n-1)(2n+1)(2n+3)}|n\rangle. \tag{7}$$

Next we define the creation and annihilation operators by the standard relations $a^{(\pm)} = \frac{1}{\sqrt{2}}(X \mp iP)$. On the elements of the basis in the space $\mathcal{H}$ these operators act by the rules

$$a^{(+)}|n\rangle = \sqrt{2}b_n|n+1\rangle, \quad a^{(-)}|n\rangle = \sqrt{2}b_{n-1}|n-1\rangle \tag{8}$$

and satisfy the commutation relations $[a^{(-)}, a^{(+)}] = \frac{1}{i}[X, P]$.

The state numbering operator N and Hamiltonian H are defined by

$$N|n\rangle = n|n\rangle, \quad H = X^2 + P^2 = a^{(+)}a^{(-)} + a^{(-)}a^{(+)}. \tag{9}$$

The eigenvalues of the operator H are equal to

$$\lambda_0 = 2{b_0}^2, \quad \lambda_n = 2({b_{n-1}}^2 + {b_n}^2). \tag{10}$$

It is natural to call the defined system as the Legendre oscillator. We define the coherent states in the space $\mathcal{H}$ for this Legendre oscillator as eigenvectors of the annihilation operator

$$a^{(-)}|z\rangle = z|z\rangle, \quad |z\rangle = \mathcal{N}^{-1}\sum_{n=0}^{\infty} \frac{z^n}{(\sqrt{2}b_{n-1})!}|n\rangle. \tag{11}$$

The normalizing factor is equal to

$$\mathcal{N}^2 = \langle z|z\rangle = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2{b_{n-1}}^2)!} = {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{3}{2}\\ 1\end{matrix}\middle| 2|z|^2\right). \tag{12}$$

After some computations one obtains

$$\begin{aligned}|z\rangle = &\left[{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{3}{2}\\ 1\end{matrix}\middle| 2|z|^2\right)\right]^{-\frac{1}{2}} \\ &\times {}_2F_1\left(\begin{matrix}\frac{3}{4},\frac{5}{4}\\ 1\end{matrix}\middle| \frac{(x^2-1)2z^2}{(1-\sqrt{2}\,xz)^2}\right)(1-\sqrt{2}\,xz)^{-\frac{3}{2}}.\end{aligned} \tag{13}$$

As the next task we will construct the measure

$$\mathrm{d}\mu(|z|^2) = W(|z|^2)\mathrm{d}^2 z, \quad \text{such that} \quad \iint_{\mathbb{C}} W(|z|^2)\,|z\rangle\langle z|\mathrm{d}^2 z = 1,$$

where $\mathrm{d}^2 z = \mathrm{d}(\mathrm{Re} z)\mathrm{d}(\mathrm{Im} z)$. It is known (see f. ex., [26]) that this problem is reduced to a solution of the following classical Stieltjes moment problem

$$\int_0^1 \tau^n W(\tfrac{1}{2}\tau)\mathrm{d}\,\tau = \frac{(\Gamma(n+1))^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}, \quad \tau = 2t = 2|z|^2, \quad n \geq 0. \tag{14}$$

The solution of the above moment problem is given [18] by

$$W(t) = \frac{(16t-5)P_{\frac{1}{2}}(4t-1) - 3P_{\frac{3}{2}}(4t-1)}{2(2t-1)} + \\ +2\delta(2t-1), \quad 0 < t \le \frac{1}{2}, \tag{15}$$

which gives for the measure the following expression $(0 < |z| \le \frac{1}{\sqrt{2}})$

$$\mathrm{d}\mu(|z|^2) = \left[\frac{(16|z|^2-5)P_{\frac{1}{2}}(4|z|^2-1) - 3P_{\frac{3}{2}}(4|z|^2-1)}{2(2|z|^2-1)} + 2\delta(2|z|^2-1)\right] \\ \times \mathrm{d}\,(\mathrm{Re}\, z)\mathrm{d}\,(\mathrm{Im}\, z). \tag{16}$$

The overlapping of two coherent states is equal [18] to

$$\langle z_1 | z_2 \rangle = \\ = \left[{}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix} \middle| 2|z_1|^2 \right) {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix} \middle| 2|z_2|^2 \right) \right]^{-\frac{1}{2}} {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 1 \end{matrix} \middle| 2\overline{z}_1 z_2 \right). \tag{17}$$

In [21] we extend this construction of coherent states to the case of ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ (the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = \beta$), which (up to normalization) coincides with Gegenbauer polynomials $C_n^{\lambda}(x)$: $(2\lambda)_n\, P^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x) = (\lambda+\frac{1}{2})_n C_n^{\lambda}(x)$, where $(a)_n$ – denotes the Pochhammer symbol (shifted factorial) defined by the relation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdot\ldots\cdot(a+n-1), \quad (a)_0 = 1; \quad n \ge 1. \tag{18}$$

Note that the CS for Gegenbauer oscillator in the cases $\alpha = 0$ and $\alpha = -\frac{1}{2}$ reproduce the CS of Legendre [18] and Chebyshev [19, 21] oscillators, respectively.

3. Coherent states for the Laguerre oscillator. Let $\{\Psi_n(x)\}_{n=0}^{\infty}$ be the orthogonal system of the Laguerre polynomials

$$\Psi_n(x) = \sqrt{\frac{\Gamma(\alpha_n+1)}{n!\Gamma(\alpha+1)}}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right), \quad n \geqslant 0. \tag{19}$$

in the Hilbert space $\mathcal{H} = \mathrm{L}^2(R^1;\mu)$ with the probability measure

$$\mu(\mathrm{d}x) = x^{\alpha} e^{-x} \frac{\mathrm{d}x}{\sqrt{\Gamma(\alpha+1)}}.$$

The Laguerre polynomials fulfill the recurrent relations

$$
\begin{gathered}
x\Psi_n(x) = b_n\Psi_{n+1}(x) + a_n\Psi_n(x) + b_{n-1}\Psi_{n-1}(x), \\
\Psi_0(x) = 1, \quad b_{-1} = 0,
\end{gathered}
\tag{20}
$$

where the coefficients a_n and b_n are equal to

$$
a_n = 2n + \alpha + 1, \quad b_n = -\sqrt{(n+1)(n+\alpha+1)}, \qquad n \geq 0. \tag{21}
$$

Repeating the above construction of the CS, we obtain the coherent states for the Laguerre oscillator

$$
|z;\alpha\rangle = \frac{1}{(\sqrt{2}\,x)^{\alpha} I_{\alpha}(\sqrt{2}\,z)} e^{z/\sqrt{2}} I_{\alpha}(2^{3/4}\sqrt{xz}), \tag{22}
$$

where

$$
I_{\alpha}(z) = \left(\frac{z}{2}\right)^{\alpha} \sum_{n=0}^{\infty} \frac{z^n}{2^n n! \Gamma(\alpha+n+1)} \tag{23}
$$

is the Bessel function of the first kind. In this case the overlapping of two coherent states is given by [20]

$$
\langle z_1;\alpha \mid z_2;\alpha\rangle = \frac{I_{\alpha}(2\sqrt{\bar{z}_1 z_2})}{\sqrt{I_{\alpha}(\sqrt{2}\,|z_1|) I_{\alpha}(\sqrt{2}\,|z_2|)}}. \tag{24}
$$

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Поступило 30 ноября 2002 г.