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SPREADING MAPS (POLYMORPHISMS), SYMMETRIES OF POISSON PROCESSES, AND MATCHING SUMMATION

ABSTRACT. The matrix of a permutation is a partial case of Markov transition matrices. In the same way, a measure preserving bijection of a space (A, α) with finite measure is a partial case of Markov transition operators. A Markov transition operator also can be considered as a map (polymorphism) $(A, \alpha) \to (A, \alpha)$, which spreads points of (A, α) into measures on (A, α).

Denote by $\mathbb{R}^*$ the multiplicative group of positive real numbers and by $\mathcal{M}$ the semigroup of measures on $\mathbb{R}^*$. In this paper, we discuss $\mathbb{R}^*$-polymorphisms and Υ-polymorphisms, who are analogues of the Markov transition operators (or polymorphisms) for the groups of bijections $(A, \alpha) \to (A, \alpha)$ leaving the measure α quasiinvariant; two types of the polymorphisms correspond to the cases, when A has finite and infinite measure respectively. For the case, when the space A itself is finite, the $\mathbb{R}^*$-polymorphisms are some $\mathcal{M}$-valued matrices.

We construct a functor from Υ-polymorphisms to $\mathbb{R}^*$-polymorphisms, it is described in terms of summations of $\mathcal{M}$-convolution products over matchings of Poisson configurations.

0.0. Notation and terminology. The subject of this paper is pure measure theory without any additional structures.

The term "*measure*" in this paper means a positive Borel measure. The term "subset" of a space with measure means a Borel measurable subset.

The term *space with measure* means a Lebesgue measure space i.e., a space which is equivalent to the union of some interval of $\mathbb{R}$ (the interval can be finite, infinite or empty) and some collection of points having nonzero measures (this collection can be finite, countable or empty). We say that the measure is *continuous*, if all points have zero measure.

We denote spaces with measure by (A, α), (B, β), (M, μ) etc., the Latin capital letter denotes the space, the Greek letter denotes the measure.

All our measures are defined on Borel σ-algebras.

The symbol $\mathbb{R}^*$ denotes the multiplicative group of positive real numbers. By $\mathcal{M}$ we denote the space of finite positive measures on $\mathbb{R}^*$. We equip this space with the weak convergence; a sequence $\mathbf{u}_j \in \mathcal{M}$ weakly converges to $\mathbf{u} \in \mathcal{M}$, if for any bounded continuous function ψ on $\mathbb{R}^*$ we have the convergence $\int \psi(x)\, d\mathbf{u}_j(x) \to \int \psi(x)\, d\mathbf{u}(x)$ (this definition forbids departure of the measure to $+\infty$ and 0). The expression $\mu * \nu$ denotes the convolution of measures on the multiplicative group $\mathbb{R}^*$.

0.1. Groups. We consider 4 groups. For a space (A, α) with a finite continuous measure, we define the following groups.

– $\mathrm{Ams}(A)$ is the group of all measure preserving bijections $A \to A$ (Ams is the abbreviation of "automorphisms of the measure space"),

– $\mathrm{Gms}(A)$ is the group of all maps $A \to A$ leaving the measure α quasiinvariant.

For a space (M, μ) with an infinite continuous measure, we define two groups:

– $\mathrm{Ams}_\infty(M)$ is the group of all measure preserving bijections $M \to M$,

– $\mathrm{Gms}_\infty(M)$ is the group of all maps $A \to A$ leaving the measure μ quasiinvariant and satisfying the condition

$$\int_M |q'(m) - 1|\, d\mu(m) < \infty.$$

Remark. The group $\mathrm{Gms}_\infty(M)$ has a homomorphism to the additive group of $\mathbb{R}$ given by

$$q \mapsto \int_M (q'(m) - 1)\, d\mu(m).$$

It turns out that all these groups admit natural embeddings to semigroups of spreading maps (or polymorphisms). The semigroup of polymorphisms related to the group $\mathrm{Ams}(A)$ is a well-known object (see [31], see also [14], [20]). Recall its definition.

0.2. The usual polymorphisms. Let (A, α), (B, β) be spaces with probability measures. Consider a probability measure $\mathfrak{P}$ on $A \times B$. We say that $\mathfrak{P}$ is a *polymorphism* or *bistochastic kernel* $\mathfrak{P} : A \to B$ if

– the image of $\mathfrak{P}$ under the projection[1] $A \times B \to A$ is the measure α;

– the image of $\mathfrak{P}$ under the projection $A \times B \to B$ is the measure β.

[1] It is also called the *marginal*.

By the Rokhlin theorem on conditional measures (see [28]), for almost all $a \in A$ there exists a probability measure $\mathfrak{P}_a$ on $a \times B$ such that

$$\mathfrak{P}(Q) = \int_A \mathfrak{P}_a(Q \cap \{a \times B\})\, d\alpha(a).$$

Remarks. 1) Let U, V be sets. Let R be a subset in $U \times V$. We can consider R as a *relation* or a *multivalued map* $U \to V$. For a point $u \in U$, its image consists of all the points $v \in V$ such that $(u, v) \in R$. For two relations $R \subset U \times V$, $S \subset V \times W$, we define their product $T = SR \subset U \times W$. It consists of all $(u, w) \in U \times W$ such that there exists $v \in V$ satisfying the conditions $(u, v) \in R$, $(v, w) \in S$. Multivalued maps appear in a natural way in various branches of mathematics. The most classical example is the definition of algebraic functions $\mathbb{C} \to \mathbb{C}$. Recall that an algebraic function is a subset in $\mathbb{C} \times \mathbb{C}$ satisfying a polynomial equation $p(x, y) = 0$.

2) Nonformally, a polymorphism $\mathfrak{P}$ is a kind of a multivalued map that spreads each point $a \in A$ into the measure $\mathfrak{P}_a$, i.e., we know not only the image of a point, but also a probability distribution on its image.

3) Polymorphisms are also continuous analogues of Markov transition matrices (see [31] for detailed explanations, see also [9]).

Example. Let $q : A \to A$ be a measure preserving bijection. Consider its graph graph(q), i.e., the subset of $A \times A$ consisting of all the points $(a, q(a))$. Consider the map $A \to A \times A$ given by $a \mapsto (a, q(a))$. The image $\mathfrak{P}_q$ of measure α with respect to this map is a measure supported by graph(A). Obviously, $\mathfrak{P}_q$ is a polymorphism.

Example. The measure $\alpha \times \beta$ is a polymorphism $(A, \alpha) \to (B, \beta)$. Nonformally, this polymorphism is the total "uniform spreading" of the set A along the set B.

Let $\mathfrak{P} : (A, \alpha) \to (B, \beta)$ and $\mathfrak{Q} : (B, \beta) \to (C, \gamma)$ be two polymorphisms. Let $\mathfrak{P}_a(b)$ and $\mathfrak{Q}_b(c)$ be the corresponding systems of conditional measures. We define the product $\mathfrak{R} = \mathfrak{Q}\mathfrak{P} : (A, \alpha) \to (C, \gamma)$ in terms of these conditional measures

$$\mathfrak{R}_a(c) = \int_B \mathfrak{Q}_b(c)\, d\mathfrak{P}_a(b). \tag{1}$$

Denote by $\mathrm{Pol}(A, B)$ the set of all polymorphisms $A \to B$.

The set $\mathrm{Pol}(A, A)$ is a semigroup with respect to the multiplication. This semigroup contains the group $\mathrm{Ams}(A)$.

Let $\mathfrak{P}_j, \mathfrak{P} : (A, \alpha) \to (B, \beta)$ be polymorphisms. We say that the sequence $\mathfrak{P}_j$ converges to $\mathfrak{P}$ if for each measurable subsets $U \subset A$, $V \subset B$ the sequence of real numbers $\mathfrak{P}_j(U \times V)$ converges to $\mathfrak{P}(U \times V)$.

It is readily seen that the space $\mathrm{Pol}(A, B)$ is compact.

It is easy to show (see [31], [20]) that the group $\mathrm{Ams}(A)$ is dense in the semigroup $\mathrm{Pol}(A, A)$.

Example. Let $q \in \mathrm{Ams}(A)$ be a mixing (i.e., for any subsets U, $V \in A$ the measure $\alpha(U \cap q^n(V))$ tends to $\alpha(U) \times \alpha(V)$ as $n \to +\infty$). Then q^n converges to the "uniform spreading" $\alpha \times \alpha$ in $\mathrm{Pol}(A, A)$. There is a wide literature on polymorphisms in the ergodic theory, see [7], [14], [31].

Remark. In fact, we have the category of polymorphisms. The objects are Lebesgue spaces with probability measure, and morphisms $A \to B$ are polymorphisms. For groups Gms, Ams_∞, Gms_∞, we also describe below some categories, whose objects are Lebesgue spaces with measure.

0.3. Closure of an invariant action and the extension problem. Consider a group G acting by measure preserving maps on a space A with a finite continuous measure α.

Extension problem. *For a given action of a group G, find the closure Γ of $G \subset \mathrm{Ams}(A)$ in the semigroup of polymorphisms of A.*

It seems that nothing interesting can happen for connected non-Abelian Lie groups G (the case of Abelian groups is another story). Nevertheless, the problem becomes very nontrivial for infinite-dimensional ("large") groups[2]. Indeed, the semigroup $\mathrm{Pol}_{\mathbb{R}^*}(A, A)$ is compact, and hence the semigroup Γ is also compact. Obviously, any compactification of a large group G essentially differs from the group G itself.

0.4. Another variant of extension problem. In many cases, the semigroup Γ is known by *a priory* reasons. Assume that G has some collection of unitary representations. Then usually *there exists a canonical semigroup $\Gamma \supset G$ such that any unitary representation of the group G admits a canonical extension to a representation of the semigroup Γ*. This statement was claimed by G. I. Olshanski in the end of 70s (see [25], [26], [18]), for more details see [20]).

[2] It seems that the term "large" group introduced by Vershik is better than "infinite-dimensional" group. For instance, our groups Ams, Ams_∞, Gms, Gms_∞ have no structure of a manifold, but they are "very large".

This is not a general theorem but an experimental fact. Nevertheless, in the most cases, there exists a constructive description of the semigroup Γ and its representations, see [20].

For many groups G, there exist also *a priory* theorems about the extension of representations to Γ.

Examples. 1) For $G = \mathrm{Ams}(A)$, the semigroup Γ is the semigroup $\mathrm{Pol}(A, A)$. The *a priory* theorem on extension of representations is obtained in [19], see also [20], Section 8.4.

2) For $G = \mathrm{Ams}_\infty$, Gms, Gms_∞, the semigroups Γ are the semigroups of polymorphisms defined below (Sections 1–2), see [19].

3) If G is the complete orthogonal group of a Hilbert space, then the semigroup Γ is the semigroup $Contr$ of all operators in the real Hilbert space with norm $\leqslant 1$, [25].

4) More interesting examples with inordinate Γ are contained in [26], [18], [20].

In many cases (see [22]), it can be easily shown that any homomorphism $G \to \mathrm{Ams}(A)$ can be extended to a homomorphism $\Gamma \to \mathrm{Pol}(A, A)$.

Thus we obtain the following variant of the extension problem (this variant is not exactly equivalent to the previous one).

Consider any case when Γ is known. For a given measure preserving action of a "large" group G, find an explicit description of the homomorphism $\Gamma \to$ Pol.

0.5. The purpose of the paper. I know only one work that can be attributed to this extension problem. Consider the well-known action of the complete infinite-dimensional orthogonal group $O(\infty)$ on the space with Gaussian measure (see [29], [30], see also [20]). The corresponding homomorphism of the semigroup of contractions $Contr$ to Pol was explicitly described by Nelson [17].

A few interesting measure preserving actions of large groups are known, and hence the polymorphism extension problem has a restricted interest. But the zoo of quasiinvariant actions is very rich (see the survey [22] and recent papers on virtual permutations and Pickrell type inverse limits of symmetric spaces [27], [11], [12], [3], [4], [24]).

It turns out that there are polymorphism-like semigroups related to all the groups Ams_∞, Gms, Gms_∞. We describe them explicitly below in Sections 1-2.

It seems that the most important of these objects is the semigroup

$\mathrm{Pol}_{\mathbb{R}^*}(A, A)$ related to the group $\mathrm{Gms}(A)$, its elements are measures on

$$A \times A \times \mathbb{R}^*$$

satisfying some additional conditions. These $\mathbb{R}^*$-polymorphisms can be considered as "spreading maps", but they spread not only points; also Radon–Nykodim derivatives at points are spreaded.

For each quasiinvariant action of a large group G on a measure space (A, α), we obtain a problem about extension of the homomorphism $G \to \mathrm{Gms}(A)$ to the homomorphism from Γ to $\mathrm{Pol}_{\mathbb{R}^*}(A)$.

The purpose of this paper is to understand the degree of the interest of this problem. We consider the simplest (for my test) nontrivial quasiinvariant action of a large group on a measure space (see the next subsection).

0.6. Poisson configurations. Let M be a space with a continuous infinite measure μ. Denote by $\Omega(M)$ the space of all countable subsets $\mathbf{m} = (m_1, m_2 \dots)$ in M. We define the *Poisson measure* ν on $\Omega(M)$ by the following conditions.

1^*. Let $A \subset M$ have finite measure. Denote by $\mathcal{S}_k(A)$ the set of all $\mathbf{m} \in \Omega(M)$ such that the set $A \cap \mathbf{m}$ consists of k points. Then

$$\nu(\mathcal{S}_k(A)) = \frac{\mu(A)^k}{k!} e^{-\mu(A)}.$$

2^*. Let sets $A_1, \dots, A_n$ be mutually disjoint. Then the events $\mathcal{S}_{k_1}(A_1), \dots, \mathcal{S}_{k_n}(A_n)$ are independent, i.e.,

$$\nu\Big(\bigcap_{j=1}^{n} \mathcal{S}_{k_j}(A_j)\Big) = \prod_{j=1}^{n} \nu(\mathcal{S}_{k_j}(A_j)).$$

It is easily shown that these conditions define a unique probability measure on $\Omega(M)$.

Theorem. *The measure ν on $\Omega(M)$ is quasiinvariant with respect to the group $\mathrm{Gms}_\infty(M)$, the Radon–Nykodim derivative of the transformation*

$$\mathbf{m} = (m_1, m_2, \dots) \mapsto q\mathbf{m} = (qm_1, qm_2, \dots), \qquad q \in \mathrm{Gms}_\infty(M), \tag{2}$$

is given by the formula

$$\exp\Big\{-\int_M (q'(m) - 1)\, d\mu(m)\Big\} \prod_{m_j \in \mathbf{m}} q'(m_j). \tag{3}$$

This quasiinvariance was obtained by Vershik, Gelfand, and Graev [33] (in their paper, q was a finitely supported diffeomorphism of a manifold), the infinitesimal version of the theorem was obtained earlier by Goldin, Grodnik, Powers, Sharp, and Menikoff [8], [16] (see also [1]); the above variant of the theorem was obtained in [19], for details see [20], Section X.4. Spherical functions on the group Gms_∞ with respect to the group Ams_∞ are discussed in [10].

0.7. The result of the paper. Thus we have the canonical homomorphism

$$\mathrm{Gms}_\infty(M) \to \mathrm{Gms}\bigl(\Omega(M)\bigr). \tag{4}$$

In this paper, we describe explicitly the homomorphism of the semigroups of polymorphisms extending (4).

In fact, we construct some canonical family of measures ($\mathbb{R}^*$-polymorphisms) on

$$\Omega(M) \times \Omega(M) \times \mathbb{R}^*.$$

They can be interpreted as "spreading maps" of the space $\Omega(M)$. Any such "map" can be obtained as a limit of the transformations (2); thus our $\mathbb{R}^*$-polymorphisms themselves are some kind of symmetries of Poisson processes. We define our $\mathbb{R}^*$-polymorphisms of $\Omega(M)$ in terms of the *matching summation* formula (18). In fact, this formula is similar to the expressions for the Taylor coefficients of

$$\sum \sigma_{kl} z^k u^l = \exp\Bigl\{\sum_{k,l} a_{kl} z_k u_l + \sum_k b_k z_k + \sum_l c_l u_l + d\Bigr\}.$$

In these expressions, the scalars a_{kl}, b_k, c_l, d are replaced by measures on $\mathbb{R}^*$ and the products of scalars are replaced by convolutions of the measures. The analogue of $\exp(d)$ in the formula (18) is a sophisticated expression.

Matching summation itself appears in mathematics in various situations (see [15], [21]), but such combinatorial expressions with measures seem unusual.

This work is a continuation of [21], but logically these two papers are independent.

0.8. Structure of the paper. Section 1 contains preliminaries on $\mathbb{R}^*$-polymorphisms, i.e., polymorphisms related to the group Gms. In Section 2, we define Υ-polymorphisms related to the group Gms_∞.

In Section 3, for any Υ-polymorphism, we construct an $\mathbb{R}^*$-polymorphism of the corresponding spaces of Poisson configurations.

The result of this paper is the formula (18) and Theorems A, B.

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1. $\mathbb{R}^*$-POLYMORPHISMS

In Sections 1–2, we apply the double coset multiplication machinery for producing semigroups of polymorphisms. In fact, we also give motivation-independent definitions of $\mathbb{R}^*$-polymorphisms and Υ-polymorphisms in 1.5–1.8 and 2.7–2.8. But it seems that the double coset motivation is really necessary in Section 2.

On double coset multiplication and similar operations, see [6], [31], [25], [26], a relatively complete list of such constructions is contained in the book [20], its Russian edition is more complete.

Consider a group G and its subgroups H, K. The *double coset space* $H\backslash G/K$ is the quotient space of G with respect to the equivalence relation

$$g \sim kgh, \qquad \text{where} \quad g \in G,\, h \in H,\, k \in K.$$

The equivalence classes are called *double cosets*.

1.1. Double coset multiplication on Ams $\backslash$ Gms/Ams**.** Fix a space (A, α) with a continuous probability measure. Let $g \in \mathrm{Gms}(A)$. Consider the map $A \to \mathbb{R}^*$ given by

$$a \mapsto g'(a).$$

Denote by $\mathbf{u}_g$ the image of this map. Obviously, $\mathbf{u}_g$ is a probability measure on $\mathbb{R}^*$; this measure also satisfies the condition

$$\int_{\mathbb{R}^*} x \, d\mathbf{u}(x) = 1. \tag{5}$$

The last property is equivalent to

$$\int_A g'(a)\, d\alpha(a) = 1.$$

Denote by $\mathcal{L}$ the set of all probability measures on $\mathbb{R}^*$ satisfying the condition (5).

Obviously, for any $h_1, h_2 \in \mathrm{Ams}(A)$, we have

$$\mathbf{u}_{h_1 g h_2} = \mathbf{u}_g,$$

i.e., the map $g \mapsto \mathbf{u}_g$ is constant on double cosets. It is readily seen that the map

$$\mathrm{Ams}(A) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A) \to \mathcal{L}$$

defined by $g \mapsto \mathbf{u}_g$ is a bijection.

We claim that there exists a natural multiplication on the double coset space $\mathrm{Ams} \setminus \mathrm{Gms}/\mathrm{Ams}$.

Consider $\mathbf{v}, \mathbf{w} \in \mathcal{L}$. Consider the representatives p, q of the corresponding double cosets, i.e., $\mathbf{u}_p = \mathbf{v}$, $\mathbf{u}_q = \mathbf{w}$. Of course, the element $\mathbf{u}_{pq}$ depends on the choice of p and q (and it is not determined by $\mathbf{u}$, $\mathbf{w}$).

Nevertheless, there exists the following nonformal reasoning. Let $h \in \mathrm{Ams}(A)$ be "as general as possible". It is clear that h "very strongly mixes" the space A, this implies that $\mathbf{u}_{phg}$ is very close to the convolution $\mathbf{u}_p * \mathbf{u}_q$. For an "absolutely generic" h, we will obtain the convolution $\mathbf{u}_p * \mathbf{u}_q$ itself. Thus the multiplication of double cosets is the convolution of the corresponding measures $\mathbf{u}_q$.

One of ways to say the same reasoning carefully is the following.

We say that a sequence $h_n \in \mathrm{Ams}(A)$ is *generic* if it converges to the uniform spreading (see 0.3) in $\mathrm{Pol}(A, A)$. The definition may be restated as follows: a sequence h_j is generic if

$$\forall B, C \subset A \qquad \lim_{n\to\infty} \alpha\big(h_n(B) \cap C\big) = \alpha(B)\alpha(C).$$

Remark. If A is a space with a finite nonprobability measure, then the definition of a *generic* sequence h_n has the form

$$\forall B, C \subset A \qquad \lim_{n\to\infty} \alpha\big(h_n(B) \cap C\big) = \frac{\alpha(B)\alpha(C)}{\alpha(A)^2}. \tag{5.a}$$

The following statement is obvious.

Lemma. *For a generic sequence h_n and any $p, q \in \mathrm{Gms}(A)$, the sequence $\mathbf{u}_{ph_nq}$ weakly converges to $\mathbf{u}_p * \mathbf{u}_q$.*

Thus we define the multiplication of double cosets as the convolution of the corresponding measures.

1.2. Partitions. Let A be a space with a probability measure. Consider its finite or countable partition

$$T : A = A_1 \cup A_2 \cup \dots$$

By A/T we denote the quotient-space, i.e., the countable space, where the measures of the points are $\alpha(A_1)$, $\alpha(A_2)$, ... Denote by $\mathrm{Ams}(A|T)$ the group

$$\mathrm{Ams}(A|T) = \mathrm{Ams}(A_1) \times \mathrm{Ams}(A_2) \times \cdots \subset \mathrm{Ams}(A).$$

1.3. Double cosets. Consider a space (A, α) with a continuous measure. Consider two partitions of A (they may coincide)

$$S : A = A_1 \cup A_2 \cup \ldots; \qquad T : A = B_1 \cup B_2 \cup \ldots$$

Consider the quotients A/S and A/T. Denote their points by a_1, a_2, ... and b_1, b_2, ..., respectively. Denote the measures of the points by α_1, α_2, ... and β_1, β_2, ...

Consider the double cosets

$$\mathrm{Ams}(A|S) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A|T).$$

Fix $p \in \mathrm{Gms}(A)$. For each pair A_i, B_j, consider the set $A_i \cap p^{-1}(B_j)$. Denote by $\mathbf{p}_{ij}$ the image of the measure α restricted to $A_i \cap p^{-1}(B_j)$ under the map

$$A_i \cap p^{-1}(B_j) \to \mathbb{R}^*.$$

Thus we obtain an $\mathcal{M}$-valued matrix

$$\mathfrak{P} = \begin{pmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} & \cdots \\ \mathbf{p}_{21} & \mathbf{p}_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{7}$$

where each $\mathbf{p}_{ij}$ is a measure on $\mathbb{R}^*$; these measures satisfy the conditions

$$\sum_i \int_{\mathbb{R}^*} x d\mathbf{p}_{ij}(x) = \beta_j, \tag{8}$$

$$\sum_j \int_{\mathbb{R}^*} d\mathbf{p}_{ij}(x) = \alpha_i. \tag{9}$$

The origin of these conditions are the identities

$$\alpha(A_i) = \sum_j \alpha\big(A_i \cap p^{-1}(B_j)\big);$$

$$\alpha(B_j) = \sum_i \alpha\big(p(A_i) \cap B_j\big) = \sum_i \int_{A_i \cap p^{-1}(B_j)} p'(a)\, d\alpha(a).$$

It is readily seen that the map $p \mapsto \mathfrak{P}$ induces a bijection from the double coset space (6) to the space of all matrices (7) satisfying the conditions (8)–(9).

1.4. The multiplication of double cosets. For a space (A, α) with a continuous probability measure, consider 3 partitions (they may coincide)

$$S : A = A_1 \cup A_2 \cup \dots ; \qquad T : A = B_1 \cup B_2 \cup \dots ; \qquad R : A = C_1 \cup C_2 \cup \dots$$

Denote by β_1, β_2, ... the measures of the sets B_1, B_2, ...

We intend to define the multiplication of double cosets

$$\mathrm{Ams}(A|S)\backslash\mathrm{Gms}(A)/\mathrm{Ams}(A|T)\times\mathrm{Ams}(A|T)\backslash\mathrm{Gms}(A)/\mathrm{Ams}(A|R) \quad \to \\ \to \mathrm{Ams}(A|S) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A|R).$$

We say that a sequence

$$h_n = (h_n^{(1)}, h_n^{(2)}, \dots) \in \mathrm{Ams}(A|T) = \prod_j \mathrm{Ams}(A_j)$$

is *generic* if for each j the sequence $h_n^{(j)}$ is generic in $\mathrm{Ams}(A_j)$.

Consider a transformation $p \in \mathrm{Gms}(A)$ and the corresponding double coset in $\mathrm{Ams}(A|S) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A|T)$, i.e., consider the matrix $\mathfrak{P} = \{\mathbf{p}_{ij}\}$. Consider a transformation $q \in \mathrm{Gms}(A)$ and the corresponding double coset in $\mathrm{Ams}(A|T) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A|R)$. Denote by $\mathfrak{Q} = \{\mathbf{q}_{jk}\}$ the corresponding matrix.

For the product qh_np, denote by $\mathfrak{R}_n$ the corresponding double coset in $\mathrm{Ams}(A|S) \setminus \mathrm{Gms}(A)/\mathrm{Ams}(A|R)$.

Lemma. *The sequence of $\mathcal{M}$-valued matrices $\mathfrak{R}_n$ converges (element-wise) to the matrix*

$$\mathfrak{R} = \begin{pmatrix} \mathbf{r}_{11} & \mathbf{r}_{12} & \dots \\ \mathbf{r}_{21} & \mathbf{r}_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} \mathbf{p}_{11} & \mathbf{p}_{12} & \dots \\ \mathbf{p}_{21} & \mathbf{p}_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \beta_1^{-1} & 0 & \dots \\ 0 & \beta_2^{-1} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathbf{q}_{11} & \mathbf{q}_{12} & \dots \\ \mathbf{q}_{21} & \mathbf{q}_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where the product of matrix elements is the convolution of measures on $\mathbb{R}^*$, i.e.,

$$\mathbf{r}_{ik} = \sum_j \frac{1}{\beta_j} \mathbf{p}_{ij} * \mathbf{p}_{jk}.$$

Formula (10) defines the required product of double cosets.

Now we will make a definition from this lemma. The definition is formal and motivation-independent.

1.5. $\mathbb{R}^*$-polymorphisms of countable spaces. Consider a countable (or finite) space A with a probability measure. We denote its points by a_1, a_2, ..., and their measures by α_1, α_2, ... Let A, B be two countable spaces. Then an $\mathbb{R}^*$-polymorphism $A \to B$ is an $\mathcal{M}$-valued matrix (7) satisfying the conditions (8)–(9).

Let A, B, C be countable (or finite) spaces with probability measures. Let $\mathfrak{P} : A \to B$, $\mathfrak{Q} : B \to C$ be $\mathbb{R}^*$-polymorphisms. Then their product $\mathfrak{R} = \mathfrak{Q}\mathfrak{P} : A \to C$ is defined by formula (10).

1.6. $\mathbb{R}^*$-polymorphisms in general case ([19], [20]). Consider spaces (A, α), (B, β) with probability measures. An $\mathbb{R}^*$-polymorphism $\mathfrak{P} : A \to B$ is a measure $\mathfrak{P}$ on $A \times B \times \mathbb{R}^*$ satisfying the following two conditions:

1. The image of the measure $\mathfrak{P}$ under the projection $A \times B \times \mathbb{R}^* \to A$ coincides with α.

2. Denote by x the coordinate on $\mathbb{R}^*$. Consider the measure $x \cdot \mathfrak{P}$. We require the image of $x \cdot \mathfrak{P}$ under the projection $A \times B \times \mathbb{R}^* \to B$ to be β.

Denote by $\mathrm{Pol}_{\mathbb{R}^*}(A, B)$ the set of all $\mathbb{R}^*$-polymorphisms $A \to B$.

Example. Consider a space A with a continuous probability measure. Consider $q \in \mathrm{Gms}(A)$. Denote by $q'(a)$ its Radon–Nykodim derivative. Consider the map $A \to A \times A \times \mathbb{R}^*$ given by

$$a \mapsto (a, q(a), q'(a)).$$

Denote by $\mathfrak{P}(q)$ the image of the measure α under this map. Then $\mathfrak{P}(q)$ is an element of $\mathrm{Pol}_{\mathbb{R}^*}(A, A)$.

1.7. Convergence. Consider general spaces (A, α), (B, β) with probability measures. Consider an arbitrary $\mathbb{R}^*$-polymorphism $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$. Fix Borel subsets $M \subset A$, $N \subset B$ and consider the restriction of the measure $\mathfrak{P}$ to $M \times N \times \mathbb{R}^*$. Denote by $\mathbf{p}[M, N]$ the image of this restriction under the map $M \times N \times \mathbb{R}^* \to \mathbb{R}^*$.

We say that the sequence $\mathfrak{P}_j \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$ converges to $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$ if for each $M \subset A$, $N \subset B$,

1) the sequence $\mathbf{p}_j[M, N]$ weakly converges to $\mathbf{p}[M, N]$;

2) the sequence $x\mathbf{p}_j[M, N]$ weakly converges to $x\mathbf{p}[M, N]$.

See examples of convergence below in 1.11.

Remark. Consider a space A with a continuous probability measure μ. It is easy to prove that the group $\mathrm{Gms}(A)$ is dense in the semigroup $\mathrm{Pol}_{\mathbb{R}^*}(A, A)$ ([19]).

1.8. Definition of product of $\mathbb{R}^*$-polymorphisms in general case. Let A be a space with a probability measure. Consider its finite or countable partition

$$T : A = A_1 \cup A_2 \cup \dots$$

By A/T we denote the quotient-space, i.e., the countable space, where the measures of points are $\alpha(A_1)$, $\alpha(A_2)$, ...

Consider also a partition of a space B

$$S : B = B_1 \cup B_2 \cup \dots$$

For any $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$, we define

$$\mathfrak{P}^{\downarrow}_{T,S} \in \mathrm{Pol}_{\mathbb{R}^*}(A/T, B/S)$$

as the matrix consisting of the measures $\mathbf{p}[A_i, B_j]$ (see 1.7).

Conversely, consider

$$\mathfrak{R} \in \mathrm{Pol}_{\mathbb{R}^*}(A/T, B/S).$$

This is a matrix whose matrix elements $\mathbf{r}_{ij}$ are measures on $\mathbb{R}^*$. For each A_i, B_j, consider the measure on $A_i \times B_j \times \mathbb{R}^*$ given by

$$\frac{\alpha}{\alpha(A_i)} \times \frac{\beta}{\beta(B_j)} \times \mathbf{r}_{ij}.$$

This defines some measure $\mathfrak{R}^{\uparrow}_{T,S}$ on

$$A \times B \times \mathbb{R}^* = \bigcup_{ij} A_i \times B_j \times \mathbb{R}^*.$$

Obviously,

$$\mathfrak{R}^{\uparrow}_{T,S} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B).$$

Also, $\left(\mathfrak{P}^{\uparrow}_{T,S}\right)^{\downarrow} = \mathfrak{P}$, and, obviously, $\left(\mathfrak{R}^{\downarrow}_{T,S}\right)^{\uparrow}$ is not $\mathfrak{R}$.

We say that a sequence $T^{(j)}$ of partitions is *approximative*, if a partition $T^{(j+1)}$ is a refinement of $T^{(j)}$ and elements of the partitions generate the Borel σ-algebra of A.

Now we are ready to define the product of $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$ and $\mathfrak{Q} \in \mathrm{Pol}_{\mathbb{R}^*}(B, C)$. Consider approximative sequences of partitions $T^{(j)}$, $S^{(j)}$,

$U^{(j)}$ of the spaces A, B, C. We define the polymorphism $\mathfrak{R} = \mathfrak{Q}\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, C)$ as

$$\lim_{j\to\infty} \left(\mathfrak{Q}^{\downarrow}_{S^{(j)},U^{(j)}} \mathfrak{P}^{\downarrow}_{T^{(j)},S^{(j)}}\right)^{\uparrow}_{T^{(j)},U^{(j)}}.$$

Remark. For any group G it is possible to define G-polymorphisms in the same way, see [19], [20]. For some groups G, there exist nontrivial functors from the category of polymorphisms to the category of Hilbert spaces and operators ([19], [20]) (for $G = SL_2(\mathbb{R})$, $O(1, n)$, $U(1, n)$; these functors extend the so-called Araki multiplicative integral construction, see [2], [32]); for some groups G there exist nontrivial central extensions of the categories of G-polymorphisms (for $G = Sp(2n, \mathbb{R})$, $U(p, q)$, $SO^*(2n)$, [23]). It seems that some polymorphism-like structures appear in the mathematical hydrodynamics, see [5].

1.9. Remark. Action of $\mathbb{R}^*$-polymorphisms on the spaces L^p. Let $w = u + iv$ be in $\mathbb{C}$, let $0 \leqslant u \leqslant 1$. Let A be a space with a continuous probability measure. The group $\mathrm{Gms}(A)$ acts in the space $L^{1/u}(A)$ by the isometries

$$T_w(q)f(a) = f(q(a))q'(a)^w.$$

Let us extend this action to an action of $\mathbb{R}^*$-polymorphisms.

Let (A, α), (B, β) be spaces with probability measures.

Proposition. *Let $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$. Then the expression*

$$S_w(\mathfrak{P}|f, g) = \iiint\limits_{A\times B\times\mathbb{R}^*} f(a)g(b)x^{u+iv}\, d\mathfrak{P}(a, b, x)$$

is a bounded bilinear form on $L^{1/(1-u)}(A) \times L^{1/u}(B)$ and moreover

$$|S_w(\mathfrak{P}|f, g)| \leqslant \|f\|_{L^{1/(1-u)}} \cdot \|g\|_{L^{1/u}}.$$

Let us define the linear operator

$$T_w(\mathfrak{P}) : L^{1/u}(B) \to L^{1/u}(A)$$

by the duality condition

$$\int\limits_A f(a)T_w(\mathfrak{P})g(a)\, d\alpha(a) = S_w(\mathfrak{P}|f, g).$$

Obviously,

$$\|T_w(\mathfrak{P})\| \leqslant 1.$$

Proposition. *For each spaces A, B, C with probability measures and each $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A,B)$, $\mathfrak{Q} \in \mathrm{Pol}_{\mathbb{R}^*}(B,C)$,*

$$T_w(\mathfrak{Q})T_w(\mathfrak{P}) = T_w(\mathfrak{Q}\mathfrak{P}).$$

1.10. Remark. Action of $\mathbb{R}^*$-polymorphisms on $\mathcal{M}$-valued functions. Let (A,α) be a space with a probability measure. Denote by $\mathcal{S}(A)$ the space of all functions $a \mapsto \nu_a$ on A taking values in $\mathcal{M}$ and satisfying the condition

$$\int_A \int_{\mathbb{R}^*} x \, d\nu_a(x) \, d\alpha(a) = 1.$$

Denote by $\bullet$ the single-point space with a probability measure. The space $\mathrm{Pol}_{\mathbb{R}^*}(A,\bullet)$ is identified in an obvious way with the space $\mathcal{S}(A)$.

Each element of $\mathrm{Pol}_{\mathbb{R}^*}(B,A)$ induces the map $\mathrm{Pol}_{\mathbb{R}^*}(A,\bullet)$ to $\mathrm{Pol}_{\mathbb{R}^*}(B,\bullet)$ given by the formula

$$\mathbf{U} \mapsto \mathbf{U}\mathfrak{P}; \qquad \mathbf{U} \in \mathrm{Pol}_{\mathbb{R}^*}(A,\bullet).$$

Thus we obtain the canonical map

$$\Theta_{\mathfrak{P}} : \mathcal{S}(A) \to \mathcal{S}(B).$$

Obviously, for any $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(B,A)$, $\mathfrak{Q} \in \mathrm{Pol}_{\mathbb{R}^*}(C,B)$, we have

$$\Theta_{\mathfrak{P}}\Theta_{\mathfrak{Q}} = \Theta_{\mathfrak{P}\mathfrak{Q}}.$$

1.11. Remarks. Examples of convergence. 1) Let $A = B$ be the interval $[0,1]$. Consider the sequence q_n of monotonic maps $[0,1] \to [0,1]$ given by

$$q_n(a) = a + \frac{1}{2\pi n}\sin(2\pi n a).$$

Then the limit $\mathfrak{P}$ of q_n is a measure on $[0,1] \times [0,1] \times \mathbb{R}^*$ supported by the set consisting of the points

$$(a,a,x); \qquad 0 < x < 2,$$

and the density of $\mathfrak{P}$ on this set is given by

$$\frac{da\,dx}{\pi\sqrt{2x - x^2}}.$$

2) Let $A = B$ be the same. Then the sequence $q_n(a) = a^n$ has no limit in $\mathbb{R}^*$-polymorphisms.

3) Let A, B be spaces with continuous measures. Let S, T be their partitions. Let $g_n \in \mathrm{Ams}(A|S)$, $B_n \in \mathrm{Ams}(A|T)$ be generic sequences. Then

$$\lim_{n\to\infty}\Big\{\lim_{m\to\infty} h_n \mathfrak{P} g_m\Big\} = \mathfrak{P}^{\downarrow}_{S,T}$$

4) Let A, B be spaces with continuous measures. Let $S^{(n)}, T^{(n)}$ be approximative sequences of their partitions. Then, for any $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, B)$,

$$\lim_{n\to\infty} \big(\mathfrak{P}^{\downarrow}_{S^{(n)},T^{(n)}}\big)^{\uparrow}_{S^{(n)},T^{(n)}} = \mathfrak{P}.$$

1.12. Remark. How to formulate the problem of limit behavior of powers of a polymorphism? For $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, A)$, denote by $\mathfrak{P}^n$ its powers. If $\mathfrak{P} \in \mathrm{Ams}(A) \subset \mathrm{Pol}(A, A)$, then the problem of limit behavior of the powers is the problem of the ergodic theory. If $A = \bullet$ is a single-point set, then the limit behavior of $\mathfrak{P}^n$ is described by the central limit theorem. The following problem is an attempt to unite the both subjects of the classical theories mentioned above.

We notice that the group $\mathbb{R}^*$ admits a one-parameter family of automorphisms $x \mapsto x^\alpha$, where $\alpha \in \mathbb{R} \setminus 0$. These automorphisms induce the one-parameter family of automorphisms of the semigroup $\mathcal{M}$, i.e.,

$$\mathbf{u}(x) \mapsto \mathbf{u}(x^\alpha), \qquad \mathbf{u} \in \mathcal{M}.$$

The latter automorphisms induce automorphisms of the semigroup of *all* $\mathcal{M}$-valued $n \times n$ matrices (7) equipped with the multiplication (10)[3]

For any $\mathfrak{P} \in \mathrm{Pol}_{\mathbb{R}^*}(A, A)$ and any $\alpha \in \mathbb{R} \setminus 0$ we define the measures $\mathfrak{P}(a, b, x^\alpha)$ on $A \times A \times \mathbb{R}^*$ as the image of $\mathfrak{P}$ under the map $(a, b, x) \mapsto (a, b, x^\alpha)$.

We obtain the following problem: *Is it possible to find a sequence α_n such that the sequence $\mathfrak{P}(a, b, x^{\alpha_n})^n$ converges to some nontrivial limit?*

2. Polymorphisms of bordered spaces

2.1. The classes $\mathcal{M}^\triangledown$, $\mathcal{M}^\blacktriangledown$ of measures on $\mathbb{R}^*$. Let $\mathbf{u}$ be a measure on $\mathbb{R}^*$. We say that $\mathbf{u}$ belongs to the class $\mathcal{M}^\triangledown$ if

$$\int_{\mathbb{R}^*} d\mathbf{u}(x) < \infty, \qquad \int_{\mathbb{R}^*} x\, d\mathbf{u}(x) < \infty.$$

[3] These automorphisms break the condition (8). But the product (10) itself exists without the conditions (8)–(9).

We say that $\mathbf{u}$ is an element of the class $\mathcal{M}^{\blacktriangledown}$ if

$$\int |x-1|\, d\mathbf{u}(x) < \infty.$$

For the class $\mathcal{M}^{\blacktriangledown}$, *we admit infinite atomic measures supported by* $x = 1$.

We also define the convergence in $\mathcal{M}^{\triangledown}$ and $\mathcal{M}^{\blacktriangledown}$. A sequence $\mathbf{u}_j$ converges to $\mathbf{u}$ in $\mathcal{M}^{\triangledown}$ if $\mathbf{u}_j$ weakly converges to $\mathbf{u}$ and $x\mathbf{u}_j$ weakly converges to $x\mathbf{u}$. A sequence $\mathbf{u}_j$ converges to $\mathbf{u}$ in $\mathcal{M}^{\blacktriangledown}$ if $|x-1|\mathbf{u}_j$ weakly converges to $|x-1|\mathbf{u}$.

2.2. Bordered spaces. Let M be a Lebesgue measure space (we admit the case when M is an empty set). We define the corresponding *bordered space*

$$M^{\curlyvee} = M \cup \xi_\infty^M,$$

where ξ_M is a formal point.

Remark. It is natural to think that $\xi_\infty = \xi_\infty^M$ is "the point of M at infinity". Also, it is natural to think that the measure of the point ξ_∞ is ∞.

We also define *measurable subsets* in $\mathcal{M}^{\curlyvee}$. Let $A \subset M$ be a Borel subset.

a) The set $A \cup \xi_\infty$ is measurable in $\mathcal{M}^{\curlyvee}$.

b) If A has a finite measure, then A is measurable in $\mathcal{M}^{\curlyvee}$.

All other subsets in $M^{\curlyvee}$ are not measurable.

2.3. Partitions of bordered spaces. Consider a bordered space $M^{\curlyvee}$. Its *good partition* $\mathcal{U}$ (we will omit the word "good") is a partition

$$\mathcal{U} : M^{\curlyvee} = M_1 \cup \dots \cup M_k \cup M_\infty$$

into mutually disjoint subsets such that $M_1, \dots, M_k$ have finite measure and $\xi_\infty \in M_\infty$. We say that M_∞ is an infinite element of the partition, all other elements are *finite.*

For the partition $\mathcal{U}$, we define the quotient space $M^{\curlyvee}/\mathcal{U}$. It consists of the points with the measures $\mu(M_1), \dots, \mu(M_k)$ and the point ξ_∞.

For the partition $\mathcal{U}$, we define the group of automorphisms of the partition

$$\mathrm{Ams}_\infty(M^{\curlyvee}|\mathcal{U}) = \mathrm{Ams}(M_1) \times \dots \times \mathrm{Ams}(M_k) \times \mathrm{Ams}_\infty(M_\infty).$$

2.4. Multiplication of double cosets $\mathrm{Ams}_\infty \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty$**.** For $q \in \mathrm{Gms}_\infty(M)$ we consider the image $\mathbf{u}_q$ of the measure μ under the map $M \to \mathbb{R}^*$ given by $m \mapsto q'(m)$. Obviously, $\mathbf{u}_q \in \mathcal{M}^\blacktriangledown$, and the map $q \mapsto \mathbf{u}_q$ defines a bijection

$$\mathrm{Ams}_\infty(M) \setminus \mathrm{Gms}_\infty(M)/\mathrm{Ams}_\infty(M) \quad \to \quad \mathcal{M}^\blacktriangledown.$$

We say that a sequence $h_n \in \mathrm{Ams}_\infty(M)$ is *generic* if for any subsets $A, B \subset M$ with finite measures, we have

$$\lim_{n\to\infty} \mu\big(h_n(A) \cap B\big) = 0.$$

Remark. See formula (5.a).

Example. Let $M = \mathbb{R}$. We can take $h_n(x) = x + n$.

Fix a generic sequence h_n. Consider $\mathbf{v}, \mathbf{w} \in \mathcal{M}^\blacktriangledown$. Consider $p, q \in \mathrm{Gms}_\infty(M)$ lying in the corresponding double cosets.

Lemma. *Denote by* $\mathbf{u}_n$ *the element of* $\mathcal{M}^\blacktriangledown$ *corresponding to* qh_np*. Then*

$$\lim_{n\to\infty} \mathbf{u}_n = \mathbf{v} + \mathbf{w}.$$

Remark. Compare with Subsection 1.1.

2.5. Double cosets. Let (M, μ) be a space with a continuous infinite measure. Fix two partitions

$$\mathcal{U} : M = M_1 \cup \dots \cup M_s \cup M_\infty, \qquad \mathcal{V} : M = N_1 \cup \dots \cup N_t \cup N_\infty$$

of M. Denote the measures of the sets M_i by μ_i and the measures of N_j by ν_j. For any $p \in \mathrm{Gms}_\infty(M)$ and any $\alpha = 1, \dots, s, \infty$ and $\beta = 1, \dots, t, \infty$, we define the measure $\mathbf{p}_{\alpha\beta}$ on $\mathbb{R}^*$ as the image of the measure μ under the map

$$M_\alpha \cap p^{-1}(N_\beta) \to \mathbb{R}^*$$

given by $m \mapsto p'(m)$. Thus we obtain the matrix

$$\mathfrak{P} = \begin{pmatrix} \mathbf{p}_{11} & \cdots & \mathbf{p}_{1t} & \mathbf{p}_{1\infty} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{p}_{s1} & \cdots & \mathbf{p}_{st} & \mathbf{p}_{s\infty} \\ \mathbf{p}_{\infty 1} & \cdots & \mathbf{p}_{\infty t} & \mathbf{p}_{\infty\infty} \end{pmatrix} \tag{11}$$

consisting of measures on the group $\mathbb{R}^*$. These measures satisfy the following equalities

$$\sum_{j=1}^{t}\int_{\mathbb{R}^*} d\mathbf{p}_{ij}(x) + \int_{\mathbb{R}^*} d\mathbf{p}_{i\infty}(x) = \mu_i, \qquad i = 1, 2, \dots, s, \tag{12}$$

$$\sum_{i=1}^{s}\int_{\mathbb{R}^*} x\, d\mathbf{p}_{ij}(x) + \int_{\mathbb{R}^*} x\, d\mathbf{p}_{\infty j}(x) = \nu_j, \qquad j = 1, 2, \dots, t, \tag{13}$$

and the conditions

$$\mathbf{p}_{ij}, \mathbf{p}_{i\infty}, \mathbf{p}_{\infty j} \in \mathcal{M}^{\triangledown}, \qquad \mathbf{p}_{\infty\infty} \in \mathcal{M}^{\blacktriangledown}. \tag{14}$$

Obviously, the map $p \mapsto \mathfrak{P}$ is constant on each double coset

$$\mathrm{Ams}_\infty(M \big| \mathcal{U}) \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty(M \big| \mathcal{V}),$$

and moreover this defines a bijection between the double coset space and the space of all matrices (11) satisfying (12)–(14).

We also will write the matrix (11) in the $(s+1)\times(t+1)$-block form

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{P}_{\mathrm{fin},\mathrm{fin}} & \mathfrak{P}_{\mathrm{fin},\infty} \\ \mathfrak{P}_{\infty,\mathrm{fin}} & \mathbf{p}_{\infty,\infty} \end{pmatrix}.$$

2.6. Product of double cosets. Now consider 3 partions of the space M (all these partitions may coincide)

$$\mathcal{U} : M = M_1 \cup \dots \cup M_s \cup M_\infty, \qquad \mathcal{V} : M = N_1 \cup \dots \cup N_t \cup N_\infty,$$
$$\mathcal{W} : K = K_1 \cup \dots \cup K_r \cup K_\infty.$$

We intend to define a multiplication of double cosets

$$\begin{aligned}\mathrm{Ams}_\infty(M \big| \mathcal{U}) \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty(M \big| \mathcal{V}) \\ \times\ \mathrm{Ams}_\infty(M \big| \mathcal{V}) \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty(M \big| \mathcal{W}) \to \\ \to \mathrm{Ams}_\infty(M \big| \mathcal{U}) \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty(M \big| \mathcal{W}),\end{aligned}$$

i.e., we want to define a multiplication of matrices (11).

We say that a sequence

$$\begin{aligned} h_n = (h_n^{(1)}, \dots, h_n^{(t)}, h_n^{(\infty)}) \in \\ \in \mathrm{Ams}_\infty(M \big| \mathcal{V}) = \mathrm{Ams}(N_1) \times \dots \times \mathrm{Ams}(N_t) \times \mathrm{Ams}_\infty(N_\infty)\end{aligned}$$

is *generic* if all the sequences $h_n^{(\beta)}$ are generic ($\beta = 1, 2, \dots, t, \infty$).

Now we repeat the double coset multiplication construction. Consider a matrix $\mathfrak{P}$ which corresponds to some element of $\mathrm{Ams}_\infty(M \,|\, \mathcal{U}) \setminus \mathrm{Gms}_\infty / \mathrm{Ams}_\infty(M \,|\, \mathcal{V})$. Consider a matrix $\mathfrak{Q}$ which corresponds to some element of $\mathrm{Ams}_\infty(M \,|\, \mathcal{V}) \setminus \mathrm{Gms}_\infty(M)/\mathrm{Ams}_\infty(M \,|\, \mathcal{W})$. Consider the representatives $p, q \in \mathrm{Gms}_\infty(M)$ of these double cosets. For a generic sequence $h_n \in \mathrm{Ams}_\infty(M \,|\, \mathcal{U})$, denote by $\mathfrak{R}_n$ the element of the double coset space

$$\mathrm{Ams}_\infty(M \,|\, \mathcal{U}) \setminus \mathrm{Gms}_\infty(M)/\mathrm{Ams}_\infty(M \,|\, \mathcal{W})$$

containing qh_np. Then the limit $\mathfrak{R}$ of $\mathfrak{R}_n$ is given by

$$\mathfrak{R} = \begin{pmatrix} \mathfrak{Q}_{\mathrm{fin},\mathrm{fin}} \cdot D \cdot \mathfrak{P}_{\mathrm{fin},\mathrm{fin}} & \mathfrak{Q}_{\mathrm{fin},\mathrm{fin}} \cdot D \cdot \mathfrak{P}_{\mathrm{fin},\infty} + \mathfrak{Q}_{\mathrm{fin},\infty} \\ \mathfrak{Q}_{\infty,\mathrm{fin}} \cdot D \cdot \mathfrak{P}_{\mathrm{fin},\mathrm{fin}} + \mathfrak{P}_{\infty,\mathrm{fin}} & \mathfrak{Q}_{\infty,\mathrm{fin}} \cdot D \cdot \mathfrak{P}_{\mathrm{fin},\infty} + \mathfrak{P}_{\infty,\infty} + \mathfrak{Q}_{\infty,\infty} \end{pmatrix}$$

where

$$D = \begin{pmatrix} \nu_1^{-1} & 0 & \dots \\ 0 & \nu_2^{-1} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

and ν_j are the measures of the elements $N_1, \dots, N_t$ of the partition $\mathcal{V}$.

The associativity of the product can be easily checked by a direct calculation.

2.7. $\curlyvee$-Polymorphisms of finite bordered spaces. Let $M^\curlyvee$, $N^\curlyvee$ be *finite* bordered spaces, and let the measures of (finite) points be μ_1, $\mu_2, \dots, \mu_s$ and $\nu_1, \nu_2, \dots, \nu_t$. An element $\mathfrak{P}$ of $\mathrm{Pol}^\curlyvee(M^\curlyvee, N^\curlyvee)$ (a $\curlyvee$-polymorphism) is a $(s+1) \times (t+1)$-matrix (11) satisfying the conditions (12)–(14). The product of the polymorphisms is given by (15).

2.8. $\curlyvee$-Polymorphisms of general bordered spaces. Let $(M^\curlyvee, \mu)$, $(N^\curlyvee, \nu)$ be bordered spaces. An element $\mathfrak{P}$ of $\mathrm{Pol}^\curlyvee(M^\curlyvee, N^\curlyvee)$ is a measure on $M^\curlyvee \times N^\curlyvee \times \mathbb{R}^*$ satisfying some conditions given below. It is natural to represent this measure as the block matrix

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{P}_{\mathrm{fin},\mathrm{fin}} & \mathfrak{P}_{\mathrm{fin},\infty} \\ \mathfrak{P}_{\infty,\mathrm{fin}} & \mathfrak{P}_{\infty,\infty} \end{pmatrix},$$

where $\mathfrak{P}_{\mathrm{fin},\mathrm{fin}}$ is a measure on $M \times N \times \mathbb{R}^*$, $\mathfrak{P}_{\mathrm{fin},\infty}$ is a measure on $\xi_\infty^M \times N \times \mathbb{R}^* \simeq N \times \mathbb{R}^*$, $\mathfrak{P}_{\infty,\mathrm{fin}}$ is a measure on $M \times \xi_\infty^N \times \mathbb{R}^* \simeq M \times \mathbb{R}^*$, and $\mathfrak{P}_{\infty,\infty}$ is a measure on $\xi_\infty^M \times \xi_\infty^N \times \mathbb{R}^* \simeq \mathbb{R}^*$.

The measure $\mathfrak{P}$ satisfies the following conditions (which repeat the conditions (12)–(14)).

1. $\mathfrak{P}_{\mathrm{fin},\mathrm{fin}}$, $\mathfrak{P}_{\mathrm{fin},\infty}$, $\mathfrak{P}_{\infty,\mathrm{fin}} \in \mathcal{M}^{\triangledown}$, and $\mathfrak{P}_{\infty,\infty} \in \mathcal{M}^{\blacktriangledown}$.

2. Let us restrict the measure $\mathfrak{P}$ to the set $M \times N^{\curlyvee} \times \mathbb{R}^*$. Then the image of this restriction under the map $M \times N^{\curlyvee} \times \mathbb{R}^* \to M$ coincides with μ.

3. Let us restrict the measure $x \cdot \mathfrak{P}$ to the set $M^{\curlyvee} \times N \times \mathbb{R}^*$. Then the image of this restriction under the map $M^{\curlyvee} \times N \times \mathbb{R}^* \to N$ coincides with ν.

The product of $\curlyvee$-polymorphisms is defined by the same formula (15). We only must define the products in each block. For instance, let us give an interpretation of $\mathfrak{Q}_{\mathrm{fin},\mathrm{fin}} \cdot D \cdot \mathfrak{P}_{\mathrm{fin},\mathrm{fin}}$. It is sufficient to use the prescription 1.8. To avoid a divergence, consider countable partitions of M, N, K into pieces with finite measures. Then we consider approximative sequences of partitions, etc.etc.

2.9. Embedding $\mathrm{Gms}_\infty \to \mathrm{Pol}^{\curlyvee}$**.** For $q \in \mathrm{Gms}_\infty(M)$, we define the matrix $\mathfrak{Q}$ by the conditions

$$\mathfrak{Q}_{\mathrm{fin},\infty} = 0, \qquad \mathfrak{Q}_{\infty,\mathrm{fin}} = 0, \qquad \mathfrak{Q}_{\infty,\infty} = 0,$$

and $\mathfrak{Q}_{\mathrm{fin},\mathrm{fin}}$ is the image of the measure μ under the map $M \to M \times M \times \mathbb{R}^*$ given by $m \mapsto (m, q(m), q'(m))$.

Remark. The group $\mathrm{Gms}_\infty(M)$ is exactly the group of all invertible $\curlyvee$-polymorphisms of $M^{\curlyvee}$.

Below *we identify elements of* $\mathrm{Gms}_\infty(M)$ *and the corresponding elements of* $\mathrm{Pol}^{\curlyvee}(M^{\curlyvee}, M^{\curlyvee})$.

2.10. A remark on the formula for product. The exotic multiplication (15) of matrices is a degeneration of the usual matrix multiplication. Indeed, let ϵ be infinitely small. Consider block $(n+1) \times (n+1)$-matrices of the form

$$\begin{pmatrix} A + o(1) & \epsilon b + o(\epsilon) \\ \epsilon c + o(\epsilon) & 1 + \epsilon^2 d + o(\epsilon^2) \end{pmatrix}.$$

Then the product of such matrices has the form

$$\begin{pmatrix} A + o(1) & \epsilon b + o(\epsilon) \\ \epsilon c + o(\epsilon) & 1 + \epsilon^2 d + o(\epsilon^2) \end{pmatrix} \begin{pmatrix} A' + o(1) & \epsilon b' + o(\epsilon) \\ \epsilon c' + o(\epsilon) & 1 + \epsilon^2 d' + o(\epsilon^2) \end{pmatrix} = \\ = \begin{pmatrix} AA' + o(1) & \epsilon(Ab' + b) + o(\epsilon) \\ \epsilon(cA' + c') + o(\epsilon) & 1 + \epsilon^2(d + d' + cb') + o(\epsilon^2) \end{pmatrix},$$

and we obtain a formula similar to (15).

2.11. Remarks on the convergence in $\mathrm{Pol}^{\curlyvee}$**.** Let $\mathfrak{P} \in \mathrm{Pol}^{\curlyvee}(M^{\curlyvee}, N^{\curlyvee})$. Let $A \subset M^{\curlyvee}$, $B \subset N^{\curlyvee}$ be measurable subsets (see 2.1). We restrict the measure $\mathfrak{P}$ to $A \times B \times \mathbb{R}^*$. Denote by $\mathbf{p}[A \times B]$ the image of this restriction under the projection $A \times B \times \mathbb{R}^* \to \mathbb{R}^*$.

Let $\mathfrak{P}_j, \mathfrak{P} \in \mathrm{Pol}^{\curlyvee}(M^{\curlyvee}, N^{\curlyvee})$. The sequence $\mathfrak{P}_j$ converges to $\mathfrak{P}$ if the following two conditions are satisfied.

a) For any measurable subsets $A \subset M^{\curlyvee}$, $B \subset N^{\curlyvee}$ the sequence $\mathbf{p}_j[A \times B]$ converges to $\mathbf{p}[A \times B]$ in $\mathcal{M}^{\blacktriangledown}$.

b) Moreover, if A or B does not contain ξ_∞, then we have convergence in $\mathcal{M}^{\triangledown}$.

Examples. a) Let $M = \mathbb{R}$. Let $y = f(x)$ be a diffeomorphism of $\mathbb{R}$. Assume

$$q_n(x) = f(x) + n.$$

Let us describe the limit $\mathfrak{P}$ of the sequence q_n in the sense of $\mathrm{Pol}^{\curlyvee}(\mathbb{R}^{\curlyvee}, \mathbb{R}^{\curlyvee})$. The measure $\mathfrak{P}$ is supported by $\mathbb{R} \times \xi_\infty \times \mathbb{R}^*$. It coincides with the image of the Lebesgue measure on $\mathbb{R}$ under the map

$$x \mapsto (x, \xi_\infty, f'(x)).$$

b) Under the same conditions, the limit $\mathfrak{Q}$ of the sequence

$$q_n(x) = f(x - n) + n$$

is supported by $\xi_\infty \times \xi_\infty \times \mathbb{R}^*$. It coincides with the image of the Lebesgue measure under the map

$$x \mapsto (\xi_\infty, \xi_\infty, f'(x)).$$

It is easy to understand that *the group* Gms_∞ *is dense in the semigroup* $\mathrm{Pol}^{\curlyvee}(M^{\curlyvee}, M^{\curlyvee})$. Thus this semigroup is some kind of a boundary of the group $\mathrm{Gms}_\infty(M)$.

2.12. Remarks on the polymorphisms related to Ams_∞**.** We discuss this case for completeness. Let $A^{\curlyvee}$, $B^{\curlyvee}$ be bordered spaces. A polymorphism $\mathfrak{P}$ is a measure on $A \times B$ satisfying the following conditions.

1. The projection of $\mathfrak{P}$ onto A is majorized by α.
2. The projection of $\mathfrak{P}$ onto B is majorized by β.

We define the product of polymorphisms by the same formula (1).

Remark. We also can define this type of polymorphisms $A^{\curlyvee} \to B^{\curlyvee}$ as $\curlyvee$-polymorphisms supported by the set

$$A^{\curlyvee} \times B^{\curlyvee} \times 1 \subset A^{\curlyvee} \times B^{\curlyvee} \times \mathbb{R}^*.$$

3. Construction of the functor.

3.1. Configurations. We say that a *configuration* on a bordered space $M^\curlyvee$ is a countable (or finite) collection

$$\mathbf{m} = \begin{bmatrix} m_1, & m_2, & m_3, \dots \\ p_1, & p_2, & p_3, \dots \end{bmatrix} \tag{17}$$

of distinct points $(m_1, m_2, m_3, \dots)$ of $M^\curlyvee$ having integral positive multiplicities $p_1, p_2, p_3, \dots$. We also assume that any configuration contains ξ_∞ with infinite multiplicity. The collection $\mathbf{m}$ is defined up to permutations of the points together with their multiplicities (i.e., up to permutations of columns of (17)).

We will also give another definition. A *configuration* is a map ϕ from a countable set Z to $M^\curlyvee$ such that the preimage $\phi^{-1}(\xi_\infty)$ of ξ_∞ contains infinitely many points. Two configurations $\phi : Z \to M^\curlyvee$, $\phi' : Z' \to M^\curlyvee$ are *equivalent* if there exists a bijection $\psi : Z \to Z'$ such that $\phi = \phi'\psi$.

Of course, these two definitions are equivalent. Indeed, consider a map $\phi : Z \to M^\curlyvee$. The set $(m_1, m_2, \dots)$ is the image of ϕ; the multiplicity p_j of a point m_j is the number of elements in $\phi^{-1}(m_j)$.

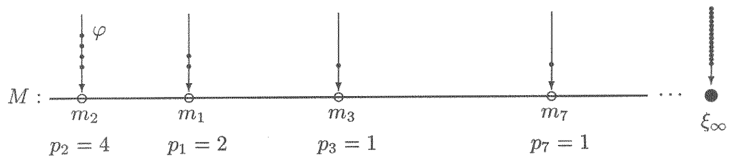

Fig. 1. A configuration. Black points are elements of Z, the map ϕ is the projection down.

Denote by $\Omega(M^\curlyvee)$ the space of configurations on $M^\curlyvee$ defined up to equivalence.

For a map $\rho : M^\curlyvee \to N^\curlyvee$ (we assume $\rho(\xi_\infty) = \xi_\infty$), we have the natural map $\Omega(M^\curlyvee) \to \Omega(N^\curlyvee)$ given by $\phi \mapsto \rho \circ \phi$. In particular, for any partition $\mathcal{U}$, we obtain the map $\Omega(M) \to \Omega(M/\mathcal{U})$.

3.2. Poisson measures: finite case. Consider a finite bordered space $M^\curlyvee$, let the measures of the points $m_1, \dots, m_k$ of M be $\mu_1, \dots, \mu_k$. For a configuration $\phi : Z \to M^\curlyvee$, we denote by p_j the number of points in the preimage $\phi^{-1}(m_j)$ (the *multiplicity* of m_j; it may be 0). Thus the space $\Omega(M^{\curlyvee})$ is identified with the space $\mathbb{Z}_+^k$. We define the Poisson measure ν_M on $\Omega(M^{\curlyvee})$ by the following condition: the measure of the point $(p_1, p_2, \dots, p_k) \in \mathbb{Z}_+$ is

$$\prod_k \frac{\mu_k^{p_k} e^{-\mu_k}}{p_k!}.$$

3.3. Poisson measures: general case. The Poisson measure ν_M on $\Omega(M)$ is defined by the following condition: for any partition $\mathcal{U}$ of $M^{\curlyvee}$, the image of ν_M under the map

$$\Omega(M^{\curlyvee}) \to \Omega(M^{\curlyvee}/\mathcal{U})$$

coinsides with the Poisson measure $\nu_{M^{\curlyvee}/\mathcal{U}}$ on $\Omega(M^{\curlyvee}/\mathcal{U})$, see [13], [33], [20], [21] for more details.

3.4. Normed exponent. Consider a measure $\psi \in \mathcal{M}^{\blacktriangledown}$ on $\mathbb{R}^*$. The function

$$r(s) = \int_{\mathbb{R}^*} (x^{is} - 1)\, d\psi(x)$$

is a well-defined conditionally positive definite function on $\mathbb{R}$. Hence $e^{r(s)}$ is the positive definite function. Hence $e^{r(s)}$ is the Fourier transform of some measure $\varkappa$:

$$e^{r(s)} = \int_0^\infty x^{is} d\varkappa(x).$$

We define the *normed exponent* $\exp_\circ[\psi]$ by

$$\exp_\circ[\psi] = \varkappa.$$

Remark. Assume $\psi \in \mathcal{M}^{\triangledown}$. Then

$$\exp_\circ[\psi] = \exp\Bigl\{-\int_{\mathbb{R}^*} d\psi(x)\Bigr\} \cdot \Bigl\{\delta_1 + \frac{\psi}{1!} + \frac{\psi * \psi}{2!} + \frac{\psi * \psi * \psi}{3!} + \dots\Bigr\},$$

where δ_1 denotes the atomic unit measure supported by the point $1 \in \mathbb{R}^*$.

3.5. Partial bijections. Let S, T be finite sets. A partial bijection $Q : S \to T$ is a bijection of a subset $A \subset S$ to a subset $B \subset T$. We say that A is the *domain* of Q (the notation is $A = \mathrm{Dom}(Q)$) and B is the *image* of Q (the notation is $B = \Im(Q)$). We denote by $\mathrm{PB}(S, T)$ the set of all partial bijections $S \to T$.

3.6. The construction. Consider finite spaces $M^\curlyvee$, $N^\curlyvee$ and the associated spaces $\Omega(M^\curlyvee)$, $\Omega(N^\curlyvee)$ equipped with the Poisson measures.

For each $\curlyvee$-polymorphism $\mathfrak{P} \in \mathrm{Pol}^\curlyvee(M^\curlyvee, N^\curlyvee)$, we will construct an $\mathbb{R}^*$-polymorphism $\omega(\mathfrak{P}) \in \mathrm{Pol}_{\mathbb{R}^*}(\Omega(M^\curlyvee), \Omega(N^\curlyvee))$.

Consider arbitrary configurations $\phi : Z \to M^\curlyvee$ and $\psi : Y \to N^\curlyvee$. Denote by $S \subset Z$, $T \subset Y$ the preimages of the sets M, N; obviously, the configuration ϕ (resp., ψ) is completely defined by the restriction to S (resp., T). Denote by p_i (resp., q_j) the multiplicities of the points of the configuration ϕ (resp., ψ). We define the measure $\omega_{\phi\psi}$ on $\mathbb{R}^*$ by

$$\omega_{\phi\psi} = C \cdot \delta_h * \exp_\circ[\mathbf{p}_{\infty\infty}] * \frac{1}{\prod p_i! \prod q_j!} \sum_{Q \in \mathrm{PB}(S,T)} \Bigl\{ \prod_{s \in \mathrm{Dom}(Q), t = Qs} \mathbf{p}_{st} * \\ * \prod_{s \notin \mathrm{Dom}(Q)} \mathbf{p}_{s\infty} * \prod_{t \notin \Im(Q)} \mathbf{p}_{\infty t} \Bigr\}, \qquad (18)$$

where the summation is over the set of all partial bijections $T \to S$,

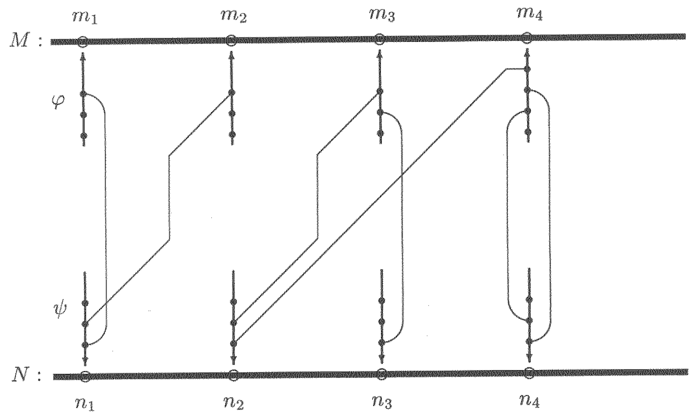

Fig. 2. A partial bijection (matching) of configurations.

$\exp_\circ[\cdot]$ denotes the normed exponent, the symbols $\prod$ denote convolutions of measures on $\mathbb{R}^*$, the scalar factor C is given by

$$C = \exp\Bigl\{ -\sum_{i,j} \int d\mathbf{p}_{ij} - \sum_i \int d\mathbf{p}_{i\infty} - \sum_j \int d\mathbf{p}_{\infty j} \Bigr\},$$

and δ_h is the unit δ-measure on $\mathbb{R}^*$ supported by the point

$$h = \exp\Big\{-\int (x-1)\, d\big[\sum_{i,j} \mathbf{p}_{ij} + \sum_i \mathbf{p}_{i\infty} + \sum_j \mathbf{p}_{\infty j} + \mathbf{p}_{\infty\infty}\big]\Big\}.$$

Theorem A. a) *The matrix* $\omega(\mathfrak{P})$ *composed from measures* $\omega_{\phi\psi}$ *is an element of* $\mathrm{Pol}_{\mathbb{R}^*}(\Omega(M^\Upsilon), \Omega(N^\Upsilon))$.

b) *The map* $\mathfrak{P} \mapsto \omega(\mathfrak{P})$ *is a functor, i.e., for each finite bordered spaces* M^Υ, N^Υ, K^Υ *and for each* Υ*-polymorphisms* $\mathfrak{P} \in \mathrm{Pol}^\Upsilon(M^\Upsilon, N^\Upsilon)$, $\mathfrak{Q} \in \mathrm{Pol}^\Upsilon(N^\Upsilon, K^\Upsilon)$,

$$\omega(\mathfrak{Q})\omega(\mathfrak{P}) = \omega(\mathfrak{Q}\mathfrak{P}). \tag{19}$$

3.7. Construction of the functor (Ω, ω) in general case. Let (M^Υ, μ), (N^Υ, ν) be arbitrary bordered spaces, and let

$$\mathcal{U} : M^\Upsilon = M_1 \cup \dots \cup M_k \cup M_\infty, \qquad \mathcal{V} : N^\Upsilon = N_1 \cup \dots \cup N_l \cup N_\infty$$

be partitions of M^Υ, N^Υ, respectively. Let $\mathfrak{P} \in \mathrm{Pol}^\Upsilon(M^\Upsilon, N^\Upsilon)$. For any $\alpha = 1, \dots, k, \infty$ and $\beta = 1, \dots, l, \infty$ we consider the map

$$M_\alpha \times N_\beta \times \mathbb{R}^* \to \mathbb{R}^*$$

and the image of the measure $\mathfrak{P}$ under this map. Thus we obtain an $\mathcal{M}$-valued matrix; it defines the element of $\mathrm{Pol}^\Upsilon(M^\Upsilon/\mathcal{U}, N^\Upsilon/\mathcal{V})$. We denote it by

$$\mathfrak{P}^{\downarrow}_{[\mathcal{U},\mathcal{V}]}.$$

Let M^Υ be a bordered space. Let $\mathcal{U}^{(j)}$ be a sequence of partitions, and let each $\mathcal{U}^{(j+1)}$ be a refinement of $\mathcal{U}^{(j)}$. We say that the sequence $\mathcal{U}^{(j)}$ of partitions is *approximative* if finite elements of all partitions $\mathcal{U}^{(j)}$ generate the Borel σ-algebra of M.

Fix $\mathfrak{P} \in \mathrm{Pol}^\Upsilon(M^\Upsilon, N^\Upsilon)$. Let $\mathcal{U}^{(j)}$, $\mathcal{V}^{(j)}$ be approximative sequences of partitions of M^Υ, N^Υ, respectively. Then we have the chain of the spaces

$$\dots \leftarrow M^\Upsilon/\mathcal{U}^{(j)} \times N^\Upsilon/\mathcal{V}^{(j)} \times \mathbb{R}^* \leftarrow M^\Upsilon/\mathcal{U}^{(j+1)} \times N^\Upsilon/\mathcal{V}^{(j+1)} \times \mathbb{R}^* \leftarrow \dots \tag{20}$$

The sequence $\mathfrak{P}^{\downarrow}_{[\mathcal{U}^{(j)},\mathcal{V}^{(j)}]}$ of bordered polymorphisms (defined in Sec. 3.1) is a projective sequence of measures with respect to the chain (20).

Theorem B. a) *Let* $\mathfrak{P} \in \mathrm{Pol}^\Upsilon(M^\Upsilon, N^\Upsilon)$. *Let* $\mathcal{U}^{(j)}$, $\mathcal{V}^{(j)}$ *be approximative sequences of partitions of* M^Υ, N^Υ. *Then the system*

$$\omega(\mathfrak{P}^{\downarrow}_{[\mathcal{U}^{(j)},\mathcal{V}^{(j)}]}) \in \mathrm{Pol}_{\mathbb{R}^*}\big(\Omega(M^\Upsilon/\mathcal{U}^{(j)}), \Omega(M^\Upsilon/\mathcal{V}^{(j)})\big)$$

is a projective system of measures with respect to the maps

$$\ldots \leftarrow \Omega(M^\Upsilon/\mathcal{U}^{(j)})\times\Omega(N^\Upsilon/\mathcal{V}^{(j)}) \leftarrow \Omega(M^\Upsilon/\mathcal{U}^{(j+1)})\times\Omega(N^\Upsilon/\mathcal{V}^{(j+1)}) \leftarrow \ldots$$

The inverse limit

$$\omega(\mathfrak{P}) \in \mathrm{Pol}_{\mathbb{R}^*}(\Omega(M), \Omega(N))$$

of this chain does not depend on the choice of the approximative sequences $\mathcal{U}^{(j)}$ and $\mathcal{V}^{(j)}$.

b) *The map $\mathfrak{P} \mapsto \omega(\mathfrak{P})$ is a functor, i.e., for each M^Υ, M^Υ, K^Υ and each Υ-polymorphisms $\mathfrak{P} \in \mathrm{Pol}^\Upsilon(M^\Upsilon, N^\Upsilon)$, $\mathfrak{Q} \in \mathrm{Pol}_{\mathbb{R}^*}(N^\Upsilon, K^\Upsilon)$,*

$$\omega(\mathfrak{Q})\omega(\mathfrak{P}) = \omega(\mathfrak{Q}\mathfrak{P}).$$

c) *Let $q \in \mathrm{Gms}_\infty(M)$. Then $\omega(q)$ is the transformation of $\Omega(M^\Upsilon)$ given by $(m_1, m_2, \ldots) \mapsto (q(m_1), q(m_2), \ldots)$, i.e., our functor (Ω, ω) extends the map (4).*

3.9. Remarks on the proofs. There are two ways to prove Theorem A. Both ways require some calculations.

The first way. Consider a space M^Υ with a continuous infinite measure. Consider a partition $\mathcal{U}$ of M^Υ. We have a map from $\Omega(M^\Upsilon)$ to the countable space $\Omega(M^\Upsilon/\mathcal{U})$, and thus we have a partition of the space $\Omega(M^\Upsilon)$. Denote this partition by $\Omega(\mathcal{U})$. For any partition $\mathcal{U}$, the map

$$\mathrm{Gms}_\infty(M) \to \mathrm{Gms}(\Omega(M))$$

induces the maps of the subgroups

$$\mathrm{Ams}_\infty(M\big|\mathcal{U}) \to \mathrm{Ams}\big(\Omega(M^\Upsilon\big|\Omega(\mathcal{U})\big). \tag{21}$$

Thus we have the map of double cosets

$$\begin{aligned}&\mathrm{Ams}_\infty(M\big|\mathcal{U}) \setminus \mathrm{Gms}_\infty(M)/\mathrm{Ams}_\infty(M\big|\mathcal{V}) \to \\ &\qquad \to \mathrm{Ams}\big(\Omega(M^\Upsilon\big|\Omega(\mathcal{U})\big) \setminus \mathrm{Gms}(\Omega(M))/\mathrm{Ams}\big(\Omega(M^\Upsilon\big|\Omega(\mathcal{V})\big).\end{aligned}$$

The map (21) transforms generic sequences to generic sequences, and this implies the product formula (19). To obtain (18), it remains to calculate this map explicitly.

Another way of proof of (19) is a direct calculation. Formula (19) is equivalent to a family of identities for some infinite sums depending on elements of $\mathcal{M}^\nabla$. The same identities for series depending on complex variables appear in the following situation.

Consider the space $\mathcal{F}_n$ of entire functions on $\mathbb{C}^n$. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator, let $b, c \in \mathbb{C}^n$. Consider the linear operator

$$U(A, b, c)f(z) = f(Az + b)\exp(\sum c_j z_j).$$

Obviously,

$$U(A, b, c)U(A', b', c') = \exp(\sum b_j c'_j)U(A'A, A'b + b', A^t c' + c). \quad (22)$$

Consider the matrix elements of this operator in the basis $z_1^{p_1} \dots z_n^{p_n}$. Explicit expressions for these matrix elements can be easily written as polynomials in A, b, c; they almost coincide with the expression (18). In this basis, the product formula (22) is a collection of identities for series of complex numbers.

The identities that are necessary for the proof of (19) are the same, but complex numbers are replaced by elements of the semigroup $\mathcal{M}^\nabla$. It remains to observe that for any $s \in \mathbb{C}$ such that $0 \leqslant \Re s \leqslant 1$, the map

$$\mathbf{u} \mapsto \int_0^\infty x^s d\mathbf{u}(x)$$

is a homomorphism of rings $\mathcal{M}^\nabla \to \mathbb{C}$ and this family of homomorphisms separates elements of $\mathcal{M}^\nabla$.

Theorem B is a corollary of Approximation Theorem for categories [20], Theorem 8.1.10.

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