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ON SAME-INVARIANT LINEAR GROUPS

Abstract. A linear group $G \leq GL(V)$ is called *same-invariant* if the subspaces of linear invariants V^g are the same for all $g \in G, g \neq 1$. Below we consider properties of such a group G under the assumption that G contains unipotent elements.

1. Introduction

Let V be a finite dimensional linear space over a field K and let $G \leq GL(V)$ be a linear group. We call G *same-invariant* if

$$V^g = V^G$$

for every $g \in G$, $g \neq 1$.

If G is finite and $V^G = 0$, we have the case of a finite linear group acting without fixed points on $V \setminus \{0\}$. These groups which are classified by H. Zassenhaus and G. Vincent in the case $K = \mathbb{C}$, $\mathbb{R}$ (see [7]) are of great importance for various questions. Clearly, the classification of finite linear groups acting without fixed points (on $V \setminus \{0\}$) for arbitrary fields K can be derived from the Zassenhaus-Vincent classification (note that if $V^g = 0$ for every $g \in G$, $g \neq 1$, then the characteristic of K does not divide the order of G and therefore the group G can be "lifted from characteristic zero").

Now let the characteristic of the field K divide the order of the group G. Then the group G contains unipotent elements and hence $V^G \neq 0$. In this case the same-invariance defined above is a natural generalization of an action without fixed points. A more general property is considered in [3–5]. Namely, the linear group $G \leq GL(V)$ is called *semi-regular* if every semisimple element $g \in G$, $g \neq 1$, has no fixed points (except 0). Note that if $G \leq GL(V)$ is same-invariant, then the natural image of G in $GL(V/V^G)$ is semi-regular. The finite semi-regular groups are classified in [3–5] up to the maximal normal p-subgroup where $p = \operatorname{char} K$. (Some properties of same-invariant finite groups considered below can be

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derived from this classification, however, we do not use this classification which is based on delicate properties of finite groups.) We cannot expect a complete classification of same-invariant finite groups. However, the results below give a description of such groups which can be useful for applications.

We also consider two classes of infinite same-invariant groups, namely, locally finite groups and groups of type $G(K)$ (where G is a linear algebraic group defined over a field K of characteristic zero) under the assumption that such groups contain unipotent elements. In both cases, the description of same-invariant groups is similar to the description of finite same-invariant groups.

2. General properties of same-invariant linear groups with unipotent elements

Let $G \leq GL(V)$ be a same-invariant group where V is a finite dimensional vector space over a field K. Further, let $U \leq G$ be a maximal unipotent subgroup.

Lemma 1. *The group U is commutative.*

Proof. There exists an element $v \in V \setminus V^U$ such that $u(v) \equiv v (\mathrm{mod}\ V^U)$ for every $u \in U$. Obviously, $[u_1, u_2](v) = v$ for every $u_1,\ u_2 \in U$ and therefore $\operatorname{rank}([u_1, u_2]) < \operatorname{rank} u$ for $u \in U$, $u \neq 1$. Since U is also same-invariant, $[u_1, u_2] = 1$. □

Lemma 2. *Let $1 \neq F \leq G$. Suppose there exists a decomposition*

$$V = V_F \oplus V^F,$$

where V_F is an F-invariant subspace of V. Then for every $u \in U$, $u \neq 1$, we have $uFu^{-1} \neq F$.

Proof. Assume $uFu^{-1} = F$ for some $u \in U$, $u \neq 1$. Then

$$u(V_F) = V_F. \tag{1}$$

Indeed, let $e_1, \ldots, e_k$ be a basis for V_F and $e_{k+1}, \ldots, e_n$ be a basis for V^F. Then the matrices corresponding to u, u^{-1} with respect to the basis $e_1, \ldots, e_n$ of V have the form:

$$u = \begin{pmatrix} \mathbb{A} & | & \mathbb{X} \\ --- & | & --- \\ \mathbb{O} & | & \mathbb{E} \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} \mathbb{A}^{-1} & | & -A^{-1}\mathbb{X} \\ --- & | & --- \\ \mathbb{O} & | & \mathbb{E} \end{pmatrix}, \tag{2}$$

where $\mathbb{A}$, $\mathbb{A}^{-1} \in M_k(K)$, $\mathbb{X} \in M_{k\times(n-k)}(K)$, and $\mathbb{E}$ is the identity in $M_{(n-k)\times(n-k)}(K)$. For every $f \in F$, $f \neq 1$,

$$f = \begin{pmatrix} \mathbb{C} & | & \mathbb{O} \\ --- & | & --- \\ \mathbb{O} & | & \mathbb{E} \end{pmatrix}, \tag{3}$$

where $\mathbb{C} \in M_k(K)$. From (2) and (3) we get

$$ufu^{-1} = \begin{pmatrix} \mathbb{A}\mathbb{C}\mathbb{A}^{-1} & | & \mathbb{X} - \mathbb{A}\mathbb{C}\mathbb{A}^{-1}\mathbb{X} \\ --- & | & --- \\ \mathbb{O} & | & \mathbb{E} \end{pmatrix}. \tag{4}$$

Since $V_F^f = \{0\}$, the matrix $\mathbb{C}$ and therefore the matrix $\mathbb{A}\mathbb{C}\mathbb{A}^{-1}$ has no eigenvalues equal to 1. Thus, the matrix $\mathbb{E} - \mathbb{A}\mathbb{C}\mathbb{A}^{-1}$ is invertible and $\mathbb{X} - \mathbb{A}\mathbb{C}\mathbb{A}^{-1}\mathbb{X} = \mathbb{O}$ if and only if $\mathbb{X} = \mathbb{O}$. Now (1) follows from (2), (3), and (4).

Since u is unipotent, (1) implies $V_F^u \neq \{0\}$ and therefore rank $(u-1) < \dim V_F$. This is a contradiction to the same-invariance of G. □

Lemma 3. *Let $F \leq G$ be a subgroup satisfying the condition of the preceding lemma. Then $fuf^{-1} \neq u$ for every $1 \neq f \in F$ and $1 \neq u \in U$.*

Proof. Indeed, the subgroup $\langle f \rangle \leq F$ also satisfies the condition of the preceding lemma. □

Lemma 4. *Let G be abelian and let $U \neq 1$. Then $G = U$.*

Proof. We may assume $K = \overline{K}$ (here $\overline{K}$ is the algebraic closure of K). Suppose $g \in G \setminus U$. Then there exists an element $v \in V$ such that $g(v) = \alpha v$ for some $\alpha \in K$, $\alpha \neq 1$. Let $u \in U$, $u \neq 1$, and let $W = K[\langle u \rangle]v$. Then W is a u-invariant vector subspace of V and therefore there exists a vector $w \in W$ such that $u(w) = w$. But $[g, u] = 1$ and therefore $g(w) = \alpha w$. This is a contradiction to the same-invariance. □

Lemma 5. *Let $\mathfrak{B} = \mathfrak{T}\mathfrak{U} \leq GL_n(K)$, $n = \dim V$, be the group of upper triangular matrices, where $\mathfrak{T}$ is the group of diagonal matrices and $\mathfrak{U}$ is the group of unipotent upper triangular matrices. Further, let $\gamma = \gamma_s\gamma_u$, $\delta = \delta_s\delta_u \in G \cap \mathfrak{B}$, where γ_s, $\delta_s \in \mathfrak{T}$, γ_u, $\delta_u \in \mathfrak{U}$. Suppose $\gamma_s = \delta_s \neq 1$ and $[\gamma, \delta] = 1$. Then $\gamma = \delta$.*

Proof. Let $e_1, \dots, e_n$ be a basis for V such that the matrices corresponding to γ, $\delta \in G$ are triangular. Further, let $\gamma_s = \delta_s = \operatorname{diag}(\alpha_1, \dots, \alpha_i, 1, \dots, 1)$ where $\alpha_i \neq 1$. Then there exists a vector

$v \in \langle e_i, \ldots, e_n \rangle$ such that $\gamma(v) = \alpha_i v$. Let $W = K[\langle \delta \rangle]v$. Then $W \leq \langle e_i, \ldots, e_n \rangle$ and there exists a vector $w \in W$ such that $\delta(w) = \alpha_i w$. Since $[\gamma, \delta] = 1$ we have $\gamma(w) = \alpha_i w$. Hence $\gamma\delta^{-1}(w) = w$. Now the same-invariance implies $\gamma = \delta$. □

Lemma 6. *Let $U \neq 1$. Further, let $M \trianglelefteq G$ be a maximal abelian normal subgroup. Assume $M \neq 1$. Then $M = U$.*

Proof. First we show that the group M consists of unipotent elements. We may (and shall) assume that $K = \overline{K}$ is an algebraically closed field.

Suppose $m \in M$ is a semisimple element. Let $u \in U$, $u \neq 1$. Consider the group

$$F = \langle m_i = u^i m u^{-i} \mid i \in \mathbb{Z} \rangle.$$

Obviously, the group F is $\langle u \rangle$-invariant. Moreover, the group F is diagonalizable and therefore $V = V_F \oplus V^F$ for some F-invariant subspace V_F. Thus we have a contradiction to Lemma 2. This means that the group M has no semisimple elements.

Let char $K = p$ and let $m \in M$. Then $m_1 = m^{p^l}$ is a semisimple element for some l. Moreover, if m is not a unipotent element, then $m_1 \neq 1$. The arguments above imply that m is a unipotent element.

Let char $K = 0$ and let $u \in U$, $u \neq 1$. Then the group $L = \langle u, M \rangle$ is triangularizable. We may assume $L \leq \mathfrak{B} = \mathfrak{T}\mathfrak{U}$ (where $\mathfrak{T}$ is the group of diagonal matrices and $\mathfrak{U}$ is the group of the upper triangular unipotent matrices). Let $m = m_s m_u \in M$, where $m_s \in \mathfrak{T}$ and $m_u \in \mathfrak{U}$. Then $m' = umu^{-1} = m_s m'_u$, where $m'_u \in \mathfrak{U}$. Suppose $m_s \neq 1$. Since $m, m' \in M$, we have $[m, m'] = 1$ and we can apply Lemma 5. Thus, $m = m'$ and hence $[m, u] = 1$. Now, applying Lemma 4 to $G = \langle m, u \rangle$, we get $m_s = 1$.

Now we may assume $M \leq U$ (for some maximal unipotent subgroup U of G). Consider the homomorphism

$$\phi : G \longrightarrow \operatorname{Aut} M$$

given by the formula $\phi(g)(m) = gmg^{-1}$. By Lemma 1, $U \leq \operatorname{Ker} \phi$. Lemma 4 shows that all elements of $\operatorname{Ker} \phi$ are unipotent. Hence $\operatorname{Ker} \phi = U = M$. □

3. Finite same-invariant groups I

Here $G \leq GL(V)$ is a finite same-invariant group, $p = \text{char } K \neq 0$.

Theorem 1. *If $p \mid |G|$ then:*

(a) *G is the semidirect product $U \cdot D$, where U is the normal Sylow p-subgroup of G;*

(b) *U is an elementary abelian p-group;*
(c) *D acts on $U \setminus \{1\}$ (by conjugation) without fixed points.*

Proof. We need the following:

Lemma 7. *The group G has a normal abelian Sylow p-subgroup.*

Proof. We show that G is not a non-abelian simple group. Assume that G is a non-abelian simple group. Then, according to the Feit–Thompson Theorem, there exists an involution in G. Moreover, since G is a non-abelian simple group, it is generated by involutions and therefore there exist non-commuting involutions. Further, the action of G on $W = V/V^G$ is faithful (because G is simple). Thus, $\operatorname{char} K \neq 2$ (because in the case $\operatorname{char} K = 2$ every involution of G acts trivially on W). Since every element of G that has an order prime to p acts without fixed points on W, the restriction of an involution of G on W is a scalar matrix $\operatorname{diag}(-1, -1, \ldots, -1)$. Hence G cannot be a non-abelian simple group.

Now let $\Gamma \trianglelefteq \Delta \trianglelefteq \Sigma \leq G$. Assume $1 \neq \Gamma_p \trianglelefteq \Gamma$, where Γ_p is a Sylow p-subgroup of Γ which is supposed to be abelian. Then Γ_p is a characteristic subgroup of Γ and therefore Δ-invariant (under conjugation). Hence there exists an abelian normal subgroup of Δ. Then Lemma 6 implies $\Delta_p \trianglelefteq \Delta$ where Δ_p is a Sylow p-subgroup of Δ that is also abelian.

Now let $G = G_0 \geq G_1 \geq G_2 \geq \ldots \geq G_d \geq 1$ be a maximal chain of subgroups such that G_{i+1} is normal in G_i. Since G_d cannot be a non-abelian simple group, it is a cyclic group of prime order. Suppose $| G_d | \neq p$. Then there exists a positive integer i such that $p \mid | G_i |$ but $(p, | G_{i+1} |) = 1$. Then we can find an element $g \in G_i$ of order p that normalizes G_{i+1}. This is a contradiction to Lemma 3. Thus $| G_d | = p$. Starting with G_d we can move through this chain of subgroups and get $G = G_p \trianglelefteq G$ and G_p is an abelian group. □

Now
(a) follows from Lemma 7;
(b) follows from Lemma 7 and the following fact: if $u \in GL_n(K)$ is a unipotent matrix and $\operatorname{char} K = p \neq 0$, then $\operatorname{rank}(u^p - 1) < \operatorname{rank}(u - 1)$;
(c) follows from Lemma 3. □

4. Connected same-invariant algebraic groups I

Let $\widetilde{G} \leq GL_n$ be a connected linear algebraic group defined over a field K of characteristic zero and let $G = \widetilde{G}(K) \leq GL_n(K)$. We identify the group $GL_n(K)$ with $GL(V)$, where V is an n-dimensional vector space over K. Thus,

$$\widetilde{G}(K) = G \leq GL(V).$$

Further, let $R_u(\widetilde{G})$ be the unipotent radical of $\widetilde{G}$. Since K is perfect, $R_u(\widetilde{G})$ is also defined over K ([6, 12.1.7]). Moreover, the group $R_u(\widetilde{G})$ is split over K ([6], 14.3.10). Recall that by U we denote a maximal unipotent subgroup of G.

Theorem 2. *Let G be same-invariant and let $U \neq 1$. Then*
(a) *$G = U \cdot D$, where $D \cong \widetilde{D}(K)$ for some connected reductive K-group $\widetilde{D}$ and $U = R_u(\widetilde{G})(K)$;*
(b) *$U \cong K^m$;*
(c) *D acts (by conjugation) on $U \setminus \{1\}$ without fixed points.*

Proof. We have

$$\widetilde{G} = R_u(\widetilde{G}) \cdot \widetilde{D}, \tag{5}$$

where $\widetilde{D}$ is a Levi factor of $\widetilde{G}$. Since $U \neq 1$, the group $R_u(\widetilde{G})$ is not trivial. (Indeed, otherwise $\widetilde{G}$ is a reductive group and therefore linearly reductive (recall that char $K = 0$). Hence $V = V_G \oplus V^G$ for some G-invariant subspace V_G. But the restriction of an operator $u \in U$, $u \neq 1$, to the subspace V_G has a stable vector and therefore $\operatorname{rank}(u-1) \leq \dim V_G$. This is a contradiction to the same-invariance.)

The group $R_u(\widetilde{G})(K)$ is abelian by Lemma 1. Let $R_u(\widetilde{G})(K) \leq M$, where M is a maximal normal abelian subgroup of G. Then $M = U$ by Lemma 6. Since U is normal in G, we have $U = R_u(\widetilde{G})(K)$. Now (a) follows from(5).

The group $R_u(\widetilde{G})$ is split over K. Since $R_u(\widetilde{G})(K)$ is abelian, we have (b).

(c) follows from Lemma 3. □

5. The maps A and R

Here $G = U \cdot D \leq GL(V)$ is a same-invariant group, where U is the maximal unipotent subgroup of G and D is a linearly reductive group. We introduce two maps

$$A : U \longrightarrow M_{k\times(m-k)}(K) \quad \text{and} \quad R : D \longrightarrow GL_k(K)$$

for some positive integers k, m. This allows us to get more information about the structure of G.

The map A. Consider the action of G on $\overline{V} = V/V^G$. Put $\overline{V}_0 = \overline{V}^U$ and let $V_0 \subset V$ be the subspace of V such that $V^G \subset V_0$ and $\overline{V}_0 = V_0/V^G$.

Since D normalizes U, we have $d(\overline{V}_0) = \overline{V}_0$ for every $d \in D$. Thus, V_0 is a G-submodule of V. Let $x_1, \dots x_k, x_{k+1}, \dots x_m$ be a basis of V_0 such that $V^G = \langle x_{k+1}, \dots, x_m \rangle$. The definition of V_0 and the definition of this basis imply that every element $u \in U$ is represented in this basis by a matrix of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & a_{1\,k+1} & \cdots & a_{1\,m} \\ 0 & 1 & 0 & \cdots & 0 & a_{2\,k+1} & \cdots & a_{2\,m} \\ \cdots & & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & a_{k\,k+1} & \cdots & a_{k\,m} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{6}$$

Denote by $A(u)$ the matrix of size $k \times (m-k)$ that is in the upper right-hand corner of (6). Further, let $M = M_{k\times 1}(K)$ be the k-dimensional vector space of k-columns. Then we may consider the K-vector space $M_{k\times(m-k)}(K)$ as the direct sum $M^{m-k} = M \oplus \cdots \oplus M$ of $m-k$ copies of M. Thus we define the map

$$A : U \longrightarrow M_{k\times(m-k)}(K) = M^{m-k}, \tag{7}$$

where the parameters m, k are defined by the G-submodule V_0 of V ($m = \dim V_0$, $m-k = \dim V^G$).

Since $V^G = V^u$ for every $u \in U$, $u \neq 1$, we have

$$\operatorname{rank} A(u) = k = \dim V_0 - \dim V^G. \tag{8}$$

From (8) we get the inequality $k \leq m-k$ which, in turn, implies

$$A(u_1u_2) = A(u_1) + A(u_2) \tag{9}$$

for every u_1, $u_2 \in U$. Thus, the map A is a monomorphism of U into $M_{k\times(m-k)}(K) = M^{m-k}$.

Lemma 8. *Let* $\operatorname{char} K = p > 0$ *and let* $F_p \subset K$ *be its prime subfield. Then for every* $f \in F_p$ *and every* $u \in U$ *there exists* $u' \in U$ *such that*

$$fA(u) = A(u').$$

Proof. This follows from (9). □

Lemma 9. *Suppose $G = \widetilde{G}(K)$, where $\widetilde{G}$ is a linear algebraic group which is defined over a field K, char $K = 0$. Then for every $f \in K$ and every $u \in U$ there exists $u' \in U$ such that*

$$fA(u) = A(u').$$

Proof. We may consider A as a regular map $\widetilde{U} \longrightarrow M_{k\times(m-k)}(\overline{K})$. Since K is perfect, the group $\widetilde{U}$ is split over K ([6] 14.3.8). Hence the group $\widetilde{U}$ is K-isomorphic to G_a^l for some l. Let $L \cong G_a \leq \widetilde{U}$. Then $A(L)$ is a K-defined algebraic subgroup of $M_{k\times(m-k)}(\overline{K})$. Obviously, $A(L) \cong G_a$. Moreover, $A(L(K)) \leq A(L)(K) = K$. Since K is perfect, this implies $A(L(K)) = A(L)(K)$. Now if $u \in L(K)$, then for every $f \in K$ there exists some $u' \in L(K)$ such that $A(u') = fA(u)$. Since $\widetilde{U} = G_a^l$, we get our assertion for every $u \in U$. □

The subspace $\mathfrak{L}$. For every $m \in M_{k\times(m-k)}(K)$ we denote by $\operatorname{rank} m$ the rank of m as a matrix. Further, let $l_1, l_2, \dots, l_{m-2k+1} \in M_{k\times(m-k)}(K)$ be the matrices:

$$l_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},$$

$$l_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1\cdots & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & 0 & 1 & \cdots & 0 \end{pmatrix}, \dots$$

$$\dots, l_{m-2k+1} = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & 0\cdots & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Put

$$\mathfrak{L} = \langle l_1, \dots, l_{m-2k+1} \rangle.$$

Note, $\operatorname{rank} l = k$ for every $l \in \mathfrak{L}$.

Lemma 10. *Let $\mathfrak{L}'$ be a K-vector subspace of $M_{k\times(m-k)}(K)$ of maximal dimension such that* $\operatorname{rank} l = k$ *for every $l \in \mathfrak{L}, l \neq 0$. Then* $\dim \mathfrak{L}' =$ *$m - 2k + 1$.*

Proof. Let $\mathfrak{I}_k$ be the ideal of $\overline{K}[x_{ij}]$, $1 \leq i \leq k$, $1 \leq j \leq m-k$, generated by k-minors of the matrix (x_{ij}). Then $\operatorname{ht}(\mathfrak{I}) \leq (m-k) - k + 1 = m -$

$2k+1$ ([1]). Hence every irreducible component Y_i of the corresponding determinantal variety has codimension $\le m-2k+1$. Since $\{0\} = (\mathfrak{L}' \cap Y_i)$ for some component Y_i, we have

$$\dim \mathfrak{L}' \le m - 2k + 1. \tag{10}$$

Since the space $\mathfrak{L}$ satisfies the condition for $\mathfrak{L}'$, we have $\dim \mathfrak{L}' \ge m - 2k + 1$. Now we get our assertion from (10). □

The map R. Further, by changing the basis $x_1, \dots, x_m$ for an appropriate one (that, however, still preserves the form (6)), we can have for every $d \in D$ a matrix of the form:

$$\begin{pmatrix} d_{11} & \cdots & d_{1k} & 0 & \cdots & 0 & 0 \\ d_{21} & \cdots & d_{2k} & 0 & \cdots & 0 & 0 \\ \cdots & & & & & & \\ d_{k1} & \cdots & d_{kk} & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{11}$$

(Indeed, the group D is linearly reducible. Thus there exists a D-module V_0' such that $V_0 = V' \oplus V^D$. Now if we take bases $y_1, \dots, y_k$ of V' and $y_{k+1}, \dots, y_m$ of $V^D = V^G$, we get both forms (6) and (11).)

Denote by $R(d)$ the $k \times k$-matrix which is in the upper left-hand corner of (11). Thus we have a representation

$$R : D \longrightarrow GL_k(K). \tag{12}$$

One can easily check that

$$A(dud^{-1}) = R(d)A(u) \tag{13}$$

for every $d \in D$, $u \in U$.

Let $F = F_p$ if G is a finite group and let $F = K$ if $G = G(K)$. Further, let $\mathfrak{D} = F[D]$ be the group algebra of D over F and let

$$\phi : \mathfrak{D} \longrightarrow M_k(K)$$

be the natural homomorphism which is induced by the representation $R : D \longrightarrow CL_k(K)$. Put $\mathfrak{C} = \phi(\mathfrak{D})$. Further, let

$$\gamma = f_1 d_1 + f_2 d_2 + \cdots + f_s d_s \in \mathfrak{D},$$

where $f_i \in F$, $d_i \in D$. We have

$$\phi(\gamma) = f_1 R(d_1) + f_2 R(d_2) + \cdots + f_s R(d_s) \in \mathfrak{C}.$$

We can define the following action of $\mathfrak{C}$ on $A(U)$ using Lemmas 8 and 9, and the equality (13):

$$\begin{aligned}\phi(\gamma)(A(u)) &= f_1 R(d_1)A(u) + \cdots + f_s R(d_s)A(u) = \\ &= f_1 A(d_1 u d_1^{-1}) + \cdots + f_s(A(d_s u d_s^{-1}).\end{aligned} \tag{14}$$

Further, the F-algebra $\mathfrak{C}$ is semisimple (because D is linearly reducible). This means

$$\mathfrak{C} = \mathfrak{C}_1 \oplus \cdots \oplus \mathfrak{C}_r,$$

where $\{\mathfrak{C}_i\}$ are simple F-algebras, that is, $\mathfrak{C}_i \cong M_{k_i}(L_i)$ where L_i is a division F-algebra. We show $r = 1$, $k_1 = 1$ and $\mathfrak{C} = L$ for some division F-algebra L. Indeed, otherwise we can find elements $\sigma, \tau \in \mathfrak{C}$ such that

$$\sigma \neq 0, \quad \tau \neq 0, \quad \sigma\tau = 0. \tag{15}$$

Since $\mathfrak{C} \subset M_k(K)$ we may consider σ, τ as matrices in $M_k(K)$. Then (15) implies

$$1 \leq \operatorname{rank} \sigma < k. \tag{16}$$

From (8), (14), and (16)

$$1 \leq \operatorname{rank}(\sigma(A(u)) < k \tag{17}$$

for every $u \in U$, $u \neq 1$, (here we consider $\sigma(A(u))$ as a matrix in $M_{k\times(m-k)}(K)$). Since $1 \leq \operatorname{rank}(\sigma(A(u))$ we have $\sigma(A(u)) \neq 0$. But $\sigma(A(u)) \in A(U)$ (by (14)). Hence (17) contradicts (8). Thus, we have proved

$$\mathfrak{C} = L, \tag{18}$$

where L is a division F-algebra.

6. Finite same-invariant groups II

Here $G = U \cdot D \leq GL(V)$ is a finite same-invariant group over a field K such that $p = \operatorname{char} K \mid | G |$ and m, k are the numbers defined in the preceding section.

Recall that $L = \mathfrak{C} = \phi(F_p[D])$ is a division algebra from (18) over a field $F = F_p$ (in the case of a finite group G). Hence L is a field that is a finite extension of F_p.

Further, consider the group U as an $F_p[D]$-module. Since the order q of the group D is prime to p, we have

$$U = \sum_{i=1}^{i=l} U_i, \tag{19}$$

where U_i are irreducible $F_p[D]$-modules. Also, we can consider each $F_p[D]$-module U_i as an L-module.

Theorem 3. *In the notation above we have:*
(a) *D is a cyclic group $\langle\sigma\rangle$;*
(b) *$L = F_p(\epsilon)$, $\epsilon = \sqrt[q]{1}$, and $\phi(\sigma) = \epsilon$;*
(c) *$U_i \cong L^+$ as L-module for every i;*
(d) *all irreducible $\{U_i\}_{i=1}^{i=l}$ in (19) are isomorphic as $F_p[D]$-modules.*

Proof.
(a) Since the restriction of the map

$$\phi : \mathfrak{D} = F_p[D] \longrightarrow M_k(K)$$

to D is injective, the group D can be identified with a subgroup of $\mathfrak{C} = L$ and, therefore, is a cyclic group.

(b) The element $\epsilon = \phi(\sigma)$ generates the algebra $L = \mathfrak{C} = \phi(F_p[D])$ over F_p.

(c) Since $0 \neq Lu_i \leq U_i$ for $u_i \in U_i$, $u_i \neq 0$, we have $Lu_i = U_i$ and thus U_i is isomorphic to L^+ as an L-module.

(d) Let $u_i \in U_i, u_i \neq 0$. Further, $LA(u_i) = \{lA(u_i) \mid l \in L\} \subset M_{k\times(m-k)}(K)$ (here $lA(u_i)$ is the multiplication of the matrices $l \in M_{k\times k}(K)$ and $A(u_i) \in M_{k\times(m-k)}(K)$) is an $F_p[D]$-module which is isomorphic to U_i (by (13) and (c)) and for every $d \in D$ the action of d on $LA(u_i)$ is defined by the formula $R(d)(lA(u_i)) = (R(d)l)A(u_i)$ (again by (13)). Thus, if we identify D with a subgroup of L^* (the representation R is faithful), the action of D on each module $U_i \cong L$ is equivalent to the left multiplication of L on elements of L^*. Hence all U_i are isomorphic as $F_p[D]$-modules. $\square$

Example 1. Let $D = \langle\sigma\rangle$ be a cyclic group of order q, $(q,p) = 1$, and let $L = F_p(\epsilon)$ where $\epsilon = \sqrt[q]{1}$ is a primitive root of unity of degree q. Further, let $U = (L^+)^a$ for some positive integer a and let $U \cdot D$ be a semidirect product, where

$$\sigma u \sigma^{-1} = \epsilon u$$

for every $u \in U$ (here we consider the componentwise multiplication by ϵ of u). By Theorems 1 and 3 every finite same-invariant group is isomorphic to such a group G and vice versa, every such finite group has a faithful representation over a field of characteristic p that has a same-invariant image. Indeed, let K be an extension of L of degree a. Then

$$\left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \right\} \mid v \in K\} \cdot \left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \cong U \cdot D.$$

Example 2. Let $\mid U \mid= p \neq 2$ and $q =\mid D \mid \mid p-1$, $q \neq 1$. Hence $U = \langle u \rangle \cong F_p^+$ and we can identify the group $D = \langle \sigma \rangle$ with a subgroup of F_p^*. Suppose V is an indecomposable same-invariant $U \cdot D$-module. Then there exists a basis for V such that

$$u = \begin{pmatrix} 1 & u_{12} & u_{13} \cdots & \cdots & u_{1n} \\ 0 & 1 & u_{23} \cdots & \cdots & u_{2n} \\ \cdots & & & & \\ 0 & \cdots & 0 & 1 & u_{n-1n} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} \epsilon^{n-1} & 0 & 0 \cdots & \cdots & 0 \\ 0 & \epsilon^{n-2} & 0 \cdots & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & 0 & \epsilon & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix},$$

where $\epsilon \in F_p^*$ is a primitive root of unity of degree q, $n < q \leq p-1$, and $(2, q) = \cdots = ((n-1), q) = 1$. Moreover, $\operatorname{rank}(u^r - 1) = n-1$ for every $r \leq p$ (here we omit the proof of this fact).

Remark 1. We cannot expect any classification of same-invariant indecomposable G-modules V in the case where U is not a cyclic group (see [2] 64.3).

The case of locally finite same-invariant groups. Let

$$G = \varinjlim G_i \leq GL(V)$$

be a same-invariant group, where $G_i \leq GL(V)$ are finite groups. Obviously, every such group is same-invariant and $V^{G_i} = V^G$ for every $G_i \neq 1$.

Assume $U \neq 1$. Then $U_i \neq 1$ for every $i \geq$ some i_0. By Theorem 1 for every $i \geq i_0$

$$G_i = U_i \cdot D_i,$$

where U_i is an elementary abelian p-group and $\mid D_i \mid = n_i$ is prime to p. Moreover, D_i acts freely on $U_i \setminus 1$.

We may choose the subgroup D_i such that $D_i \geq D_{i-1}$ for every $i \geq i_0$. Indeed, consider the group $G_{i-1} = U_{i-1} \cdot D_{i-1} \leq G_i = U_i \cdot D_i$. From Theorem 3, $D_{i-1} = \langle \sigma_{i-1} \rangle$, $D_i = \langle \sigma_i \rangle$. Further, $\sigma_i^l = \sigma_{i-1} u$ for some $u \in U_i$ and some positive integer l. Since σ_{i-1} acts freely on $U_i \setminus \{1\}$, there exists an element $v \in U_i$ such that $v \sigma_{i-1} u v^{-1} = \sigma_{i-1}$. Now we can take the group $\langle v \sigma_i v^{-1} \rangle$ instead of D_i.

Thus, we may assume that $\{D_i\}_{i \geq i_0}$ and $\{U_i\}_{i \geq i_0}$ are inductive systems of finite subgroups of $GL(V)$. Put

$$U = \varinjlim U_i \leq GL(V), \quad D = \varinjlim D_i \leq GL(V).$$

Then we have $G = U \cdot D$.

7. Connected same-invariant algebraic groups II

We keep here the notation and assumptions of Sec. 4.

Now let $\overline{K}$ be the algebraic closure of K and $W = V \otimes_K \overline{K}$. Further, let $GL(W)$ be a K-defined algebraic group that is K-isomorphic to GL_n, $n = \dim V$, and such that $GL(W)(K) = GL(V)$. Thus, we consider the group $\widetilde{G}$ (see Sec. 4) as a K-subgroup of the K-group $GL(V)$.

Further, let L be the division algebra defined by (18). Recall that $L \subset \mathrm{End}_K(V') \cong M_k(K)$, where $V' \leq V$ is a subspace that is D-invariant (see Sec. 5). Hence L is a finite dimensional K-algebra.

Further, if $W' = V' \otimes_K \overline{K}$ then $GL(W')$ is a K-subgroup of $GL(W)$ and $GL(W')(K) = GL(V')$. There exists a K-defined subgroup $GL_1(L)$ of the K-group $GL(W')$ such that $GL_1(L)(K) = L^* = L \setminus \{0\}$.

Further, the subspace W' is D-invariant. Hence W' is also $\widetilde{D}$-invariant (note that $D = \widetilde{D}(K)$ is dense in $\widetilde{D}$). The restriction of $\widetilde{D}$ to W' gives us a faithful K-rational representation $\widetilde{D} \longrightarrow GL(W')$ (see the definition of the map R in Sec. 5). Thus we may consider the group $\widetilde{D}$ as a K-subgroup of $GL(W')$. The definition of the algebra L implies

$$D = \widetilde{D}(K) \leq L^* = GL_1(L)(K). \tag{20}$$

Further, let

$$\widetilde{U} = \sum_{i=1}^{i=l} \widetilde{U}_i \tag{21}$$

be a decomposition (over K) of the K-rational $\overline{K}(\widetilde{D})$-module $\widetilde{U}$ (we consider the action of $\widetilde{D}$ on $\widetilde{U}$ by conjugation) into the sum of irreducible (over K) K-rational $\widetilde{D}$-modules. Let $U_i = \widetilde{U}_i(K)$. Then

$$U = \sum_{i=1}^{i=l} U_i \tag{22}$$

is a decomposition of the $K[D]$-module U into the sum of irreducible $K[D]$-modules (this follows from the density of D in $\widetilde{D}$).

Theorem 4. *In the notation above we have:*
(a) *$\widetilde{D}$ is a K-subgroup of the K-group $GL_1(L)$;*
(b) *$U_i \cong L^+$ as L-module for every i;*
(c) *all irreducible components $\{U_i\}_{i=1}^{i=l}$ in (22) are isomorphic as $K[D]$-modules;*
(d) *all components $\{\widetilde{U}_i\}_{i=1}^{i=l}$ in (21) are K-isomorphic as $\overline{K}[\widetilde{D}]$-modules.*

Proof. (a) The intersection of K-subgroups of a K-group is also a K-subgroup. Hence $\widetilde{H} = \widetilde{D} \cap GL_1(L)$ is a K-subgroup of $GL(W')$ and also a K-subgroup of $GL_1(L)$. Further, (20) implies $D = \widetilde{D}(K) \in \widetilde{H}$ and therefore $\widetilde{H} = \widetilde{D}$.

(b) The same arguments as in (c) of Theorem 3.

(c) The same arguments as in (d) of Theorem 3.

(d) This follows from (c) and the density of D in $\widetilde{D}$. □

Theorem 5. *Let $Y \leq V$ be a minimal non-trivial G-submodule. Then:*
(a) *$2\dim_K Y^G \geq \dim_K Y$;*
(b) *$\dim_K U \leq 2\dim_K Y^G - \dim_K Y + 1$.*

Proof. Since Y is non-trivial, the action of G on Y is faithful and same-invariant (this follows from the same-invariance of the G-module V). Thus, we may assume that $Y = V$ and the G-module V has no non-trivial G-submodules. Hence $V_0 = V$ where V_0 is the G-submodule of V which is defined in Sec. 5 (see the definition of the map A). Now

$$\dim V = m, \quad \dim V^G = m - k. \tag{23}$$

Since $m - k \geq k$ (see Sec. 5), we get (a) from (23). Further, (8), and (23), and Lemma 10 give us (b). □

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