Записки научных
семинаров ПОМИ
Том 288, 2002 г.

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ESTIMATES OF DEVIATIONS FOR GENERALIZED NEWTONIAN FLUIDS

Abstract. The paper is concerned with deriving extimates of deviations from exact solutions for stationary models of viscous incompressible fluids. it is shown that if a function compared with exact solution is subject to the incompressibility condition, then the deviation majorant consists of terms that penalize inaccuracy in the equilibrium equation and rheological relation defined by a ceratin dissipative potential. If such a function does not satisfy the incompressibility condition, then an additional term arises. The factor of this term depends on the constant in the Ladyzhenskaya–Babuška–Brezzi.

Dedicated to dear
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1. Introduction

We consider slow stationary flow of a generalized Newtonian incompressible fluid with nonlinear viscosity. The classical statement of this problem is as follows: find a vector-valued function u (velocity), a scalar-valued function p (pressure) and a tensor-valued function σ^* (stress deviator) such that

$$-\operatorname{div}\sigma^* = f - \nabla p \quad \text{in} \;\; \Omega, \tag{1.1}$$

$$\operatorname{div} u = 0 \text{ in } \;\Omega, \tag{1.2}$$

$$\sigma^* \in \partial g(\varepsilon(u)) \text{ in } \;\Omega, \tag{1.3}$$

$$u = u_0 \text{ on } \;\partial\Omega, \tag{1.4}$$

where Ω is a connected bounded domain in $\mathbb{R}^n$ having Lipschitz continuous boundary $\partial\Omega$, f and u_0 are given functions and g is the dissipative potential that defines physical properties of the fluid considered and ∂g denotes the subdifferential of g (if g is Gateaux differentiable, then it is replaced by the derivatiive g').

Hereafter, it is assumed that

$$f \in L_2(\Omega, \mathbb{R}^n), \quad u_0 \in W_2^1(\Omega, \mathbb{R}^n), \quad \operatorname{div} u_0 = 0. \tag{1.5}$$

and

$$g(\varepsilon) = \frac{\nu}{2} \mid \varepsilon \mid^2 + \psi(\varepsilon), \tag{1.6}$$

where $\psi : \mathbb{M}_s^{n\times n} \to \mathbb{R}_+$ is a convex function such that

$$\psi(0) = 0, \quad \psi(\varepsilon) \leq c_1 \mid \varepsilon \mid^2 + c_2, \quad c_1 > 0. \tag{1.7}$$

The cases

$$\psi \equiv 0, \tag{1.8}$$

and

$$\psi(\varepsilon) = k_* \mid \varepsilon \mid \tag{1.9}$$

correspond to the classical Newtonian and Bingham models, respectively. Here k_* is a positive constant depending on plasticity properties of the fluid. Other sensible examples are given by the power laws

$$\psi(\varepsilon) = k_* |\varepsilon|^\alpha \qquad 1 < \alpha \leq 2, \tag{1.10}$$

$$\psi(\varepsilon) = k_* \left(1 + |\varepsilon|^2\right)^{\alpha/2} \tag{1.11}$$

and the Powell–Eyring models, in which

$$\psi(\varepsilon) = k_* \mid \varepsilon \mid \ln\left(1 + \mid \varepsilon \mid\right). \tag{1.12}$$

In all the models presented above, the stress tensor can be split into two terms associated with two parts of the dissipative potential so that the basic system admits the following form:

$$-\operatorname{div}(\sigma_1^* + \sigma_2^*) = f - \nabla p \quad \text{in } \Omega \tag{1.13}$$

$$\operatorname{div} u = 0 \text{ in } \Omega \tag{1.14}$$

$$\sigma_1^* = \nu\varepsilon(u) \text{ in } \Omega \tag{1.15}$$

$$\sigma_2^* \in \partial\psi(\varepsilon(u)) \text{ in } \Omega \tag{1.16}$$

$$u = u_0 \text{ on } \partial\Omega, \tag{1.17}$$

Investigation of mathematical properties of solutions to such type problems started from the works of O. A. Ladyzhenskaya [12, 13, 14]. Various types of dissipative potentials associated with generalized Newtonian fluids, their physical justifications and properties of the respective mathematical problems were considered by many authors. At this point we refer e.g. to [1, 5, 15, 16]. Regularity of solutions of problems with non Newtonian dissipative was investigated in [25, 26, 8, 9, 10]. These results has been systematically exposed in the book [7].

In this paper, we are focused on a different problem important for the numerical analysis of problems in question. Our aim is to obtain computable functionals that majorize the quantity $\|\varepsilon(v-u)\|$, where u is the generalized solution of (1.13)–(1.17), v is a function from the natural functional space containing u, and $\|\cdot\|$ is the respective energy norm. Having such a majorant one can verify the quality of approximate solutions. To derive deviation majorants (they may be also called functional type a posteriori error estimates) we apply the method earlier used by the author in various elliptic type boundary–value problems (see [20, 21, 22] and the references therein). In [23], this method was applied to the Stokes problem. The approach proposed in these papers differs from the methods earlier used for a posteriori error control in the finite element analysis of flow problems (see, e.g., [3, 19, 27]). The main difference consists of that deviation majorants are independent of the particular method and approximation used. They are derived on the functional level and contain only external data and certain global constants depending on the domain.

For the problem (1.13)–(1.17) the majorants come in two main forms:

(a) for divergence-free (solenoidal) velocity fields satisfying the given boundary condition (1.17),

(b) for any vector-valued functions having first generalized derivatives and satisfying (1.17).

In the case (a), the majorant is the sum of three quantities that measure residuals in Eqs. (1.13), (1.15), and (1.16) (see (4.9)). In the case (b), the majorant has a more complicated structure and includes an additional term that measures the error in the incompressibility condition (1.14). This term can be estimated if the constant $\mathbb{C}_\Omega$ in the Ladyzhenskaya–Babuska–Brezzi condition ([2, 4, 12])

$$\inf_{\substack{q\in \overset{\circ}{L}_2\\ q\neq 0}} \sup_{\substack{w\in V_0\\ w\neq 0}} \frac{\int\limits_\Omega q \operatorname{div} w\, dx}{\|q\|\,\|\nabla w\|} \geq \mathbb{C}_\Omega \tag{1.18}$$

is known or if we have an estimate of it.

2. Preliminaries

2.1. Notation. We begin with introducing functional spaces used in the subsequent analysis. Let $V = W_2^1(\Omega, \mathbb{R}^n)$, and $\Sigma = L_2(\Omega, \mathbb{M}_s^{n\times n})$, where $\mathbb{M}_s^{n\times n}$ is the space of real-valued symmetric matrixes. The scalar product in this space we denote by two dots and the norm by $\|\cdot\|$. The unit matrix is denoted by $\mathbb{I}$. If $\tau \in \mathbb{M}_s^{n\times n}$, then $\tau^D = \tau - \frac{1}{n}\mathbb{I}$ is the deviator of τ. V_0 is the subset of V containing functions with zero traces on $\partial\Omega$ and $\overset{\circ}{L}_2$ is the space of all square–summable scalar functions such that

$$\int_\Omega p\,dx = 0\,.$$

The set of smooth solenoidal vector fields is denoted by $\dot{J}^\infty$, i.e.,

$$\dot{J}^\infty(\Omega) := \{v \in C_0^\infty(\Omega, \mathbb{R}^n)\,\|\operatorname{div} v = 0,\ \operatorname{supp} v \subset\subset \Omega\}.$$

The space $\mathcal{H}^1$ is the closure of the set $\dot{J}^\infty$ in the norm $\|\nabla v\|$. The set $V_0 + u_0$ consists of functions from V having the form $w + u_0$, where $w \in V_0$. Analogously, the set $\mathcal{H}^1 + u_0$ contains functions from $\mathcal{H}^1$ having the form $w + u_0$, where $w \in \mathcal{H}^1$. Topologically dual spaces are denoted by stars. Formally, the strains ε and the stresses σ^*

Belong to a pair of dual spaces. For this reason, all tensor-valued functions presenting stresses are marked by stars. These functions are viewed as elements of Σ and Σ^*, respectively. By $\Sigma^*_{\mathrm{div}}(\Omega)$ we denote a subspace of Σ^*, which contains functions having square summable divergence. Σ^*_{div} is a Hilbert space with respect to the scalar product

$$(\tau^*, \eta^*) := \int_\Omega (\tau^* : \eta^* + \operatorname{div}\tau^* \cdot \operatorname{div}\eta^*)\,dx\,.$$

Also, we will use the following two affine subsets of Σ^*:

$$\Sigma^*_f(\Omega, \mathcal{H}^1) := \left\{ \tau^* \in \Sigma^*(\Omega)\,\|\int_\Omega \varepsilon(w) : \tau^*\,dx = \int_\Omega f\cdot w dx \quad \forall\, w \in \mathcal{H}^1 \right\}.$$

and

$$\Sigma^*_f(\Omega, V_0) := \left\{ \tau^* \in \Sigma^*(\Omega)\,\|\int_\Omega \varepsilon(w) : \tau^*\,dx = \int_\Omega f\cdot w dx \quad \forall\, w \in V_0 \right\}.$$

Since $\mathcal{H}^1 \subset V_0$, we see that $\Sigma_f^*(\Omega, V_0) \subset \Sigma_f^*(\Omega, \mathcal{H}^1)$. In what follows, we use the norm

$$\|g\|_{(-1)} := \sup_{w \in V_0} \frac{\int\limits_\Omega g w \, dx}{\|\varepsilon(w)\|},$$

which is a norm in the space V_0^*.

2.2. Variational problem. A generalized solution of (1.13)–(1.17) satisfies the variational inequality

$$\int\limits_\Omega \nu\varepsilon(u) : \varepsilon(v-u)dx + \Psi(\varepsilon(v)) - \Psi(\varepsilon(u)) \geq \int\limits_\Omega f \cdot (v-u)dx, \ \forall v \in \mathcal{H}^1 + u_0, \tag{2.1}$$

where

$$\Psi(\varepsilon) := \int\limits_\Omega \psi(\varepsilon)\, dx \, .$$

This problem has a variational form.

Problem $\mathcal{P}$. *Find $u \in \mathcal{H}^1 + u_0$ such that*

$$J(u) = \inf \mathcal{P} := \inf_{v \in \mathcal{H}^1 + u_0} J(v),$$

where

$$J(v) = \int\limits_\Omega g(\varepsilon(v))\, dx - \int\limits_\Omega f \cdot v \, dx, \quad g(\varepsilon) = \frac{\nu}{2} \mid \varepsilon \mid^2 + \psi(\varepsilon).$$

Theorem 2.1. *Problem $\mathcal{P}$ is uniquely solvable.*

This assertion easily follows from general results of the calculus of variations (see, e.g., [6]).

Let $\psi^* : \mathbb{M}_s^{n\times n} \to \mathbb{R}$ be the function conjugate to ψ in the sense of Young–Fenchel, i.e.,

$$\psi^*(\kappa^*) = \sup_{\kappa \in \mathbb{M}_s^{n\times n}} \left\{ \kappa^* : \kappa - \psi(\kappa) \right\}. \tag{2.2}$$

Define the Lagrangian

$$L(v;\tau_1^*,\tau_2^*) := \int\limits_\Omega \left(\varepsilon(v) : (\tau_1^* + \tau_2^*) - \frac{1}{2\nu} \mid \tau_1^* \mid^2 - \psi^*(\tau_2^*)\right) dx - \int\limits_\Omega f \cdot v \, dx.$$

It is directly verified that

$$\sup_{\tau_1^* \in \Sigma^*} \int\limits_\Omega (\tau : \tau_1^* - \frac{1}{2\nu} \mid \tau_1^* \mid^2) \, dx = \int\limits_\Omega \frac{\nu}{2} \mid \tau \mid^2 \ dx. \tag{2.3}$$

It is easy to verify that $\psi^*(\kappa^*) \geq \frac{1}{4c_1} \mid \kappa^* \mid^2 - c_2$. Hence, any τ^* such that $\int_\Omega \psi^*(\tau^*) dx \leq +\infty$ belongs to Σ^* and, therefore,

$$\Psi(\tau) = \sup_{\tau_2^* \in \Sigma^*} \int\limits_\Omega \{\tau(x) : \tau_2^* - \psi^*(\tau_2^*)\} dx \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$J(v) = \sup_{(\tau_1^*,\tau_2^*) \in \Sigma^* \times \Sigma^*} L(v;\tau_1^*,\tau_2^*)$$

By taking infimum of L over v, we obtain another functional:

$$\inf_{v \in \mathcal{H}^1 + u_0} L(v;\tau_1^*,\tau_2^*) = \begin{cases} I^*(\tau_1^*,\tau_2^*), & \text{if } (\tau_1^*,\tau_2^*) \in T_f^*(\Omega,\mathcal{H}^1), \\ -\infty & \text{if } (\tau_1^*,\tau_2^*) \notin T_f^*(\Omega,\mathcal{H}^1), \end{cases}$$

where

$$I^*(\tau_1^*,\tau_2^*) := \int\limits_\Omega \left(\varepsilon(u_0) : (\tau_1^* + \tau_2^*) - \frac{1}{2\nu} \mid \tau_1^* \mid^2 - \psi^*(\tau_2^*)\right) dx - \int\limits_\Omega f \cdot u_0 \, dx$$

and

$$T_f^*(\Omega,\mathcal{H}^1) := \{(\tau_1^*,\tau_2^*) \in \Sigma^* \times \Sigma^* \,\|(\tau_1^* + \tau_2^*) \in \Sigma_f^*(\Omega,\mathcal{H}^1)\}.$$

The following problem associated with the functional I^* is called *dual* to $\mathcal{P}$.

Problem $\mathcal{P}^*$. *Find* $(\sigma_1^*,\sigma_2^*) \in T_f^*(\Omega,\mathcal{H}^1)$ *such that*

$$I^*(\sigma_1^*,\sigma_2^*) = \sup \mathcal{P}^* := \sup_{(\tau_1^*,\tau_2^*) \in T_f^*(\Omega,\mathcal{H}^1)} I^*(\tau_1^*,\tau_2^*).$$

Since $\psi^*(\kappa^*) \geq \frac{1}{4c_1} \mid \kappa^* \mid^2 - c_2$, the functional $(-I^*)$ is coercive on $\Sigma^* \times \Sigma^*$. The set $T_f^*(\Omega,\mathcal{H}^1)$ is closed in the topology of $\Sigma^* \times \Sigma^*$. These facts imply the existence theorem.

Theorem 2.2. *Problem $\mathcal{P}^*$ has a solution.*

The first step in deriving estimates of deviation from exact solutions is given by the assertion below.

Theorem 2.3. *For any $v \in \mathcal{H}^1 + u_0$,*

$$\int\limits_{\Omega} \frac{\nu}{2} \mid \varepsilon(v-u) \mid^2 \, dx \leq J(v) - J(u). \tag{2.5}$$

Proof. This estimate follows from (2.1) and the identity

$$J(v) - J(u) = \int\limits_{\Omega} \Big(\frac{\nu}{2} \mid \varepsilon(v-u) \mid^2 + \nu\varepsilon(u) : \varepsilon(v-u) \Big) \, dx$$
$$+ \Psi(\varepsilon(v)) - \Psi(\varepsilon(u)) - \int\limits_{\Omega} f \cdot (v-u) \, dx.$$

3. Perturbed problems

Let $q \in L_2(\Omega)$ and $\tau_2^* \in \Sigma^*$ be two given functions. Introduce the perturbed functional

$$\widetilde{J}(v) := \int\limits_{\Omega} \Big(\frac{\nu}{2} \mid \varepsilon(v) \mid^2 + \tau_2^* : \varepsilon(v) - \psi^*(\tau_2^*) \Big) \, dx$$
$$- \int\limits_{\Omega} (f \cdot v + q \operatorname{div} v) \, dx \tag{3.1}$$

and the respective variational problem.

Problem $\widetilde{\mathcal{P}}$. *Find $\widetilde{u} \in V_0 + u_0$ such that*

$$\widetilde{J}(\widetilde{u}) = \inf \widetilde{\mathcal{P}} := \inf_{v \in V_0 + u_0} \widetilde{J}(v) \, .$$

Certainly, $\widetilde{J}$ and $\widetilde{u}$ depend on q and τ_2^*. However, for simplicity, we do not put these arguments explicitly.

Theorem 3.1. *Problem $\widetilde{\mathcal{P}}$ is uniquely solvable and*

$$\inf \widetilde{\mathcal{P}} \leq \inf \mathcal{P} \tag{3.2}$$

Proof. Existence and uniqueness of $\widetilde{u}$ follows from the properties of $\widetilde{J}$ and $V_0 + u_0$. For any $v \in \mathcal{H}^1 + u_0$,

$$\widetilde{J}(v) = \int\limits_\Omega \left(\frac{\nu}{2} \mid \varepsilon(v) \mid^2 + \tau_2^* : \varepsilon(v) - \psi^*(\tau_2^*)\right) dx - \int\limits_\Omega f \cdot v dx \leq J(v).$$

Therefore,

$$\inf_{v \in V_0 + u_0} \widetilde{J}(v) \leq \inf_{v \in \mathcal{H}^1 + u_0} \widetilde{J}(v) \leq \inf_{v \in \mathcal{H}^1 + u_0} J(v) = \inf \mathcal{P} \tag{3.3}$$

and (3.2) follows.

The perturbed problem has a dual counterpart. To present it we consider the perturbed Lagrangian

$$\widetilde{L}(v; \tau_1^*, \tau_2^*) := \int\limits_\Omega \left(\varepsilon(v) : (\tau_1^* + \tau_2^*) - \frac{1}{2\nu} \mid \tau_1^* \mid^2 - \psi^*(\tau_2^*)\right) dx$$
$$- \int\limits_\Omega (f \cdot v + q \cdot \operatorname{div} v) dx.$$

It is obvious that

$$\widetilde{J}(v) = \sup_{\tau_1^* \in \Sigma^*} \widetilde{L}(v; \tau_1^*, \tau_2^*).$$

Since

$$\inf_{v \in V_0 + u_0} \widetilde{L}(v; \tau_1^*, \tau_2^*) = \begin{cases} I^*(\tau_1^*, \tau_2^*), & \text{if } (\tau_1^*, \tau_2^*) \in T^*_{f - \nabla q}(\Omega, V_0), \\ -\infty & \text{if } (\tau_1^*, \tau_2^*) \notin T^*_{f - \nabla q}(\Omega, V_0), \end{cases},$$

where

$$T^*_{f - \nabla q}(\Omega, V_0) := \left\{ (\tau_1^*, \tau_2^*) \in \Sigma^* \times \Sigma^* \| (\tau_1^* + \tau_2^*) \in \Sigma^*_{f - \nabla q}(\Omega, V_0) \right\}.$$

Note that

$$(\tau_1^*, \tau_2^*) \in T^*_{f - \nabla q}(\Omega, V_0) \Rightarrow (\tau_1^*, \tau_2^*) \in T^*_f(\Omega, \mathcal{H}^1).$$

Now, we arrive at the following problem dual to $\widetilde{\mathcal{P}}$.

Problem $\widetilde{\mathcal{P}}^*$. *Given $\tau_2^* \in \Sigma^*$ and $q \in L_2(\Omega)$, find $\widetilde{\sigma_1^*} \in \Sigma^*$ such that*

$$(\widetilde{\sigma_1^*}, \tau_2^*) \in T^*_{f - \nabla q}(\Omega, V_0)$$

and

$$I^*(\widetilde{\sigma_1^*}, \tau_2^*) = \sup \widetilde{\mathcal{P}}^* := \sup_{\tau_1^*} I^*(\tau_1^*, \tau_2^*),$$

where the supremum is taken over all $\tau_1^* \in \Sigma^*$ satisfying the condition $(\tau_1^*, \tau_2^*) \in T^*_{f - \nabla q}(\Omega, V_0)$.

Theorem 3.2. *Problem $\widetilde{\mathcal{P}}^*$ has a unique solution $\widetilde{\sigma_1^*}$, which satisfies the following relations:*

$$\inf \widetilde{\mathcal{P}} = \widetilde{J}(\widetilde{u}) \ = \ \sup \widetilde{\mathcal{P}}^* \ = I^*(\widetilde{\sigma_1^*}, \tau_2^*)\,, \tag{3.4}$$

$$\nu\varepsilon(\widetilde{u}) \ = \widetilde{\sigma_1^*}. \tag{3.5}$$

Proof. These assertions can be proved by straightforwardly. Really, the function $\widetilde{u}$ satisfies the condition

$$\int\limits_\Omega (\nu\varepsilon(\widetilde{u}) : \varepsilon(w) + \tau_2^* : \varepsilon(w)) \ dx = \int\limits_\Omega (f \cdot w + q\mathrm{div} w)dx \qquad \forall w \in V_0. \tag{3.6}$$

Hence,

$$\int\limits_\Omega f \cdot \widetilde{u} dx = \int\limits_\Omega (\nu \mid \varepsilon(\widetilde{u}) \mid^2 - \nu\varepsilon(\widetilde{u}) : \varepsilon(u_0) + \tau_2^* : \varepsilon(\widetilde{u} - u_0) + \\ + f \cdot u_0 + q\mathrm{div}(\widetilde{u} - u_0))dx,$$

and, therefore,

$$\inf \widetilde{\mathcal{P}} = \widetilde{J}(\widetilde{u}) = \\ = \int\limits_\Omega \left(\frac{\nu}{2} \mid \varepsilon(\widetilde{u}) \mid^2 + \tau_2^* : \varepsilon(\widetilde{u}) - \psi^*(\tau_2^*)\right) \ dx - \int\limits_\Omega q\mathrm{div}(\widetilde{u} - u_0)dx - \int\limits_\Omega f \cdot \widetilde{u}\, dx \\ = \int\limits_\Omega \left(\tau_2^* : \varepsilon(u_0) - \frac{\nu}{2} \mid \varepsilon(\widetilde{u}) \mid^2 - \psi^*(\tau_2^*) + \nu\varepsilon(u_0) : \varepsilon(\widetilde{u}))\right) dx - \int\limits_\Omega f \cdot u_0 dx\,. \tag{3.7}$$

In virtue of (3.6), $\nu\widetilde{u} + \tau_2^* \in T^*_{f - \nabla q}(\Omega, V_0)$, so that

$$I^*(\nu\widetilde{u}, \tau_2^*) \leq \sup \widetilde{\mathcal{P}}^* \tag{3.8}$$

and, thus,

$$I^*(\nu\varepsilon(\widetilde{u}), \tau_2^*) \\ = \int\limits_\Omega \left(\varepsilon(u_0) : (\nu\varepsilon(\widetilde{u}) + \tau_2^*) - \frac{\nu}{2} \mid \varepsilon(\widetilde{u}) \mid^2 - \psi^*(\tau_2^*) - f \cdot u_0\right) \ dx \\ \leq \sup \widetilde{\mathcal{P}}^* \leq \inf \widetilde{\mathcal{P}} \\ = \int\limits_\Omega \left(\tau_2^* : \varepsilon(u_0) - \frac{\nu}{2} \mid \varepsilon(\widetilde{u}) \mid^2 - \psi^*(\tau_2^*) + \nu\varepsilon(u_0) : \varepsilon(\widetilde{u}))\right) dx - \int\limits_\Omega f \cdot u_0\, dx\,.$$

This means that $\nu\varepsilon(\widetilde{u}) = \widetilde{\sigma_1^*}$ and $\inf \widetilde{\mathcal{P}} = \sup \widetilde{\mathcal{P}}^*$.

4. Majorants of deviations for solenoidal fields

First we consider the case $v \in \mathcal{H}^1 + u_0$, i.e. it is assumed that the exact solution u is compared with a function v subject to the condition $\operatorname{div} v = 0$.

From (3.2), (3.4), and (2.5) it follows that

$$\int_\Omega \frac{\nu}{2} \mid \varepsilon(v-u) \mid^2 \, dx \le J(v) - \inf \widetilde{\mathcal{P}} =$$

$$= J(v) - \sup \widetilde{\mathcal{P}}^* \le J(v) - I^*(\tau_1^*, \tau_2^*). \quad (4.1)$$

This relation is valid for any $v \in \mathcal{H}^1 + u_0$ and $(\tau_1^*, \tau_2^*) \in T^*_{f-\nabla q}(\Omega, V_0)$. The right-hand side of (4.1) is estimated as follows

$$J(v) - I^*(\tau_1^*, \tau_2^*) \le \int_\Omega \left(\frac{\nu}{2} \mid \varepsilon(v) \mid^2 + \frac{1}{2\nu} \mid \tau_1^* \mid^2 - \varepsilon(u_0) : \tau_1^* \right) dx +$$

$$+ \int_\Omega (\psi(\varepsilon(v)) + \psi^*(\tau_2^*) - \varepsilon(u_0) : \tau_2^*) \, dx + \int_\Omega f \cdot (u_0 - v) dx \, .$$

Since $(\tau_1^*, \tau_2^*) \in T^*_{f-\nabla q}(\Omega, V_0)$ and $v \in \mathcal{H}^1 + u_0$, we have

$$\int_\Omega f \cdot (u_0 - v) \, dx$$

$$= \int_\Omega (f \cdot (u_0 - v) + q \operatorname{div}(u_0 - v)) dx = \int_\Omega \varepsilon(u_0 - v) : (\tau_1^* + \tau_2^*) \, dx,$$

so that the estimate comes in the form

$$\frac{\nu}{2} \|\varepsilon(v-u)\|^2 \le \int_\Omega \left(\tfrac{\nu}{2} \mid \varepsilon(v) \mid^2 + \tfrac{1}{2\nu} \mid \tau_1^* \mid^2 - \varepsilon(v) : \tau_1^* \right) dx +$$

$$+ \int_\Omega (\psi(\varepsilon(v)) + \psi^*(\tau_2^*) - \varepsilon(v) : \tau_2^*) \, dx \, .$$

To make the nature of this estimate more transparent, we introduce two compound functionals

$$\mathrm{D}_1(\varepsilon(v),\tau_1^*) := \int\limits_\Omega \left(\tfrac{\nu}{2} \mid \varepsilon(v) \mid^2 + \tfrac{1}{2\nu} \mid \tau_1^* \mid^2 - \varepsilon(v) : \tau_1^*\right) dx = \tfrac{1}{2\nu}\|\nu\varepsilon(v) - \tau_1^*\|^2;$$

and

$$\mathrm{D}_2(\varepsilon(v),\tau_2^*) := \int\limits_\Omega (\psi(\varepsilon(v)) + \psi^*(\tau_2^*) - \varepsilon(v) : \tau_2^*)\, dx.$$

It is not difficult to verify that they are nonnegative and

$$\mathrm{D}_1(\varepsilon(v),\tau_1^*) = 0 \quad \text{if and only if} \quad \tau_1^* = \nu\varepsilon(v), \tag{4.2}$$

$$\mathrm{D}_2(\varepsilon(v),\tau_2^*) = 0 \quad \text{if and only if} \quad \tau_1^* \in \partial\psi(\varepsilon(v)), \tag{4.3}$$

so that the compound functionals vanishes if and only if (1.15) and (1.16) hold. Then, we have

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \le \mathcal{M}_1(v,\tau_1^*,\tau_2^*) := \mathrm{D}_1(\varepsilon(v),\tau_1^*) + \mathrm{D}_2(\varepsilon(v),\tau_2^*)\,. \tag{4.4}$$

For $v \in \mathcal{H}^1 + u_0$ and $(\tau_1^*,\tau_2^*) \in T^*_{f-\nabla q}(\Omega, V_0))$ relations (1.13), (1.14) and (1.17) are satisfied. Estimate (4.4) shows that for such fields the energy norm of the deviation is controlled by the quantities $\mathrm{D}_1(\varepsilon(v),\tau_1^*)$ and $\mathrm{D}_2(\varepsilon(v),\tau_2^*)$, which measure errors in the two remaining relations (1.15) and (1.16). Therefore, the functional $\mathcal{M}_1(v,\tau_1^*,\tau_2^*)$ attains its lower bound if and only if $v = u$, $\tau_1^* = \sigma_1^*$, and $\tau_2^* = \sigma_2^*$.

The requirement $(\tau_1^*,\tau_2^*) \in T^*_{f-\nabla q}(\Omega, V_0)$ can be avoided. Really, take an arbitrary $\eta_1^* \in \Sigma^*$. Then, the inclusion $(\eta_1^*,\tau_2^*) \in T^*_{f-\nabla q}(\Omega, V_0)$ may not hold. However,

$$\mathrm{D}_1(\varepsilon(v),\tau_1^*) = \mathrm{D}_1(\varepsilon(v),\eta_1^*) +$$
$$+ \int\limits_\Omega \left(\tfrac{1}{2\nu} \mid \tau_1^* - \eta_1^* \mid^2 - 2\sqrt{\mathrm{D}_1(\varepsilon(v),\eta_1^*)} : \sqrt{\frac{1}{2\nu}}(\tau_1^* - \eta_1^*)\right) dx \le$$
$$\le (1+\beta)\mathrm{D}_1(\varepsilon(v),\eta_1^*) + \tfrac{1}{2\nu}(1+\frac{1}{\beta})\|\tau_1^* - \eta_1^*\|^2,$$

where β is an arbitrary positive number. Then, we have

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \le (1+\beta)\mathrm{D}_1(\varepsilon(v),\eta_1^*) + \mathrm{D}_2(\varepsilon(v),\tau_2^*) + \left(1+\tfrac{1}{\beta}\right)\frac{1}{2\nu}\|\tau_1^* - \eta_1^*\|^2.$$

Since $\|\tau_1^* - \eta_1^*\| = \|\tau_1^* + \tau_2^* - \eta_1^* - \tau_2^*\|$ and $\tau_1^* + \tau_2^* \in T^*_{f-\nabla q}(\Omega, V_0)$, we rewrite this estimate as follows

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \leq (1+\beta)\mathrm{D}_1(\varepsilon(v), \eta_1^*) + \mathrm{D}_2(\varepsilon(v), \tau_2^*) + \\ + \left(1+\tfrac{1}{\beta}\right)\tfrac{1}{2\nu} \inf_{\eta^* \in \Sigma^*_{f-\nabla q}(\Omega, V_0)} \|\eta_1^* + \tau_2^* - \eta^*\|^2. \quad (4.5)$$

The value of the infimum in (4.5) can be estimated by the quantity

$$\mathrm{R}_f(\eta_1^* + \tau_2^*, q) = \|\mathrm{div}\,(\eta_1^* + \tau_2^*) + f - \nabla q\|_{(-1)}. \quad (4.6)$$

If $\eta_1^* + \tau_2^* \in \Sigma^*_{\mathrm{div}}(\Omega)$ and $q \in W^1_2(\Omega)$, then

$$\mathrm{R}_f(\eta_1^* + \tau_2^*, q) \leq C_\Omega \|\mathrm{div}\,(\eta_1^* + \tau_2^*) + f - \nabla q\|, \quad (4.7)$$

where C_Ω is the constant in the Friederichs-Poincare inequality

$$\|v\| \leq C_\Omega \|\nabla v\| \qquad \forall v \in V_0. \quad (4.8)$$

Thus, we obtain the required estimates in two forms:

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \leq (1+\beta)\mathrm{D}_1(\varepsilon(v), \eta_1^*) + \mathrm{D}_2(\varepsilon(v), \tau_2^*) \\ + \left(1+\tfrac{1}{\beta}\right)\tfrac{1}{2\nu}(\mathrm{R}_f(\eta_1^* + \tau_2^*, q))^2, \quad (4.9)$$

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \leq \tfrac{1}{2\nu}\left(\|\nu\varepsilon(v) - \eta_1^*\| + \mathrm{R}_f(\eta_1^* + \tau_2^*, q)\right)^2 + \mathrm{D}_2(\varepsilon(v), \tau_2^*), \quad (4.10)$$

which hold for any $v \in \mathcal{H}^1 + u_0$, $(\eta_1^*, \tau_2^*) \in \Sigma^* \times \Sigma^*$, and any $\beta > 0$. These estimates gives other forms of the deviation majorant, which involves vector-valued functions η_1^* and τ_2^* that may not satisfy the condition $\eta_1^* + \tau_2^* \in T^*_{f-\nabla q}(\Omega, V_0)$. As a result, the estimate has a new term $\mathrm{R}_f(\eta_1^* + \tau_2^*, q)$ that can be viewed as a measure the distance from $\eta_1^* + \tau_2^*$ to the set $T^*_{f-\nabla q}(\Omega, V_0)$.

Let the right-hand side of (4.9) (or (4.10)) be equal to zero. This means that the equality

$$-\mathrm{div}\,(\eta_1^* + \tau_2^*) = f - \nabla q \quad (4.11)$$

holds in the sense of distributions (if R_f is given by (4.6)) or in L_2 (if R_f is given by (4.7)) and, in addition,

$$\eta_1^* = \nu\varepsilon(v) \text{ a.e. in } \Omega, \quad (4.12)$$

$$\tau_2^* \in \partial\psi(\varepsilon(v)) \text{ a.e. in } \Omega. \quad (4.13)$$

Since $v \in \mathcal{H}^1 + u_0$, the relations (4.11)–(4.13) lead to the conclusion that v coincides with exact solution u, $\eta_1^* = \sigma_1^*$, and $\tau_2^* = \sigma_2^*$.

Denote the right-hand side of (4.9) by $\mathcal{M}_2(\beta, v, \eta_1^*, \tau_2^*, q)$. It is a functional defined on $\mathbb{R}_+ \times (V_0 + u_0) \times \Sigma^* \times \Sigma^* \times \overset{\circ}{L}_2$ with values in $\mathbb{R}_+$. Now, all said above is summarized as follows.

Theorem 4.1. (i). *For any $\beta \in \mathbb{R}_+$, $v \in \mathcal{H}^1 + u_0$ $\tau_1^* \in \Sigma^*$, $\tau_2^* \in \Sigma^*$, and $q \in \overset{\circ}{L}_2$ the functional $\mathcal{M}_2(\beta, v, \eta_1^*, \tau_2^*, q)$ majorizes the quantity $\|\varepsilon(v - u)\|^2$.*

(ii). *For any $\beta \in \mathbb{R}_+$, infimum of this functional on $(\mathcal{H}^1 + u_0) \times \Sigma^* \times \Sigma^* \times \overset{\circ}{L}_2$ is equal to zero and it is attained if and only if $v = u$, $\tau_1^* = \sigma_1^*$, $\tau_2^* = \sigma_2^*$, and $q = p$.*

We conclude this section with an example, where the estimates (4.9) or (4.10) give a complete answer to the error estimation problem. It concerns the problem of anti-plane slow flow of the Bingham fluid in a pipe is reduced to the variational problem (see, e.g. [5, 16])

$$\inf_{\widehat{v} \in \widehat{V}_0} \int\limits_{\Omega} \left(\frac{\nu}{2} \mid \nabla \widehat{v} \mid^2 + k_* \mid \nabla \widehat{v} \mid - f \widehat{v} \right) dx, \tag{4.14}$$

where $\Omega \in \mathbb{R}^2$ is a cross-section of the pipe, $\widehat{v}_0$ is the subspace of $W_2^1(\Omega)$ that contains functions with zero traces on $\partial\Omega$, ν and k_* are positive material constants, and f is the pressure difference (usually it is constant). In this case, the condition $\mathrm{div} v = 0$ is satisfied due to the plane structure of the motion, v has only one nonzero component $\widehat{v}$, so that $\varepsilon(v)$ is replaced by $\nabla \widehat{v}$ and $\psi(\xi) = k_* \mid \xi \mid$ for any $\xi \in \mathbb{R}^2$. Then η_1^* and τ_2^* are vector-valued functions and the respective functional ψ^* is given by the relation

$$\psi^*(\xi^*) = \begin{cases} 0, & \text{if } \mid \xi^* \mid \leq k_*, \\ +\infty, & \text{if } \mid \xi^* \mid > k_*. \end{cases}$$

Therefore, we obtain

$$\begin{aligned} \frac{\nu}{2} \|\nabla(\widehat{v} - \widehat{u})\|^2 \leq (1 + \beta) \mathrm{D}_1(\nabla(\widehat{v}), \eta_1^*) + \mathrm{D}_2(\nabla(\widehat{v}), \tau_2^*) \\ + \left(1 + \tfrac{1}{\beta}\right) \tfrac{1}{\nu} C_\Omega^2 \|\mathrm{div}\,(\eta_1^* + \tau_2^*) + f\|^2, \end{aligned} \tag{4.15}$$

where $\widehat{u}$ is the nonzero component of the exact solution and

$$\mathrm{D}_2(\nabla(\widehat{v}), \tau_2^*) = \begin{cases} \int\limits_{\Omega} (k_* \mid \nabla \widehat{v} \mid - \nabla \widehat{v} \cdot \tau_2^*), & \text{if } \mid \tau_2^*(x) \mid \leq k_* \text{ for a.e. } x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now, estimate (4.15) comes in the form

$$\frac{\nu}{2}\|\nabla(\widehat{v}-\widehat{u})\|^2 \leq \int\limits_{\Omega}\Big(\tfrac{(1+\beta)}{2\nu}(\nu\nabla\widetilde{v}-\eta_1^*)^2 + k_* \mid \nabla\widehat{v}\mid - \nabla\widehat{v}\cdot\tau_2^*\Big)dx +$$
$$+\left(1+\tfrac{1}{\beta}\right)\tfrac{1}{\nu}C_\Omega^2\|\operatorname{div}(\eta_1^*+\tau_2^*)+f\|^2. \qquad (4.16)$$

In (4.16), the functions η_1^* and τ_2^* must be taken such that

$$\eta_1^*, \tau_2^* \in L_2(\Omega,\mathbb{R}^2), \qquad \operatorname{div}(\eta_1^*+\tau_2^*) \in L_2(\Omega),$$
$$\mid \tau_2^*(x)\mid \leq k_* \text{ for a.e. } x\in\Omega.$$

5. Majorants of deviations for general–type velocity fields

The requirement $v \in \mathcal{H}^1 + u_0$ imposes rather obligatory conditions on the set of admissible velocity fields. However, very often approximate solutions does not exactly satisfy the incompressibility condition. In this section, we obtain majorants of the quantity $\|\varepsilon(\widetilde{v}-u)\|$ valid for any $\widetilde{v}\in V_0+u_0$.

Let v be an arbitrary function in $\mathcal{H}^1 + u_0$. Then

$$\|\varepsilon(\widetilde{v}-u)\|^2 \leq \frac{(1+\gamma)}{\gamma})\|\varepsilon(\widetilde{v}-v)\|^2 + (1+\gamma)\|\varepsilon(v-u)\|^2, \qquad (5.1)$$

where $\gamma > 0$. Estimate the second term in the right-hand side of (5.1) by the majorant obtained in the previous section. Then, we obtain

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \leq \frac{\nu(1+\gamma)}{2\gamma}\|\varepsilon(\widetilde{v}-v)\|^2$$
$$+(1+\gamma)\Big(\tfrac{1}{2\nu}\big(\|\nu\varepsilon(v)-\eta_1^*\| + \mathrm{R}_f(\eta_1^*+\tau_2^*,q)\big)^2 + \mathrm{D}_2(\varepsilon(v),\tau_2^*)\Big). (5.2)$$

Take a function η_2^* such that

$$\eta_2^*(x)\in\partial\psi(\varepsilon(v(x))) \qquad \text{for a.e. } x\in\Omega\,. \qquad (5.3)$$

Then

$$\psi^*(\eta_2^*) + \psi(\varepsilon(v)) - \eta_2^* : \varepsilon(v) = 0,$$
$$\psi^*(\eta_2^*) + \psi(\varepsilon(\widetilde{v})) - \eta_2^* : \varepsilon(\widetilde{v}) \geq 0$$

and we have

$$\psi(\varepsilon(v)) \leq \psi(\varepsilon(\widetilde{v})) + \eta_2^* : \varepsilon(v-\widetilde{v})\,. \qquad (5.4)$$

By (5.4), we deduce the estimate

$$\mathrm{D}_2(\varepsilon(v),\tau_2^*) \le \mathrm{D}_2(\varepsilon(\widetilde{v}),\tau_2^*) + \int\limits_\Omega (\eta_2^* - \tau_2^*) : \varepsilon(v-\widetilde{v})dx. \tag{5.5}$$

Take $\widetilde{\eta_2^*} \in \partial\psi(\varepsilon(\widetilde{v}))$. Then

$$\mathrm{D}_2(\varepsilon(v),\tau_2^*) \le \mathrm{D}_2(\varepsilon(\widetilde{v}),\tau_2^*) + \mathcal{R}(v,\widetilde{v}) + \int\limits_\Omega (\widetilde{\eta_2^*} - \tau_2^*) : \varepsilon(v-\widetilde{v})\,dx, \tag{5.6}$$

where

$$\mathcal{R}(v,\widetilde{v}) := \int\limits_\Omega (\eta_2^* - \widetilde{\eta_2^*}) : \varepsilon(v-\widetilde{v})\,dx. \tag{5.7}$$

Since ψ is a convex functional, the term $\mathcal{R}(v,\widetilde{v})$ is nonnegative. For any $\delta > 0$,

$$\int\limits_\Omega (\widetilde{\eta_2^*} - \tau_2^*) : \varepsilon(v-\widetilde{v})dx \le \frac{\delta}{2\nu}\|\widetilde{\eta_2^*} - \tau_2^*\|^2 + \frac{\nu}{2\delta}\|\varepsilon(v-\widetilde{v})\|^2. \tag{5.8}$$

Combining (5.2), (5.3), (5.6), and (5.8), we obtain

$$\begin{aligned}
\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \;&\le \frac{\nu(1+\gamma)}{2\gamma}\|\varepsilon(\widetilde{v}-v)\|^2 \\
+(1+\gamma)\Big(\tfrac{1}{2\nu}\big(\nu\|\varepsilon(v-\widetilde{v})\| &+ \|\nu\varepsilon(\widetilde{v}) - \eta_1^*\| + \mathrm{R}_f(\eta_1^*+\tau_2^*,q)\big)^2 + \\
+\mathrm{D}_2(\varepsilon(\widetilde{v}),\tau_2^*) + \mathcal{R}(v,\widetilde{v}) &+ \frac{\delta}{2\nu}\|\widetilde{\eta_2^*}-\tau_2^*\|^2 + \frac{\nu}{2\delta}\|\varepsilon(v-\widetilde{v})\|^2\Big).
\end{aligned} \tag{5.9}$$

Estimate (5.9) yields the following basic estimate

$$\begin{aligned}
\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)^2 \le c_1\mathrm{D}_1(\varepsilon(\widetilde{v}),\eta_1^*) \\
+ c_2\big(\mathrm{D}_2(\varepsilon(\widetilde{v}),\tau_2^*) + \tfrac{\delta}{2\nu}\|\widetilde{\eta_2^*}-\tau_2^*\|^2\big) \\
+ c_3(\mathrm{R}_f(\eta_1^*+\tau_2^*,q))^2 + c_4\mathrm{R}_{div}(\widetilde{v}),
\end{aligned} \tag{5.10}$$

where $c_1 = (1+\gamma)(1+\alpha_1+\frac{1}{\alpha_3})$, $c_2 = 1+\gamma$, $c_3 = (1+\alpha_2+\alpha_3)\,\frac{1+\gamma}{2\nu}$, and

$$\mathrm{R}_{div}(\widetilde{v}) = \inf_{v\in\mathcal{H}^1}\left\{\tfrac{\mu}{2}\|\varepsilon(\widetilde{v}-v)\|^2 + \mathcal{R}(v,\widetilde{v})\right\}.$$

Here α_1, α_2, α_3, β, γ, and δ are positive constants and $\mu = \nu\left(1+\frac{1}{\gamma}+\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\delta}\right)$.

Evidently, $\mathrm{R}_{div}(\widetilde{v})$ is a certain measure of the distance between $\widetilde{v}$ and the set $\mathcal{H}^1$.

It is easy to see that the right-hand side of (5.9) vanishes if and only if the velocity field $v \in V_0 + u_0$ and the stress fields η_1^* and τ_2^* satisfy the conditions

$$\begin{aligned} \operatorname{div}(\eta_1^* + \tau_2^*) + f - \nabla q = 0 \quad &\text{in} \ \ \Omega \\ \operatorname{div} \widetilde{v} = 0 \ &\text{in} \ \ \Omega \\ \eta_1^* = \nu(\varepsilon(\widetilde{v})) \ &\text{in} \ \ \Omega \\ \tau_2^* = \partial\psi(\varepsilon(\widetilde{v})) \ &\text{in} \ \ \Omega\,, \end{aligned}$$

which show that (cf. (1.13)–(1.17))

$$\widetilde{v} = u, \quad \eta_1^* = \sigma_1^*, \quad \tau_2^* = \sigma_2^*.$$

Two first terms in the right hand side of (5.9) are explicitly computable. The third term can be estimated by (4.7). It remains to obtain a computable estimate for $\mathrm{R}_{div}(\widetilde{v})$. In what follows we will derive such estimates for the Stokes, Bingham and power models.

Estimate (5.10) can be simplified if set $\tau_2^*(\widetilde{v}) \in \partial\psi(\varepsilon(\widetilde{v}))$. Then

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v} - u)\|^2 \leq c_1 \mathrm{D}_1(\varepsilon(\widetilde{v}), \eta_1^*) + c_3(\mathrm{R}_f(\eta_1^* + \tau_2^*(\widetilde{v}), q))^2 + c_4 \mathrm{R}_{div}(\widetilde{v}), \tag{5.11}$$

with $\mu = \nu\left(1 + \frac{1}{\gamma} + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right)$. Assume that ψ is differentiable. Then (5.11) implies the estimate

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v} - u)\|^2 \leq c_1 \mathrm{D}_1(\varepsilon(\widetilde{v}), \eta_1^*) + c_3(\mathrm{R}_f(\eta_1^* + \psi'(\varepsilon(\widetilde{v})), q))^2 + c_4 \mathrm{R}_{div}(\widetilde{v}). \tag{5.12}$$

Right-hand sides of (5.11) and (5.12) contain free functions η_1^* and q, so that one should minimize them over $\eta_1^* \in \Sigma^*(\Omega)$ and $q \in L_2(\Omega)$.

Denote the right-hand side of (5.10) by $\mathcal{M}_3(\Upsilon, v, \eta_1^*, \tau_2^*, q)$, where $\Upsilon \in \mathbb{R}^6$ is the vector $\{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma, \delta\}$. $\mathcal{M}_3$ is a functional defined on $(\mathbb{R}_+)^6 \times (V_0 + u_0) \times \Sigma^* \times \Sigma^* \times \overset{\circ}{L}_2$ and has values in $\mathbb{R}_+$. Now, all said above can be summarized as follows.

Theorem 5.1. (i). *For any* $\Upsilon \in (\mathbb{R}_+)^6$, $v \in V_0 + u_0$ $\tau_1^* \in \Sigma^*$, $\tau_2^* \in \Sigma^*$, *and* $q \in \overset{\circ}{L}_2$ *the functional* $\mathcal{M}_3(\Upsilon, v, \eta_1^*, \tau_2^*, q)$ *majorizes the quantity* $\|\varepsilon(v - u)\|^2$.

(ii). *For any* $\Upsilon \in (\mathbb{R}_+)^6$, *infimum of this functional on the set* $(V_0 + u_0) \times \Sigma^* \times \Sigma^* \times \overset{\circ}{L}_2$ *is equal to zero and it is attained if and only if* $v = u$, $\tau_1^* = \sigma_1^*$, $\tau_2^* = \sigma_2^*$, *and* $q = p$.

6. Particular cases

6.1. Stokes problem. Consider the stationary Stokes problem: find u, p and σ^* such that

$$-\operatorname{div}\sigma^* = f - \nabla p \qquad \text{in} \;\; \Omega \tag{6.1}$$

$$\operatorname{div} u = 0 \text{ in } \;\Omega \tag{6.2}$$

$$\sigma^* \;=\; \nu\varepsilon(u) \text{ in } \;\Omega \tag{6.3}$$

$$u = u_0 \text{ on } \;\partial\Omega, \tag{6.4}$$

which is a particular case of (1.13)–(1.17) where $\psi \equiv 0$, $\sigma_1^* = \sigma^*$, and $\sigma_2^* = 0$. If u possesses an additional regularity, then (6.1) and (6.3) give the classical Stokes equation $-\frac{\nu}{2}\Delta u = f - \nabla p$.

In the case $v \in \mathcal{H}^1 + u_0$, we may use estimates (4.4) and (4.5). Estimate (4.4) has the form

$$\frac{\nu}{2}|\varepsilon(v-u)||^2 \leq \mathrm{D}_1(\varepsilon(v), \tau^*) \qquad \forall \tau^* \in \Sigma_f^*(\Omega, \mathcal{H}^1),\; q \in L_2(\Omega). \tag{6.5}$$

This estimate measures the deviation via the quantity $D_1(\varepsilon(v), \tau_1^*)$, which shows the error in the relation (6.3). Since v satisfies (6.2) and (6.4), the right–hand side of (6.5) vanishes if and only if there exists $q \in L_2(\Omega)$ such that the set $\Sigma_f^*(\Omega, \mathcal{H}^1)$ contains an element $\tau^* = \nu\varepsilon(v)$. The latter means that $v = u$, $q = p$, and $\tau^* = \sigma^*$.

If τ^* is taken from a wider set, we use estimate (4.9)

$$\frac{\nu}{2}\|\varepsilon(v-u)\|^2 \leq (1+\beta)\mathrm{D}_1(\varepsilon(v), \eta^*) + \left(1 + \tfrac{1}{\beta}\right) \tfrac{1}{2\nu}(\mathrm{R}_f(\eta^*, q))^2, \tag{6.6}$$

in which $\eta^* \in \Sigma^*(\Omega)$, $q \in L_2(\Omega)$, and the relation (6.1) is accounted by the term $\mathrm{R}_f(\eta^*, q)$. It is not difficult to verify that the right-hand side of (6.6) vanishes if and only if $v = u$ and two other functions are taken such that $\tau^* = \sigma^*$ and $q = p$.

If the velocity field $\widetilde{v}$ does not belong to $\mathcal{H}^1 + u_0$, we can estimate the quantity $\|\widetilde{v} - u\|$ by applying (5.2) with $\tau_2^* = 0$. This gives

$$\begin{aligned}\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \leq & \frac{\nu(1+\gamma)}{2\gamma}\|\varepsilon(\widetilde{v}-v)\|^2 + \\ & + (1+\gamma)\tfrac{1}{2\nu}\left(\|\nu\varepsilon(v) - \eta^*\| + \mathrm{R}_f(\eta^*, q)\right)^2.\end{aligned} \tag{6.7}$$

and, therefore, (5.9) has comes in a simplified form

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \leq \frac{\nu(1+\gamma)}{2\gamma}\|\varepsilon(\widetilde{v}-v)\|^2 + \\ + (1+\gamma)\tfrac{1}{2\nu}\Big(\nu\|\varepsilon(v-\widetilde{v})\| + \|\nu\varepsilon(\widetilde{v})-\eta^*\| + \mathrm{R}_f(\eta^*,q)\Big)^2. \tag{6.8}$$

By taking infimum over γ, we obtain the estimate

$$\boldsymbol{\rho}_{div}(\widetilde{v}) \leq \|\varepsilon(\widetilde{v}-u)\| \leq 2\boldsymbol{\rho}_{div}(\widetilde{v}) + \frac{1}{\nu}\big(\|\nu\varepsilon(\widetilde{v})-\eta^*\| + \mathrm{R}_f(\eta^*,q)\big). \tag{6.9}$$

Here $\mathrm{R}_f(\eta^*,q) = \|\mathrm{div}\eta^* + f - \nabla q\|_{(-1)}$ and the term $\mathrm{R}_{div}(\widetilde{v})$ is replaced by

$$\boldsymbol{\rho}_{\mathrm{div}}(\widetilde{v}) = \inf_{v\in\mathcal{H}^1}\|\varepsilon(\widetilde{v}-v)\|.$$

The quantity $\boldsymbol{\rho}_{\mathrm{div}}(\widetilde{v})$ is the distance between $\widetilde{v}$ and the set of solenoidal functions subject to zero boundary conditions.

Estimate (6.9) has been earlier derived in the paper [23] devoted to the Stokes problem. There were also derived a posteriori error estimates for the pressure field p. This estimate is valid for any $\widetilde{v} \in V_0+u_0$, $\eta^* \in \Sigma^*(\Omega)$, $q \in L_2(\Omega)$. It is clear that the upper estimate remains valid if replace $\boldsymbol{\rho}_{\mathrm{div}}(\widetilde{v})$ by

$$\boldsymbol{\rho}^N_{\mathrm{div}}(\widetilde{v}) = \inf_{v\in\mathcal{H}^1_N}\|\varepsilon(\widetilde{v}-v)\|,$$

where $\mathcal{H}^1_N$ is any finite-dimensional subspace of $\mathcal{H}^1$. An alternative way is to use the estimate

$$\boldsymbol{\rho}_{\mathrm{div}}(w) \leq \frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}w\| \qquad \forall w \in V_0, \tag{6.10}$$

where $\mathbb{C}_\Omega$ is the constant in the Ladyzhenskaja–Babuška–Bressi condition (1.18).

If $\eta^* \in \Sigma^*_{\mathrm{div}}$, then it is also possible to present the last term as $C_\Omega\|\mathrm{div}\eta^* + f - \nabla q\|$. Hence, deviations from the solution of the Stokes problem satisfy the estimate

$$\|\varepsilon(\widetilde{v}-u)\| \leq 2\frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}\widetilde{v}\| + \frac{1}{\nu}\left(\|\nu\varepsilon(\widetilde{v})-\eta^*\| + C_\Omega\|\mathrm{div}\eta^* + f - \nabla q\|\right), \tag{6.11}$$

which has a slightly different form

$$\|\varepsilon(\widetilde{v}-u)\| \leq 2\frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}\widetilde{v}\| + \frac{1}{\nu}\left(\|\nu\varepsilon(\widetilde{v})-\eta^* - q\mathbb{I}\| + C_\Omega\|\mathrm{div}\eta^* + f\|\right). \tag{6.12}$$

6.2 Bingham model. Given a function $\widetilde{v}$, we define the sets

$$\Omega_0(\widetilde{v}) := \{x \in \Omega \| \, \varepsilon(\widetilde{v}) = 0\}, \quad \Omega_1(\widetilde{v}) = \Omega \setminus \Omega_0(\widetilde{v}).$$

The set Ω_0 is the set of "rigid" motion of the visco-plastic fluid. In the considered model, the potential ψ is given by (1.9), then

$$\widetilde{\eta}_2^*(\widetilde{v}) = \partial\psi(\varepsilon(\widetilde{v})) = \begin{cases} k_* \dfrac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid} & \text{in } \Omega_0, \\ \kappa \in \mathbb{M}_s^{n\times n}, \ \mid \kappa \mid \leq k_* & \text{in } \Omega_1. \end{cases} \tag{6.13}$$

In this case,

$$\mathcal{R}(v,\widetilde{v}) = \int\limits_\Omega (\eta_2^*(v) - \widetilde{\eta}_2^*(\widetilde{v})) : \varepsilon(v-\widetilde{v})dx \leq 2k_* \int\limits_\Omega \mid \varepsilon(v-\widetilde{v}) \mid \, dx$$

$$\leq 2k_* \mid \Omega \mid^{1/2} \|\varepsilon(v-\widetilde{v})\|, \tag{6.14}$$

so that

$$\mathrm{R}_{\,div}(\widetilde{v}) = \frac{\mu}{2}\boldsymbol{\rho}^2_{div}(\widetilde{v}) + 2k_* \mid \Omega \mid^{1/2} \boldsymbol{\rho}_{div}(\widetilde{v}) \tag{6.15}$$

and

$$\mathrm{D}_2(\varepsilon(\widetilde{v}),\tau_2^*) = \int\limits_\Omega (k_* \mid \varepsilon(\widetilde{v}) \mid - \varepsilon(\widetilde{v}) : \tau_2^*) \tag{6.16}$$

provided that $\mid \tau_2^* \mid \leq k_*$ a.e. in Ω. Also,

$$\|\widetilde{\eta}_2^*(\widetilde{v}) - \tau_2^*\|^2 = \int\limits_{\Omega_0(\widetilde{v})} (\kappa - \tau_2^*)^2 \, dx + \int\limits_{\Omega_1(\widetilde{v})} \big(k_* \frac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid} - \tau_2^*\big)^2 dx,$$

where κ can be an arbitrary function satisfying the condition $\mid \kappa \mid \leq k_*$. By taking $\kappa = \tau_2^*(x)$ for almost all $x \in \Omega_0$, we vanish the first term in the right-hand side.

Therefore, (5.10) implies the estimate

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \leq c_1 \frac{1}{2\nu}\|\nu\varepsilon(\widetilde{v}) - \eta_1^*)\|^2 +$$

$$+c_2 \int\limits_{\Omega_1(\widetilde{v})} \Big(\big(k_* \mid \varepsilon(\widetilde{v}) \mid - \varepsilon(\widetilde{v}) : \tau_2^*\big) + \big(k_* \frac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid} - \tau_2^*\big)^2 \Big) dx +$$

$$+c_3 C_\Omega^2 \|\mathrm{div}(\eta_1^* + \tau_2^*) + f - \nabla q\|^2$$

$$+c_4 \left(\frac{\mu}{2\mathbb{C}_\Omega^2}\|\mathrm{div}\widetilde{v}\|^2 + 2k_* \mid \Omega \mid^{1/2} \frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}\widetilde{v}\| \right), \tag{6.17}$$

where $\eta_1^* + \tau_2^* \in \Sigma^*_{\rm div}(\Omega)$, $q \in W_2^1(\Omega)$ and the constants c_1, c_2, c_3, and c_4 are defined as in (5.10).

6.3 Power model. If ψ is given by (1.10), then

$$\widetilde{\eta_2^*}(\widetilde{v}) = \partial\psi(\varepsilon(\widetilde{v})) = k_*\alpha\frac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid^{\theta}}, \tag{6.18}$$

where $\theta = 2 - \alpha$. To obtain the required estimate we use the following matrix inequality

$$\left(\frac{\varepsilon_1}{\mid \varepsilon_1 \mid^{\theta}} - \frac{\varepsilon_2}{\mid \varepsilon_2 \mid^{\theta}}\right) : (\varepsilon_1 - \varepsilon_2) \leq 2^{\theta}(\theta + 1) \mid \varepsilon_1 - \varepsilon_2 \mid^{2-\theta}, \tag{6.19}$$

where $\varepsilon_1, \varepsilon_2 \in \mathbb{M}_s^{n\times n}$ and $\theta \in [0,1]$. By (5.8) we obtain

$$\begin{aligned}\int\limits_{\Omega}(\eta_2^*(v) - \widetilde{\eta_2^*}(\widetilde{v})) : \varepsilon(v - \widetilde{v})dx &\leq 2^{\theta}(\theta+1)(2-\theta)k_*\int\limits_{\Omega} \mid \varepsilon(v-\widetilde{v}) \mid^{2-\theta}\, dx \\ &\leq 2^{\theta}(\theta+1)(2-\theta)k_* \mid \Omega \mid^{\theta/2} \|\varepsilon(v-\widetilde{v})\|^{2-\theta},\end{aligned} \tag{6.20}$$

so that

$$\mathrm{R}_{div}(\widetilde{v}) \leq \tfrac{\mu}{2}\boldsymbol{\rho}_{div}(\widetilde{v})^2 + C_{\theta}\boldsymbol{\rho}_{div}^{\theta-2}(\widetilde{v}),$$

where

$$C_{\theta} = 2^{\theta}(\theta+1)(2-\theta)k_* \mid \Omega \mid^{\theta/2}.$$

Let $\alpha^* = (2-\theta)/(1-\theta)$. Then

$$\mathrm{D}_2(\varepsilon(\widetilde{v}), \tau_2^*) = \int\limits_{\Omega}(k_* \mid \varepsilon(\widetilde{v}) \mid^{\alpha} + (\alpha k^*)^{1-\alpha_*}\frac{1}{\alpha_*} \mid \tau_2^* \mid^{\alpha_*} - \varepsilon(\widetilde{v}) : \tau_2^*))\, dx$$

and

$$\|\widetilde{\eta_2^*}(\widetilde{v}) - \tau_2^*\|^2 = \int\limits_{\Omega}\left((2-\theta)k_*\frac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid^{\theta}} - \tau_2^*\right)^2 dx.$$

Now, the above relations lead to the estimate

$$\frac{\nu}{2}\|\varepsilon(\widetilde{v}-u)\|^2 \le c_1\frac{1}{2\nu}\|\nu\varepsilon(\widetilde{v})-\eta_1^*)\|^2$$

$$+c_2\int\limits_\Omega \Big(k_* \mid \varepsilon(\widetilde{v}) \mid^\alpha +(\alpha k^*)^{1-\alpha_*}\frac{1}{\alpha_*} \mid \tau_2^* \mid^{\alpha_*} -\varepsilon(\widetilde{v}) : \tau_2^*)$$

$$+\big((2-\theta)k_*\frac{\varepsilon(\widetilde{v})}{\mid \varepsilon(\widetilde{v}) \mid^\theta} - \tau_2^*\big)^2\Big)dx$$

$$+c_3 C_\Omega^2\|\mathrm{div}(\eta_1^*+\tau_2^*)+f-\nabla q\|^2$$

$$+c_4\left(\frac{\mu}{2\mathbb{C}_\Omega^2}\|\mathrm{div}\widetilde{v}\|^2 + C_\theta(\frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}\widetilde{v}\|)^{2-\theta}\right), \tag{6.21}$$

where $\eta_1^*+\tau_2^* \in \Sigma^*_{\mathrm{div}}(\Omega)$, $q \in W_2^1(\Omega)$ and the constants c_1, c_2, c_3, and c_4 are defined as in (5.10).

7. On the estimates of $\boldsymbol{\rho}(\widetilde{v})$

It is not difficult to see that

$$\tfrac{1}{2}\boldsymbol{\rho}^2_{\mathrm{div}}(\widetilde{v}) \;=\; \frac{1}{2}\inf_{v\in\mathcal{H}^1}\|\varepsilon(\widetilde{v}-v)\|^2 \;=$$

$$=\inf_{v\in V_0}\sup_{\phi\in \overset{\circ}{L}_2}\int\limits_\Omega\left(\frac{1}{2}\mid\varepsilon(\widetilde{v}-v)\mid^2 -\phi\,\mathrm{div}\,v\right)dx. \tag{7.1}$$

Let $v_o = v_o(\widetilde{v})$ be an element such that

$$\tfrac{1}{2}\boldsymbol{\rho}^2_{\mathrm{div}}(\widetilde{v}) \;=\; \frac{1}{2}\|\varepsilon(\widetilde{v}-v_o)\|^2\,.$$

The function v_o is a projection of $\widetilde{v}$ on a closed subspace $\mathcal{H}^1$, which satisfies the condition

$$\int\limits_\Omega \varepsilon(\widetilde{v}-v_o):\varepsilon(w)dx = 0, \quad \forall w\in\mathcal{H}^1\,. \tag{7.2}$$

The left-hand side of (7.2) is a linear functional $l_o : V_0 \to \mathbb{R}$, which vanishes on functions from $\mathcal{H}^1$. Then (see, e.g. [11], Sec. 2.2), we conclude that $l_o(w) =< g, w >$, where $g \in H^{-1}$ and $g = \nabla\phi$ for some $\phi \in L_2(\Omega)$. Hence, there is a function $\phi_o \in \overset{\circ}{L}_2$ such that

$$\int\limits_\Omega \varepsilon(\widetilde{v}-v_o):\varepsilon(w)dx + \int\limits_\Omega \phi_o \mathrm{div}\,w\,dx = 0, \quad \forall w\in V_0\,. \tag{7.3}$$

Then, we have

$$\inf_{v\in V_0}\sup_{\phi\in \overset{\circ}{L}_2}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(\widetilde{v}-v)\mid^2 +\phi\,\mathrm{div}\, v\right)dx \geq$$

$$\geq \sup_{\phi\in \overset{\circ}{L}_2}\inf_{v\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(\widetilde{v}-v)\mid^2 +\phi\,\mathrm{div}\, v\right)dx \geq$$

$$\geq \inf_{v\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(\widetilde{v}-v)\mid^2 +\phi_o\,\mathrm{div}\, v\right)dx. \tag{7.4}$$

Consider the problem in the right-hand side of (7.4). It has a minimizer, which is v_0 (it satisfies the respective Euler equation (7.3)). This means that

$$\inf_{v\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(\widetilde{v}-v)\mid^2 +\phi_o\,\mathrm{div}\, v\right)dx \geq \int\limits_\Omega \frac{1}{2}\mid \varepsilon(\widetilde{v}-v_o)\mid^2 dx = \tfrac{1}{2}\boldsymbol{\rho}^2_{\mathrm{div}}(\widetilde{v})$$

and, therefore,

$$\tfrac{1}{2}\boldsymbol{\rho}^2_{div}(\widetilde{v}) = \sup_{\phi\in \overset{\circ}{L}_2}\inf_{v\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(\widetilde{v}-v)\mid^2 -\phi\,\mathrm{div}\, v\right)dx =$$

$$= \sup_{\phi\in \overset{\circ}{L}_2}\left[\inf_{w\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(w)\mid^2 -\phi\,\mathrm{div}\,(w+\widetilde{v})\right)dx\right]. \tag{7.5}$$

First, we obtain an upper bound of the term in the square brackets. Let w_0 be a nonzero element of the space V_0. Since $tw_0 \in V_0$ for any $t\in\mathbb{R}$, we have

$$\inf_{w\in V_0}\int\limits_\Omega \left(\frac{1}{2}\mid \varepsilon(w)\mid^2 -\phi\,\mathrm{div}\,(w+\widetilde{v})\right)dx \leq$$

$$\leq \frac{1}{2}t^2\,\|\varepsilon(w_0)\|^2 - t\int\limits_\Omega \phi\,\mathrm{div}\,w_0\,dx - \int\limits_\Omega \phi\,\mathrm{div}\,\widetilde{v}\,dx. \tag{7.6}$$

Setting $t=\left(\int\limits_\Omega \phi\,\mathrm{div}\,w_0\,dx\right)\|\varepsilon(w_0)\|^{-2}$, we minimize the right-hand side of (7.6) and obtain

$$\inf_{w\in V_0}\int\limits_\Omega\left(\frac{1}{2}\mid\varepsilon(w)\mid^2-\phi\,\mathrm{div}\,(w+\widetilde{v})\right)\,dx\;\leq$$

$$\leq-\frac{\left(\int\limits_\Omega\phi\,\mathrm{div}\,w_0\,dx\right)^2}{2\|\varepsilon(w_0)\|^2}-\int\limits_\Omega\mathrm{div}\phi\widetilde{v}\,dx. \tag{7.7}$$

Now we refer to (1.18). Since the norms $\|\nabla w\|$ and $\|\varepsilon(w)\|$ are equivalent on V_0, we use it in the form

$$\sup_{\substack{w\in V_0\\ w\neq 0}}\frac{\int\limits_\Omega\phi\,\mathrm{div}\,w\,dx}{\|\varepsilon(w)\|}\;\geq\;\mathbb{C}_\Omega\|\phi\|\quad\text{ for all }\phi\in\overset{\circ}{L}_2. \tag{7.8}$$

Pass to the infimum (with respect to w_o) in (7.7) and use (7.8). We have

$$\inf_{w\in V_0}\int\limits_\Omega\left(\frac{1}{2}\mid\varepsilon(w)\mid^2-\phi\,\mathrm{div}\,(w+\widetilde{v})\right)\,dx\leq$$

$$\leq-\frac{\mathbb{C}_\Omega^2}{2}\|\phi\|^2-\int\limits_\Omega\phi\,\mathrm{div}\,\widetilde{v}\,dx\leq$$

$$\leq\;-\frac{\mathbb{C}_\Omega^2}{2}\|\phi\|^2+\|\phi\|\|\mathrm{div}\,\widetilde{v}\|\leq\frac{\|\mathrm{div}\,\widetilde{v}\|^2}{2\mathbb{C}_\Omega^2}.$$

Thus, we find that

$$\boldsymbol{\rho}_{\mathrm{div}}(\widetilde{v})\;\leq\;\frac{1}{\mathbb{C}_\Omega}\|\mathrm{div}\,\widetilde{v}\| \tag{7.9}$$

Note that the upper bound of $\mathbb{C}_\Omega$ can be expressed via the constant C_Ω and the constant $\overset{\circ}{C}_\Omega$ in the Poincaré inequality for functions with zero average. Really,

$$\mathbb{C}_\Omega=\inf_{q\in\overset{\circ}{L}_2}\mathcal{F}(q),\qquad\mathcal{F}(q)=\sup_{w\in V_0}\frac{\int_\Omega q\,\mathrm{div}w\,dx}{\|q\|\,\|\nabla w\|}$$

For any $q \in \overset{\circ}{\mathfrak{L}}(\Omega) := \overset{\circ}{L}_2(\Omega) \cap W_1^2(\Omega)$,

$$\mathcal{F}(q) = \sup_{w \in V_0} \frac{\int_\Omega \nabla q \cdot w \, dx}{\|q\| \, \|\nabla w\|} \leq \frac{\|\nabla q\|}{\|q\|} \sup_{w \in V_0} \frac{\|w\|}{\|\nabla w\|} \leq C_\Omega \frac{\|\nabla q\|}{\|q\|}$$

Let $\overset{\circ}{C}_\Omega$ be the lowest constant in the inequality

$$\|q\| \leq \overset{\circ}{C}_\Omega \|\nabla q\| \qquad \forall q \in \overset{\circ}{\mathfrak{L}}(\Omega),$$

what means that

$$\inf_{q \in \overset{\circ}{\mathfrak{L}}(\Omega)} \frac{\|\nabla q\|}{\|q\|} = \frac{1}{\overset{\circ}{C}_\Omega}.$$

Then, we find that

$$\mathbb{C}_\Omega = \inf_{q \in \overset{\circ}{L}_2} \mathcal{F}(q) \leq \inf_{q \in \overset{\circ}{\mathfrak{L}}_2(\Omega)} \mathcal{F}(q) \leq \frac{C_\Omega}{\overset{\circ}{C}_\Omega}.$$

From the practical viewpoint this estimate can give a certain presentation of the value of $\mathbb{C}_\Omega$ for a given domain Ω. To obtain an upper bound for $\boldsymbol{\rho}_{\rm div}$ we need a lower estimate of $\mathbb{C}_\Omega$, what imposes a much more difficult task. For some special cases (e.g. rectangular domains) lower bounds has been derived in [18]. However, to the best of our knowledge, finding computable lower bounds of $\mathbb{C}_\Omega$ for general type domains still remains an unsolved problem.

8. Conclusion

The majorants $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are new variational functionals defined on functions that approximate u, σ^*, and p. They are weighted sums of quantities that give certain measures of errors in relations (1.13)–(1.16). If approximations of u are taken from the set of divergence free functions, then the wieghts contain only one global constant, which is the constant C_Ω in the Friederichs–Poincare inequality. If a more wide set of approximations is considered, then there arises an additional term containing the constant $\mathbb{C}_\Omega$ in the Ladyzhenskaya–Babuska–Brezzi condition (1.18). Exact lower bound of each majorant $\mathcal{M}_i$ (taken over the set of its definition) is zero and it is attained if and only if the above variables coincide with the respective parts of the exact solution.

Note that similar results take place in a number of other boundary-value problems. For example, for problems associated with the biharmonic operator, the majorants in addition to C_Ω contain constants in the inequalities $||w|| \leq C'_\Omega||\nabla\nabla w||$ and $||\nabla w|| \leq C''_\Omega||\nabla\nabla w||$ that holds for functions vanishing on $\partial\Omega$ together with first derivatives (see [17]). If it is required to have a majorant that accounts possible violation of the Dirichlet boundary condition, then there arises the "boundary" term that is estimated by the $H^{1/2}$ norm of the "boundary residual" with a constant in the embedding inequality for traces (see [24]). ummarizing this experience we may formulate the following *conjecture:*

Majorants of the energy norms of $||v - u||$, where v is an arbitrary function from the functional class containing a generalized solution, consist of terms that can be thought of as penalties for the unconformity in each of the basic relation with multipliers defined by the constants arising in the embedding inequalities for spaces arising in the mathematical formulation of the problem considered.

References

1. G. Astarita and G. Marrucci, *Principles of non Newtonian fluid mechanics*, McGraw Hill, London (1974).
2. I. Babuška, "The finite element method with Lagrangian multipliers," *Numer. Math.*, **20**, 179–192 (1973).
3. R. E. Bank and B. D. Welfert, "A posteriori error estimates for the Stokes problem," *SIAM J. Numer. Anal.*, **28**, 591–623 (1991).
4. F. Brezzi, "On the existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers," *R.A.I.R.O., Annal. Numer.* **R2**, 129–151 (1974).
5. G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer, Berlin (1976).
6. I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland, New York (1976).
7. M. Fuchs and G.A. Seregin, "Variational methods for problems from plasticity theory and for generalized Newtonian fluids," *Lect. Notes Math.* **1749**, Springer, Berlin (2000).
8. M. Fuchs and G. A. Seregin, Regularity results for the quasi static "Bingham variational inequality in dimension two and three," *Math. Z.*, 227 (1998), 525–541.
9. M. Fuchs and G.A. Seregin, "A regularity theory for variational integrals with *LlnL* growth," *Calc. of Var.*, 6 (1998) 2, 171–187.
10. M. Fuchs and G.A. Seregin, "Variational methods for fluids of Prandtl–Eyring type and plastic materials with logarithmic hardening," *Math. Meth. Appl. Sciences* 22 (1999), 317–351.
11. V. Girault and P. A. Raviart, *Finite element approximation of the Navier–Stokes equations*, Springer, Berlin (1986).
12. O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York (1969).
13. O. A. Ladyzhenskaya, "On nonlinear problems of continuum mechanics," Proc. Internat. Congr. Math. (Moscow 1966), Amer. Math.Soc. Transl. (2), 70 (1968).

14. O. A. Ladyzhenskaya, "On some modifications of the Navier–Stokes equations for large gradients of velocity," *Zap. Nauchn. Sem. LOMI*, 7 (1968), 126–154 (in Russian); English translation in J. Soviet Math., 10, N2 (1978).
15. J. Malek, J. Nečas, J. Rokuta and M. Ružička. "Weak and measure valued solutions to evolution partial differential equations," Applied Mathematic and Mathematical Computation vol 13., Chapman and Hall, 1996.
16. P.P. Mosolov and V.P. Mjasnikov, *Mechanics of rigid plastic media*, Nauka, Moscow (1981) (in Russian).
17. P. Neittaanmäki and S. I. Repin, "A posteriori error estimates for boundary-value problems related to the biharmonic operator," *East-West J.Numer. Math.*, 9(2001), 2, 157-178.
18. M.A. Olshanskii and E.V. Chizhonkov, "On the best constant in the *inf sup* condition for prolonged rectangular domains," *Matematicheskie Zametki* 67 (2000) 3, 387–396 (in Russian).
19. J. T. Oden, W. Wu and M. Ainthworth, "An a posteriori error estimate for finite element approximations of the Navier–Stokes equations," *Comput. Methods. Appl. Mech. Engrg.*, **111**, 185–202 (1994).
20. S. I. Repin, "A posteriori error estimation for nonlinear variational problems by duality theory," *Zap. Nauchn. Semin. POMI*, **243**, 201–214 (1997).
21. S. I. Repin, "A posteriori error estimation for variational problems with uniformly convex functionals," *Math. Comput.*, v.69(230), 2000, 481–500.
22. S. I. Repin, "A unified approach to a posteriori error estimation based on duality error majorants," *Mathematics and Computers in Simulation* 50(1999), 313–329.
23. S. I. Repin, "Aposteriori estimates for the Stokes problem," *Journal of Math. Sciences* 109 (2002), 5, 1950–1964.
24. S. I. Repin, S. Sauter and A. A. Smolianski, "A posteriori error estimates for the Dirichlet problem with account of the error in the Approximation of boundary conditions," Univ. of Zurich, Institute of Mathematics, Preprint 03-2002.
25. G.A. Seregin, "Continuity for the strain velocity tensor in two dimensional variational problem from the theory of the Bingham fluid," Preprint No. 402, SFB 256, Universitt Bonn (1999).
26. G.A. Seregin, "Some remarks on variational problems for functionals with *LlnL* growth," *Zap. Nauchn. Semin. POMI*, 213 (1994), 164–174.
27. R. Verfürth, "A posteriori error estimators for the Stokes equations," *Numer. Math.*, **55**, 309–326 (1989).

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Поступило 31 мая 2002 г.