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SOME ESTIMATES NEAR THE BOUNDARY FOR SOLUTIONS TO THE NON-STATIONARY LINEARIZED NAVIER–STOKES EQUATIONS

To the memory of A. I. Ivanov

ABSTRACT. Some new local estimates near the boundary for solutions to the non-stationary linearized Navier–Stokes equations are established.

Interior estimates for solutions to the non-stationary linearized Navier–Stokes equations are one of the main counterpart of the proof of interior partial regularity for suitable weak solutions to the Navier–Stokes equations (see [1, 3, 6]). Our goal is to establish similar estimates near the boundary. We hope to use them for the investigation of partial regularity near the boundary for suitable weak solutions to the three-dimensional Navier–Stokes equations. Other application of the obtained estimates is sufficient conditions for Hölder continuity near the boundary for a class of solutions to the three dimensional Navier–Stokes equations (see Proposition 3).

In this paper we are going to exploit the following notation:

$\mathbb{M}^{3\times 3}$ is the space of all real 3×3-matrices;

$u\cdot v=u_iv_i$, $|u|=\sqrt{u\cdot u}$, $u=(u_i)\in\mathbb{R}^3$, $v=(v_i)\in\mathbb{R}^3$ (the convention on summation over lower repeated Latin indices, running from 1 to 3, is adopted);

$A:B=A_{ij}B_{ij}$, $|A|=\sqrt{A:A}$, $A^*=(A_{ji})$, $A=(A_{ij})\in\mathbb{M}^{3\times 3}$, $B=(B_{ij})\in\mathbb{M}^{3\times 3}$;

$Aa=(A_{ij}a_j)\in\mathbb{R}^3$, $A=(A_{ij})\in\mathbb{M}^{3\times 3}$, $a=(a_i)\in\mathbb{R}^3$;

$B(x_0,R)=\{x\in\mathbb{R}^3\,\|\,|x-x_0|<R\}$, $B(\theta)=B(0,\theta)$, $B=B(1)$;

$B^+(\theta)=\{x\in B(\theta)\,\|\,x_3>0\}$, $B^+=B^+(1)$;

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$z = (x, t) \in \mathbb{R}^3 \times \mathbb{R}$ is a point in the space-time;

$Q(z_0, R) = B(x_0, R) \times]\, t_0 - R^2, t_0\, [$, $z_0 = (x_0, t_0)$, $Q(\theta) = Q(0, \theta)$, $Q = Q(1)$, $Q^+(\theta) = B^+(\theta) \times]\, -\theta^2, 0\, [$, $Q^+ = Q^+(1)$;

$\Gamma(\theta) = \{x \in B(\theta) \,\|\, x = (x_1, x_2, 0)\}$, $\Gamma = \Gamma(1)$;

$Q_{t_0,T} = \Omega \times]\, t_0, T\, [$, $\Omega \subset \mathbb{R}^3$.

We also use the Lebesgue and Sobolev spaces $L_m(\Omega; \mathbb{R}^l)$ and $W^1_m(\Omega; \mathbb{R}^l)$. The norm in $L_m(\Omega; \mathbb{R}^l)$ is denoted by

$$\|f\|_{m,\Omega} = \Big(\int\limits_{\Omega} |f|^m \, dx \Big)^{1/m}.$$

For functions defined in the space-time. we use the following spaces:

$L_{m,n}(Q_{t_0,T}; \mathbb{R}^l)$ with the norms

$$\|g\|_{m,n,Q_{t_0,T}} = \Big(\int\limits_{t_0}^{T} \|g(\,\cdot\,, t)\|^n_{m,\Omega} \, dt \Big)^{1/n}$$

and

$$\|g\|_{m,Q_{t_0,T}} = \|g\|_{m,m,Q_{t_0,T}};$$

$W^{1,0}_{m,n}(Q_{t_0,T}; \mathbb{R}^3)$ with the norm

$$\|u\|_{W^{1,0}_{m,n}(Q_{t_0,T};\mathbb{R}^3)} = \|u\|_{m,n,Q_{t_0,T}} + \|\nabla u\|_{m,n,Q_{t_0,T}};$$

$W^{2,1}_{m,n}(Q_{t_0,T}; \mathbb{R}^3)$ with the norm

$$\begin{aligned} \|u\|_{W^{2,1}_{m,n}(Q_{t_0,T};\mathbb{R}^3)} = \|u\|_{m,n,Q_{t_0,T}} + \|\nabla u\|_{m,n,Q_{t_0,T}} + \\ + \|\partial_t u\|_{m,n,Q_{t_0,T}} + \|\nabla^2 u\|_{m,n,Q_{t_0,T}}. \end{aligned}$$

Here, ∇ stands for the spatial gradient, i.e., $\nabla u = (u_{i,j})$, where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, ∂_t stands for the partial derivative in time, i.e., $\partial_t\, u = \frac{\partial u}{\partial t}$.

Our first result is formulated as follows.

Proposition 1. *Suppose that we are given numbers satisdying the conditions*

$$1 < m < +\infty, \quad 1 < n < +\infty, \quad 0 < \theta \leqslant 1. \tag{1}$$

Suppose that the functions

$$v \in W^{2,1}_{m,n}\big(Q^+(\theta);\mathbb{R}^3\big), \quad p \in W^{1,0}_{m,n}\big(Q^+(\theta)\big), \quad \text{and} \quad f \in L_{m,n}\big(Q^+(\theta);\mathbb{R}^3\big)$$

satisfy the equations

$$\left.\begin{aligned} &\partial_t v - \Delta v = f - \nabla p \\ &\operatorname{div} v = 0 \end{aligned}\right\} \quad \text{in} \quad Q^+(\theta), \tag{2}$$

and the boundary condition

$$v = 0 \quad \text{on} \quad \Gamma(\theta)\times]-\theta^2, 0[. \tag{3}$$

Then, for any $\theta_1 \in]\,0,\theta\,[$, the estimate

$$\begin{aligned} &\big\|\partial_t v\big\|_{m,n,Q^+(\theta_1)} + \big\|\nabla^2 v\big\|_{m,n,Q^+(\theta_1)} + \big\|\nabla p\big\|_{m,n,Q^+(\theta_1)} \leqslant \\ &\leqslant c_1(m,n)\frac{1}{\theta-\theta_1}\bigg\{\big\|f\big\|_{m,n,Q^+(\theta)} + \frac{1}{(\theta-\theta_1)^{l-1}}\Big[\,\big\|v\big\|_{m,n,Q^+(\theta)} + \\ &\qquad + \big\|\nabla v\big\|_{m,n,Q^+(\theta)} + \big\|p\big\|_{m,n,Q^+(\theta)}\Big]\bigg\} \end{aligned} \tag{4}$$

is valid. Here, $l = \max\{m', 4\}$, $m' = \frac{m}{m-1}$.

Proof. Let us take and fix a cut-off function φ with the properties

$$\begin{aligned} &\varphi \in C^\infty_0(\mathbb{R}^3), \quad 0 \leqslant \varphi \leqslant 1 \quad \text{in} \quad \mathbb{R}^3, \\ &\varphi \equiv 1 \quad \text{in} \quad B(\rho), \quad \varphi = 0 \quad \text{outside of} \quad B(r), \\ &\big|\nabla^s \varphi\big| \leqslant \frac{c_2}{(r-\rho)^s} \quad \text{in} \quad \mathbb{R}^3. \end{aligned}$$

Here, $\theta_1 \leqslant \rho < r \leqslant \theta$, $s = 1, 2$.

Clearly, by (2), the functions $w = \varphi v$ and $p_* = \varphi p$ satisfy the system of equations

$$\left.\begin{aligned} &\partial_t w - \Delta w + \nabla p_* = g \\ &\operatorname{div} w = v \cdot \nabla\varphi \end{aligned}\right\} \quad \text{in} \quad Q^+, \tag{5}$$

where

$$g = -2\nabla v \nabla\varphi - v\Delta\varphi + p\nabla\varphi + \varphi f.$$

Next, we choose and fix a domain Ω_0 with smooth boundary so that

$$\Gamma \subset \partial\Omega_0, \quad B^+ \subset \Omega_0.$$

Consider the following boundary value problem, depending on $t \in]-\theta^2, 0[$ as a parameter,

$$\left.\begin{aligned} &\Delta \widehat{w}(\,\cdot\,,t) - \nabla \widehat{q}(\,\cdot\,,t) = 0,\\ &\operatorname{div} \widehat{w}(\,\cdot\,,t) = v(\,\cdot\,,t)\cdot\nabla\varphi(\,\cdot\,)\end{aligned}\right\} \text{ in } \Omega_0,$$
$$\widehat{w}(\,\cdot\,,t) = 0 \text{ on } \partial\Omega_0. \tag{6}$$

By (5) and (6), functions $V = w - \widehat{w}$ and $P = p_* - \widehat{q}$ meet the relations

$$\left.\begin{aligned} &\partial_t V - \Delta V + \nabla P = G,\\ &\operatorname{div} V = 0\end{aligned}\right\} \text{ in } \Omega_0\times]-\theta^2, 0[,$$
$$V = 0 \text{ on } \partial\Omega_0\times]-\theta^2, 0[, \tag{7}$$

where $G = g - \partial_t \widehat{w}$.

We start with evaluation of G. First, we have

$$\begin{aligned}\big\|g(\,\cdot\,,t)\big\|_{m,\Omega_0} \leqslant c_3(m)\bigg[&\frac{1}{r-\rho}\,\big\|\nabla v(\,\cdot\,,t)\big\|_{m,B^+(\theta)} +\\ &+\frac{1}{(r-\rho)^2}\,\big\|v(\,\cdot\,,t)\big\|_{m,B^+(\theta)} +\\ &+\frac{1}{r-\rho}\,\big\|p(\,\cdot\,,t)\big\|_{m,B^+(\theta)} + \big\|f(\,\cdot\,,t)\big\|_{m,B^+(\theta)}\bigg].\end{aligned} \tag{8}$$

To estimate $\big\|\partial_t \widehat{w}(\,\cdot\,,t)\big\|_{m,\Omega_0}$, we observe that, by (6), the function $\partial_t \widehat{w}(\,\cdot\,,t)$ is a solution to the following boundary value problem

$$\left.\begin{aligned} &\Delta\partial_t \widehat{w}(\,\cdot\,,t) - \nabla\partial_t \widehat{q}(\,\cdot\,,t) = 0,\\ &\operatorname{div} \partial_t\widehat{w}(\,\cdot\,,t) = \nabla\varphi(\,\cdot\,)\cdot\partial_t v(\,\cdot\,,t)\end{aligned}\right\} \text{ in } \Omega_0,$$
$$\partial_t\widehat{w}(\,\cdot\,,t) = 0 \text{ on } \partial\Omega_0.$$

By (2), we have

$$\nabla\varphi\cdot\partial_t v = \nabla\varphi\cdot\big(\Delta v - \nabla p + f\big) = \operatorname{div} a + b,$$

where

$$a=-p\nabla\varphi+(\nabla v)^*\nabla\varphi,\quad b=-\nabla v:\nabla^2\varphi+p\Delta\varphi+\nabla\varphi\cdot f.$$

Then, using duality arguments (see [8, Theorem 2.4]), one can show that

$$\begin{gathered}\big\|\partial_t\widehat{w}\,(\,\cdot\,,t)\big\|_{m,\Omega_0}\leqslant c_4(m)\big(\big\|a(\,\cdot\,,t)\big\|_{m,\Omega_0}+\big\|a_3(\,\cdot\,,t)\big\|_{m,\Gamma(\theta)}+\\ +\big\|b(\,\cdot\,,t)\big\|_{m,\Omega_0}\big)\leqslant c_4'(m)\bigg[\frac{1}{(r-\rho)^2}\;\big\|\nabla v(\,\cdot\,,t)\big\|_{m,B^+(\theta)}+ \qquad (9)\\ +\frac{1}{(r-\rho)^2}\;\big\|p(\,\cdot\,,t)\big\|_{m,B^+(\theta)}+\frac{1}{(r-\rho)}\;\big\|f(\,\cdot\,,t)\big\|_{m,B^+(\theta)}+\big\|a_3(\,\cdot\,,t)\big\|_{m,\Gamma(\theta)}\bigg].\end{gathered}$$

By the definition of a,

$$\big|a_3\big|\leqslant\big|\nabla\varphi\big|\big(\big|\nabla v\big|+\big|p\big|\big).$$

On the other hand, for any $\delta\in]\,0,1\,[$, we have

$$\begin{gathered}\int\limits_{\Gamma(\theta)}\big|\nabla\varphi\big|^m\big(\big|\nabla v\big|^m+\big|p\big|^m\big)dx_1\,dx_2\leqslant\\ \leqslant c_5(m)\int\limits_{B^+(\theta)}\Big\{\big|\nabla\varphi\big|^m\big(\big|\nabla v\big|^{m-1}\big|\nabla v_{,3}\big|+\big|p\big|^{m-1}\big|p_{,3}\big|\big)+\\ +\big(\big|\nabla\varphi\big|^{m-1}\big|\nabla\varphi_{,3}\big|+\big|\nabla\varphi\big|^m\big)\big(\big|\nabla v\big|^m+\big|p\big|^m\big)\Big\}\leqslant\\ \leqslant c_5'(m)\Big\{\delta^{m/n}\int\limits_{B^+(r)}\big(\big|\nabla^2 v\big|^m+\big|\nabla p\big|^m\big)\,dx+ \qquad (10)\\ +\Big(\frac{1}{\delta}\Big)^{m'/n}\int\limits_{B^+(\theta)}\Big[\big|\nabla\varphi\big|^m+\big|\nabla\varphi\big|^{m-1}\big|\nabla^2\varphi\big|+\big|\nabla\varphi\big|^{mm'}\Big]\big(\big|\nabla v\big|^m+\big|p\big|^m\big)dx\Big\}.\end{gathered}$$

Combining estimates (8)–(10), we obtain

$$\begin{gathered}\big\|G(\,\cdot\,,t)\big\|_{m,\Omega_0}\leqslant c_6(m)\Big[\delta^{1/n}\Big(\big\|\nabla^2 v(\,\cdot\,,t)\big\|_{m,B^+(r)}+\\ +\big\|\nabla p(\,\cdot\,,t)\big\|_{m,B^+(r)}\Big)+\Big(\frac{1}{\delta}\Big)^{\frac{m'}{nm}}\Big(\frac{1}{r-\rho}\Big)^{l'}\Big(\big\|v(\,\cdot\,,t)\big\|_{m,B^+(\theta)}+ \qquad (11)\end{gathered}$$

$$+\|\nabla v(\,\cdot\,,t)\|_{m,B^+(\theta)}+\|p(\,\cdot\,,t)\|_{m,B^+(\theta)}\Big)+\frac{1}{r-\rho}\|f(\,\cdot\,,t)\|_{m,B^+(\theta)}\Big],$$

where $l'=\max\{m',2\}$ and $t\in]-\theta^2,0[$.

Let

$$\chi(t)=\begin{cases}0, & -2\leqslant t\leqslant -r^2,\\ \dfrac{t+r^2}{r^2-\rho^2}, & -r^2\leqslant t\leqslant -\rho^2,\\ 1, & -\rho^2\leqslant t\leqslant 0.\end{cases}$$

Then, from (7) it follows that

$$\left.\begin{aligned}&\partial_t V_0-\Delta V_0+\nabla P_0=G_0,\\ &\operatorname{div} V_0=0\end{aligned}\right\}\quad\text{in}\quad Q_0=\Omega_0\times]-2,0[,$$

$$V_0=0\quad\text{on}\quad \Big\{\partial\Omega_0\times[-2,0]\Big\}\cup\Big\{\Omega_0\times\{t=-2\}\Big\}. \tag{12}$$

Here, $V_0=\chi V$, $P_0=\chi P$, $G_0=\chi G+V\partial_t\chi$. For G_0, we have

$$\|G_0\|_{m,n,Q_0}\leqslant\|\chi G\|_{m,n,Q_0}+c_7(m,n)\frac{1}{(r-\rho)^2}\Bigg[\|v\|_{m,n,Q^+(\theta)}+$$

$$\Bigg(\int\limits_{-\theta^2}^{0}\|\widehat{w}(\,\cdot\,,t)\|^n_{m,\Omega_0}\,dt\Bigg)^{1/n}\Bigg]. \tag{13}$$

Since a unique solution $\widehat{w}(\,\cdot\,,t)$ to boundary value problem (6) satisfies the so-called L_m-estimate (see [7])

$$\|\widehat{w}(\,\cdot\,,t)\|_{m,\Omega_0}+\|\nabla\widehat{w}(\,\cdot\,,t)\|_{m,\Omega_0}+\|\nabla^2\widehat{w}(\,\cdot\,,t)\|_{m,\Omega_0}+\|\nabla\widehat{q}(\,\cdot\,,t)\|_{m,\Omega_0}\leqslant$$

$$\leqslant c_8(m)\big\|\nabla\big(\nabla\varphi(\,\cdot\,)\cdot v(\,\cdot\,,t)\big)\big\|_{m,\Omega_0}\leqslant \tag{14}$$

$$\leqslant c_8'(m)\Bigg[\frac{1}{(r-\rho)^2}\|v(\,\cdot\,,t)\|_{m,B^+(\theta)}+\frac{1}{r-\rho}\|\nabla v(\,\cdot\,,t)\|_{m,B^+(\theta)}\Bigg],$$

we observe that, by (11) and (13)

$$G_0\in L_{m,n}(Q_0;\mathbb{R}^3).$$

It is known that initial boundary-value problem (12) has a unique solution of the class $W^{2,1}_{m,n}(Q_0,\mathbb{R}^3)\times W^{1,0}_{m,n}(Q_0)$, and this solution satisfies the $L_{m,n}$-estimate (see [2, 5])

$$\|\partial_t V_0\|_{m,n,Q_0}+\|\nabla^2 V_0\|_{m,n,Q_0}+\|\nabla P_0\|_{m,n,Q_0}\leqslant c_9'(m,n)\|G_0\|_{m,n,Q_0}. \tag{15}$$

Here and in what follows we do not indicate dependness of our constants on Ω_0.

Let

$$\Phi(r)=\|\partial_t v\|^n_{m,n,Q^+(r)}+\|\nabla^2 v\|^n_{m,n,Q^+(r)}+\|\nabla p\|^n_{m,n,Q^+(r)}.$$

Now, one can derive from (15) the estimate

$$\|\chi\partial_t w\|^n_{m,n,Q_0}+\|\chi\nabla^2 w\|^n_{m,n,Q_0}+\|\chi\nabla p_*\|^n_{m,n,Q_0}\leqslant$$

$$\leqslant c_{10}(m,n)\Big[\|G\|^n_{m,n,Q_0}+\|w\partial_t\chi\|^n_{m,n,Q_0}+\|\widehat{w}\partial_t\chi\|^n_{m,n,Q_0}+$$

$$+\|\chi\nabla^2\widehat{w}\|^n_{m,n,Q_0}+\|\chi\partial_t\widehat{w}\|^n_{m,n,Q_0}+\|\chi\nabla\widehat{q}\|^n_{m,n,Q_0}\Big].$$

Taking into account the definition of w, we obtain from (9)–(11), (14), and the last estimate the following inequality

$$\Phi(\rho)\leqslant c_{11}(m,n)\Big\{\delta\Phi(r)+\Big(\frac{1}{\delta}\Big)^{\frac{1}{m-1}}\frac{1}{(r-\rho)^{ln}}\Big(\|v\|^n_{m,n,Q^+(\theta)}+$$

$$+\|\nabla v\|^n_{m,n,Q^+(\theta)}+\|p\|^n_{m,n,Q^+(\theta)}\Big)+\frac{1}{(r-\rho)^n}\|f\|^n_{m,n,Q^+(\theta)}\Big\}.$$

Setting $c_{11}(m,n)\delta=1/2$ and using the known iterative procedure, we arrive at the estimate

$$\Phi(\theta_1)\leqslant c_{12}(m,n)\frac{1}{|\theta_1-\theta|^n}\Big[\|f\|^n_{m,n,Q^+(\theta)}+\frac{1}{(\theta_1-\theta)^{(l-1)n}}\Big(\|v\|^n_{m,n,Q^+(\theta)}+$$

$$+\|\nabla v\|^n_{m,n,Q^+(\theta)}+\|p\|^n_{m,n,Q^+(\theta)}\Big)\Big].$$

The latter inequality implies (4). Proposition 1 is proved.

Proposition 2. *Suppose that we are given numbers*

$$1<m\leqslant 2,\quad 1<n\leqslant 2,\quad m<s<+\infty. \tag{15_1}$$

Assume that the functions

$$v\in W^{2,1}_{m,n}(Q^+;\mathbb{R}^3),\quad p\in W^{1,0}_{m,n}(Q^+),\quad f\in L_{s,n}(Q^+;\mathbb{R}^3) \tag{15_2}$$

satisfy the equations

$$\left.\begin{aligned}\partial_t v-\Delta v&=f-\nabla p\\ \operatorname{div} v&=0\end{aligned}\right\}\quad in\quad Q^+, \tag{16}$$

and the boundary condition

$$v=0 \quad on \quad \Gamma\times]-1,0[. \tag{17}$$

Then

$$v\in W^{2,1}_{s,n}\big(Q^+(1/2);\mathbb{R}^3\big),\quad p\in W^{1,0}_{s,n}\big(Q^+(1/2)\big), \tag{18}$$

and the estimate

$$\Big\|v\Big\|_{W^{2,1}_{s,n}(Q^+(1/2);\mathbb{R}^3)}+\Big\|p\Big\|_{W^{1,0}_{s,n}(Q^+(1/2))}\leqslant$$

$$\leqslant c_{14}(m,n,s)\Big\{\big\|p\big\|_{m,n,Q^+}+\big\|v\big\|_{W^{1,0}_{m,n}(Q^+;\mathbb{R}^3)}+\big\|f\big\|_{s,n,Q^+}\Big\} \tag{19}$$

is valid.

If, in addition,

$$v\in L_2(Q^+;\mathbb{R}^3)\cap L_{k',l'}\big(Q^+;\mathbb{R}^3\big),\ \ p\in L_{k,l}(Q^+),\quad f\in L_2(Q^+;\mathbb{R}^3), \tag{19_1}$$

where $k'=\frac{k}{k-1}$, $l'=\frac{l}{l-1}$, *and*

$$1<k<+\infty,\quad 1<l<+\infty, \tag{19_2}$$

then

$$\Big\|v\Big\|_{W^{2,1}_{s,n}(Q^+(1/2);\mathbb{R}^3)}+\Big\|p\Big\|_{W^{1,0}_{s,n}(Q^+(1/2))}\leqslant$$

$$\leqslant c'_{14}(m,n,s,k,l)\Big\{\big\|p\big\|_{m,n,Q^+}+\|p\|_{k,l,Q^+}+\|v\|_{2,Q^+}+ \tag{19_3}$$

$$\|v\|_{k',l',Q^+}+\big\|f\big\|_{s,n,Q^+}+\big\|f\big\|_{2,Q^+}\Big\}.$$

Proof. *Step 1.* Let

$$A=\big\|v\big\|_{W^{1,0}_{m,n}(Q^+(\frac34);\mathbb{R}^3)}+\big\|p\big\|_{m,n,Q^+(\frac34)}+\big\|f\big\|_{s,n,Q^+}.$$

Setting

$$m_1=\frac{2m}{3-m},\quad m_1^*=m'(m_1-1),$$

we observe that

$$m_1^*=\frac{3m}{3-m}. \tag{20}$$

We fix numbers

$$\frac12<\theta_1<\theta<\frac34.$$

By Proposition 1, for θ instead θ_1 and for $3/4$ instead of θ

$$\|v\|_{W^{2,1}_{m,n}(Q^+(\theta);\mathbb{R}^3)}+\|p\|_{W^{1,0}_{m,n}(Q^+(\theta))}\leqslant c_{15}(m,n,\theta)\,A. \tag{21}$$

Thus it follows from the embedding theorem that

$$\begin{aligned}\|v\|_{W^{1,0}_{m_1,n}(Q^+(\theta);\mathbb{R}^3)}+\|p\|_{m_1,n,Q^+(\theta)}\leqslant\\ \leqslant c_{16}(m,n,\theta)\Big[\|v\|_{W^{1,0}_{m_1^*,n}(Q^+(\theta);\mathbb{R}^3)}+\|p\|_{m_1^*,n,Q^+(\theta)}\Big]\leqslant\\ \leqslant c_{16}'(m,n,\theta)\,A.\end{aligned} \tag{22}$$

Now, let us repeat arguments of Proposition 1 for m_1 instead of m, and for $r=\theta$ and $\rho=\theta_1$. The only difference concerns the evaluation of $\partial_t\widehat{w}$. It is worthy to note that in order to estimate the left hand side of (10) we do not need the assumptions $v\in W^{2,1}_{m_1,n}(Q^+(\theta);\mathbb{R}^3)$ and $p\in W^{1,0}_{m,n}(Q^+(\theta))$ any more. Indeed, we have

$$\begin{aligned}&\int\limits_{\Gamma(\theta)}|\nabla\varphi|^{m_1}\left(|\nabla v|^{m_1}+|p|^{m_1}\right)dx_1\,dx_2\leqslant\\ &\leqslant c_{17}(m,\theta)\int\limits_{B^+(\theta)}\Big\{|\nabla\varphi|^{m_1}\Big(|\nabla v|^{m_1-1}|\nabla^2 v|+|p|^{m_1-1}|\nabla p|\Big)+\\ &+\Big(|\nabla\varphi|^{m_1-1}|\nabla^2 v|+|\nabla\varphi|^{m_1}\Big)\Big(|\nabla v|^{m_1}+|p|^{m_1}\Big)\Big\}\,dx\leqslant\\ &\leqslant c_{17}'(m,\theta)\bigg\{\frac{1}{|\theta-\theta_1|^{m_1}}\bigg(\int\limits_{B^+(\theta)}\Big(|\nabla v|^{(m_1-1)m'}+|p|^{(m_1-1)m'}\Big)dx\bigg)^{1/m'}\times\\ &\times\bigg(\int\limits_{B^+(\theta)}\Big(|\nabla^2 v|+|\nabla p|\Big)^m dx\bigg)^{1/m}+\bigg(\frac{1}{|\theta-\theta_1|^{m_1+1}}+\frac{1}{|\theta-\theta_1|^{m_1}}\bigg)\times\\ &\times\int\limits_{B^+(\theta)}\Big(|\nabla v|^{m_1}+|p|^{m_1}\Big)\,dx\bigg\}\end{aligned}$$

and, therefore,

$$\|a_3\|_{m_1,n,\Gamma(\theta)}\leqslant c_{18}(m,n,\theta)\bigg\{\frac{1}{\theta-\theta_1}\times$$

$$\times\Big(\int\limits_{\theta^2}^{0}\Big(\int\limits_{B^+(\theta)}\Big(\big|\nabla v\big|^{m_1^*}+\big|p\big|^{m_1^*}\Big)\,dx\Big)^{\frac{n}{m' m_1}}\times$$

$$\times\Big(\int\limits_{B^+(\theta)}\Big(\big|\nabla^2 v\big|+\big|\nabla p\big|\Big)^{m}\,dx\Big)^{\frac{n}{m m_1}}dt\Big)^{1/n}+$$

$$+\Big(\frac{1}{\theta-\theta_1}\Big)^{\frac{m_1+1}{m_1}}\Big(\big\|\nabla v\big\|_{m_1,n,Q^+(\theta)}+\big\|p\big\|_{m_1,n,Q^+(\theta)}\Big)\Big\}\leqslant$$

$$\leqslant c'_{18}(m,n,\theta)\frac{1}{(\theta-\theta_1)^2}\Big\{\Big[\int\limits_{-\theta^2}^{0}dt\Big(\big\|\nabla v(\,\cdot\,,t)\big\|_{m_1^*,B^+(\theta)}+$$

$$+\big\|p(\,\cdot\,,t)\big\|_{m_1^*,B^+(\theta)}\Big)^{\frac{m_1^* n}{m' m_1}}\Big(\big\|\nabla^2 v(\,\cdot\,,t)\big\|_{m,B^+(\theta)}+$$

$$+\big\|\nabla p(\,\cdot\,,t)\big\|_{m,B^+(\theta)}\Big)^{n/m_1}\Big]^{1/n}+\big\|\nabla v\big\|_{m_1,n,Q^+(\theta)}+\big\|p\big\|_{m_1,n,Q^+(\theta)}\Big\}\leqslant$$

$$\leqslant c''_{18}(m,n,\theta)\frac{1}{(\theta-\theta_1)^2}\Big\{\Big(\big\|\nabla^2 v\big\|_{m,n,Q^+(\theta)}+\big\|\nabla p\big\|_{m,n,Q^+(\theta)}\Big)^{1/m_1}\times$$

$$\times\Big(\big\|\nabla v\big\|_{m_1^*,n,Q^+(\theta)}+\big\|p\big\|_{m_1^*,n,Q^+(\theta)}\Big)^{\frac{m_1-1}{m_1}}+\big\|\nabla v\big\|_{m_1,n,Q^+(\theta)}+$$

$$+\big\|p\big\|_{m_1,n,Q^+(\theta)}\Big\}.$$

Thus, by (21) and (22), we obtain

$$\big\|a_3\big\|_{m_1,n,\Gamma(\theta)}\leqslant c_{19}(m,n,\theta,\theta_1)\,A.$$

The latter estimate together with arguments of Proposition 1 implies the inequality

$$\big\|v\big\|_{W^{2,1}_{m_1,n}(Q^+(\theta_1);\mathbb{R}^3)}+\big\|v\big\|_{W^{1,0}_{m_1,n}(Q^+(\theta_1))}\leqslant c_{20}(m,n,\theta,\theta_1)\,A.$$

If $m_1\geqslant s$, then inequality (19) is proved. So, assume now that $s>m_1$ and pass to the second part of the proof.

Step 2. We fix a number θ_2 so that

$$\frac{1}{2} < \theta_2 < \theta_1 < \theta < \frac{3}{4}.$$

We consider two cases. In the first of them

$$m_1 < 3.$$

Then, we let

$$m_2 = \frac{2m_1}{3 - m_1}, \quad m_2^* = (m_2 - 1)m_1',$$

and repeat arguments of Step 1, replacing θ by θ_1, θ_1 by θ_2, m_1 by m_2, and m_1^* by m_2^*. As a result, we obtain the estimate

$$\|v\|_{W^{2,1}_{m_2,n}(Q^+(\theta_2);\mathbb{R}^3)} + \|v\|_{W^{1,0}_{m_2,n}(Q^+(\theta_2))} \leqslant c_{21}(m,n,\theta,\theta_1,\theta_2)\,A. \qquad (23)$$

If $m_2 \geqslant s$, then (19) is proved.

In the second case

$$m_1 \geqslant 3.$$

Setting

$$m_2 = s, \quad m_2^* = (m_2 - 1)\,m_1',$$

we repeat the arguments of Step 1 with θ_1 instead of θ, θ_2 instead of θ_1, m_2 instead of m_1, and m_2^* instead of m_1^*, and arrive at (23) for $m_2 = s$, and (19) is proved.

If $m_1 < 3$ and $m_2 < s$, we proceed our proof within Step 2 until $m_k \geqslant 3$ or $m_{k+1} \geqslant s$. Clearly, that the number of iterations of Step 2 is finite and depends only on m and s. So, (19) is completely proved.

Step 3. Assume that conditions (19_1) and (19_2) hold. The result of Steps 1 and 2 is the following estimate

$$\|v\|_{W^{2,1}_{s,n}(Q^+(1/2);\mathbb{R}^3)} + \|p\|_{W^{1,0}_{s,n}(Q^+(1/2))} \leqslant c_{22}(m,n,s)\,A. \qquad (24)$$

Obviously, the same arguments, that were used in Steps 1 and 2, provide the estimate similar to (24)

$$\|v\|_{W^{2,1}_{s,n}(Q^+(8/9);\mathbb{R}^3)} + \|p\|_{W^{1,0}_{s,n}(Q^+(8/9))} \leqslant$$

$$\leqslant c_{23}(m,n,s)\Big(\|v\|_{W^{1,0}_{m,n}(Q^+;\mathbb{R}^3)} + \|p\|_{m,n,Q^+} + \|f\|_{s,n,Q^+}\Big). \qquad (25)$$

Let us take two cut-off functions:

$$\varphi \in C_0^\infty(\mathbb{R}^3), \quad 0 \leqslant \varphi \leqslant 1 \quad \text{in} \quad \mathbb{R}^3,$$

$$\varphi \equiv 1 \quad \text{in} \quad B\Big(\frac{3}{4}\Big), \quad \varphi \equiv 0 \quad \text{outside of} \quad B\Big(\frac{7}{8}\Big),$$

and

$$\chi(t) = \begin{cases} 0, & 0 \geqslant t \geqslant -\varepsilon^2, \\ -\dfrac{t+\varepsilon^2}{\varepsilon^2}, & -\varepsilon^2 \geqslant t \geqslant -2\varepsilon^2, \\ 1, & -2\varepsilon^2 \geqslant t \geqslant -\Big(\dfrac{3}{4}\Big)^2, \\ \dfrac{t+\left(\frac{7}{8}\right)^2}{\left(\frac{7}{8}\right)^2 - \left(\frac{3}{4}\right)^2}, & -\Big(\dfrac{3}{4}\Big)^2 \geqslant t \geqslant -\Big(\dfrac{7}{8}\Big)^2, \\ 0, & -\Big(\dfrac{7}{8}\Big)^2 \geqslant t. \end{cases}$$

$$\text{for} \quad \varepsilon \in]\,0, 1/2\,[.$$

Next, let $K_\rho(t)$, $t \in \mathbb{R}$, be the usual smoothing kernel. We define smoothing in t as follows.

$$(v)_\rho(z) = \int\limits_{-1}^{0} K_\rho(t-t')\, v\,(z')\, dt', \quad z = (x,t), \quad z' = (x', t').$$

Setting for $\rho < \min\Big\{\varepsilon^2, \frac{1}{2}\Big(1 - \Big(\frac{8}{9}\Big)^2\Big)\Big\}$.

$$w = \Big(\chi\varphi^2 (v)_\rho\Big)_\rho,$$

we observe that, by (16) and (25),

$$\int\limits_{Q^+(8/9)} \Big(\partial_t v - \Delta v + \nabla p - f\Big) \cdot w\, dz = 0.$$

Taking into account properties of smoothing kernel, one can derive from the last identity the following relation

$$\int\limits_{Q^+(8/9)} \Big(\partial_t(v)_\rho\cdot\chi\,\varphi^2(v)_\rho+\chi\,\varphi^2\big|\nabla(v)_\rho\big|^2\Big)\,dz=$$
$$=\frac12\int\limits_{B^+(8/9)}\big|(v)_\rho(\,\cdot\,,t)\big|^2\chi(t)\,\varphi^2(\,\cdot\,)dx\Big|_{t=-(8/9)^2}^{t=0}-$$
$$-\frac12\int\limits_{Q^+(8/9)}\big|(v)_\rho\big|^2\partial_t\chi dz+\int\limits_{Q^+(8/9)}\chi\,\varphi^2\big|\nabla(v)_\rho\big|^2\,dz= \tag{26}$$
$$=-\int\limits_{Q^+(8/9)}\chi\nabla(v)_\rho:\Big((v)_\rho\otimes\nabla\varphi^2\Big)\,dz+\int\limits_{Q^+(8/9)}(f)_\rho\cdot\chi\,\varphi^2(v)_\rho\,dz+$$
$$+\int\limits_{Q^+(8/9)}(p)_\rho\nabla\varphi^2\cdot(v)_\rho\,dz.$$

It is easy to show that (26) implies the estimate

$$\int\limits_{Q^+(3/4)}\big|\nabla v\big|^2\,dz\leqslant c_{24}\Big[\int\limits_{Q^+}|v|^2\,dz+\int\limits_{Q^+}\Big(|f|\,|v|+|p|\,|v|\Big)\,dz\Big]$$

and, therefore, we have the estimate

$$\big\|\nabla v\big\|_{m,n,Q^+(3/4)}\leqslant c_{25}(m,n)\big\|\nabla v\big\|_{2,Q^+(3/4)}\leqslant$$
$$\leqslant c_{25}'(m,n)\Big[\big\|v\big\|_{k',l',Q^+}+\big\|p\big\|_{k,l,Q^+}+\big\|v\big\|_{2,Q^+}+\big\|f\big\|_{2,Q^+}\Big].$$

The latter together with (24) leads to the final estimate (19_3). Proposition 2 is proved.

Lemma 1 (Hölder continuity near the boundary). *Suppose that conditions (15_1), (15_2), (16), and (17) hold. Let, in addition, the number s is so big that*

$$\lambda=2-\frac{2}{n}-\frac{3}{s}>0. \tag{27}$$

Then

$$\big|v(z)-v(z')\big|\leqslant c_{26}(m,n,s)\Big[\,|x-x'|+|t-t'|^{1/2}\Big]^\lambda A(v,p,f) \tag{28}$$

for all $z = (x,t) \in Q^+(1/4)$ *and for all* $z' = (x',t') \in Q^+(1/4)$. *Here,*

$$A(v,p,f) = \|v\|_{W^{1,0}_{m,n}(Q^+;\mathbb{R}^4)} + \|p\|_{m,n,Q^+} + \|f\|_{s,n,Q^+}.$$

Proof. We let

$$u(x_1,x_2,x_3,t) = \begin{cases} v(x_1,x_2,x_3,t), & x_3 \geqslant 0, \\ -v(x_1,x_2,-x_3,t), & x_3 < 0. \end{cases} \tag{29}$$

Then $u \in W^{2,1}_{s,n}(Q(1/2);\mathbb{R}^3)$ and, moreover,

$$\|u\|_{W^{2,1}_{s,n}(Q(1/2);\mathbb{R}^3)} \leqslant c_{27}(s,n)\|v\|_{W^{2,1}_{s,n}(Q(1/2);\mathbb{R}^3)}. \tag{30}$$

We are going to use the following version of the Poincarè–Sobolev inequality

$$\begin{aligned} &\frac{1}{R}\Big(\fint_{Q(z_0,R)} \big|u - (\nabla u)_{z_0,R}(x-x_0) - (u)_{z_0,R}\big|^n\,dz\Big)^{1/n} + \\ &\qquad +\Big(\fint_{Q(z_0,R)} \big|\nabla u - (\nabla u)_{z_0,R}\big|^n dz\Big)^{1/n} \leqslant \\ &\leqslant c_{28}(n)R\Big(\fint_{Q(z_0,R)} \Big(\big|\nabla^2 u\big| + \big|\partial_t u\big|\Big)^n\,dz\Big)^{1/n}, \quad Q(z_0,R) \subseteq Q(1/2). \end{aligned} \tag{31}$$

Here, we have used the abbreviations

$$(f)_{z_0,R} = \fint_{Q(z_0,R)} f\,dz = \frac{1}{|Q(z_0,R)|}\int_{Q(z_0,R)} f\,dz.$$

From condition (27) it follows that

$$s > 3.$$

Thus, by the embedding theorem,

$$\|\nabla u\|_{\infty,n,Q(1/2)} \leqslant c_{29}(n,s)\Big(\|\nabla u\|_{s,n,Q(1/2)} + \|\nabla^2 u\|_{s,n,Q(1/2)}\Big). \tag{32}$$

On the other hand, we obtain

$$R\Big(\fint_{Q(z_0,R)} \Big(\big|\partial_t u\big| + \big|\nabla^2 u\big|\Big)^n\,dz\Big)^{1/n} \leqslant$$

$$\leqslant c_{30}(n,s)\,R^{\lambda-1}\Big[\big\|\partial_t u\big\|_{s,n,Q^+(1/2)}+\big\|\nabla^2 u\big\|_{s,n,Q^+(1/2)}\Big], \tag{33}$$
$$Q(z_0,R)\subseteq Q(1/2).$$

Estimate (33) together with (19), (31), and (32) leads to the inequality

$$\Big(\,\fint\limits_{Q(z_0,R)}\big|u-(u)_{z_0,R}\big|\,dz\Big)^{1/n}\leqslant$$

$$\leqslant c_{31}(n,m,s)\,R^{\lambda}\,A(v,p,f),\quad Q(z_0,R)\subseteq Q(1/2). \tag{34}$$

We obtain (28) from (29), (30), and (34) in the known way. Lemma 1 is proved.

Now, we are going to consider a special example connected with investigations of partial regularity near the boundary for suitable weak solutions to the three-dimensional Navier–Stokes equations. We shall consider functions

$$\begin{aligned} &v\in L_3(Q^+;\mathbb{R}^3), && p\in L_{3/2}(Q^+),\\ &v\in W^{2,1}_{9/8,3/2}(Q^+(3/4);\mathbb{R}^3), && p\in W^{1,0}_{9/8,3/2}(Q^+(3/4)), \end{aligned} \tag{35}$$

that satisfy the equations

$$\left.\begin{aligned}&\partial_t v-\Delta v+\nabla p=0,\\ &\operatorname{div} v=0\end{aligned}\right\}\quad\text{in}\quad Q^+(3/4), \tag{36}$$
$$v=0\quad\text{on}\quad \Gamma(3/4)\times]-(3/4)^2,0[.$$

It is assumed also that

$$\big[p(\,\cdot\,,t)\big]_{0,1}=0,\quad t\in]-1,0[. \tag{37}$$

We are intersected in the behaviour of the functional

$$Y^+(v,p;\theta)=\Big(\,\fint\limits_{Q^+(\theta)}|v|^3\,dz\Big)^{1/3}+\theta\Big(\,\fint\limits_{Q^+(\theta)}\big|p-[p]_{0,\theta}\big|^{3/2}\,dz\Big)^{2/3},$$

where

$$\big[p(\,\cdot\,,t)\big]_{0,\theta}=\fint\limits_{B^+(\theta)}p(y,t)\,dy=\tfrac{1}{|B^+(\theta)|}\int\limits_{B^+(\theta)}p(y,t)\,dy.$$

Lemma 2. *Suppose that conditions* (34)–(37) *hold and let, in addition,*

$$\lambda \in]\,0, 2/3\,[.$$

Then

$$Y^+(v,p;\theta) \leqslant c_{32}(\lambda)\theta^\lambda\, Y^+(v,p;1) \tag{38}$$

for all $\theta \in]\,0,1]$.

Proof. We note that all conditions of Proposition 2 and Lemma 1 are fulfilled for

$$n = k = l = \frac{3}{2}, \quad m = \frac{9}{8}, \quad f = 0, \quad \text{and} \quad s = \frac{9}{2-3\lambda}$$

in the cylinder $Q^+(3/4)$. Then, we have

$$\big|v(x) - v(z')\big| \leqslant c_{33}(\lambda)\Big[\,|x-x'| + |t-t'|^{1/2}\Big]^\lambda A(v,p,0)$$

for all $z=(x,t) \in Q^+(1/4)$ and for all $z'=(x',t') \in Q^+(1/4)$, and

$$\big\|\nabla p\big\|_{s,3/2,Q^+(1/4)} \leqslant c_{34}(\lambda)\, A(v,p,0).$$

Using the last inequality in the proof of Proposition 2, we see that

$$A(v,p,o) \leqslant c'_{34} B(v,p).$$

Here,

$$B(v,p) = \|v\|_{3,Q^+} + \|p\|_{3/2,Q^+}. \tag{39}$$

From the last three estimates we derive the inequality

$$Y^+(v,p;\theta) \leqslant c_{35}(\lambda)\theta^\lambda\, B(v,p) \tag{40}$$

for all $\theta \in]\,0,1/4\,]$. The final estimate (38) easily follows from (37), (39), and (40). Lemma 2 is proved.

Our last example concerns sufficient conditions of Hölder continuity near the boundary for weak solutions to the three dimensional Navier–Stokes equations.

Proposition 3. *Suppose we are given numbers* n, q, m, *and functions*

$$\begin{aligned} &v \in W^{2,1}_{m,n}(Q^+;\mathbb{R}^3), \quad p \in W^{1,0}_{m,n}(Q^+), \\ &g \in L_{q,n}(Q^+;\mathbb{R}^3), \end{aligned} \tag{41}$$

such that

$$1 < m \leqslant \frac{3}{2}\,\frac{n}{2n-1}, \quad 1 < n < 2, \quad m \leqslant q. \tag{42}$$

Assume also that

$$(\nabla v)v \in L_{m,n}(Q^+;\mathbb{R}^3), \tag{43}$$

$$\left.\begin{array}{l}\partial_t v + (\nabla v)v - \Delta v = g - \nabla p,\\ \operatorname{div} v = 0\end{array}\right\} \quad in \quad Q^+, \tag{44}$$

$$v = 0 \quad on \quad \Gamma\times\,]-1,0[. \tag{45}$$

Let, in addition,

$$\lambda = 2 - \frac{2}{n} - \frac{3}{q} > 0, \tag{46}$$

and

$$v \in L_{q,\infty}(Q^+;\mathbb{R}^3). \tag{47}$$

Then, there is a constant c_{36}, depending only on n, q, $\|v\|_{W^{1,0}_{m,n}(Q^+,\mathbb{R}^3)}$, $\|p\|_{m,n,Q^+}$, $\|g\|_{q,n,Q^+}$, and $\|v\|_{q,\infty,Q^+}$, such that

$$|v(z) - v(z')| \leqslant c_{36}\Big(|x - x'| + |t - t'|^{1/2}\Big)^{\lambda} \tag{48}$$

for all $z = (x,t) \in Q^+(1/2)$ and for all $z' = (x',t') \in Q^+(1/2)$.

Remark 1. As it was shown in [3] (see Theorem 1.1) conditions (41)–(43) are quite natural for Hopf's solutions to initial boundary-value problems for the three-dimensional Navier–Stokes equations.

Proof of Propositions 3. *Step 1.* We let

$$m_0 = m < 3, \quad l_0 = \frac{3m_0}{3 - m_0}, \quad m_1 = \frac{l_0 q}{l_0 + q}, \tag{49}$$

and

$$f = g + g_1, \quad g_1 = -(\nabla v)\,v. \tag{50}$$

It is easy to verify that, by (46),

$$q > 3 \tag{51}$$

and, therefore,

$$m_0 < m_1. \tag{52}$$

By condition (43),

$$\|f\|_{m_0,n,Q^+} \leqslant c_{37}(n,q)\Big(\|g\|_{q,n,Q^+} + \|(\nabla v)v\|_{m_0,n,Q^+}\Big). \tag{53}$$

According to Proposition 1, we have the estimate

$$\|\nabla^2 v\|_{m_0,n,Q^+(7/8)} + \|\nabla p\|_{m_0,n,Q^+(7/8)} \leqslant$$
$$\leqslant c_{38}(n,q)\Big[\|v\|_{W^{1,0}_{m_0,n}(Q^+;\mathbb{R}^3)} + \|p\|_{m_0,n,Q^+} + \|f\|_{m_0,n,Q^+}\Big]. \tag{54}$$

On the other hand, by the embedding theorem,

$$\|\nabla v\|_{l_0,n,Q^+(7/8)} \leqslant c'_{38}(n)\Big(\|\nabla^2 v\|_{m_0,n,Q^+(7/8)} + \|\nabla v\|_{m_0,n,Q^+}\Big). \tag{55}$$

Then, with the help of the definition of the number m_1 (see (49)) and Hölder's inequality, we obtain

$$\|g_1\|_{m_1,n,Q^+(7/8)} \leqslant \|\nabla v\|_{l_0,n,Q^+(7/8)}\|v\|_{q,\infty,Q^+}. \tag{56}$$

Now, from (50), (53)–(56) it follows that

$$\|f\|_{m_1,n,Q^+(7/8)} \leqslant c_{39}. \tag{57}$$

In the latter inequality the constant c_{39} depends on the same arguments as the constant c_{36}.

Now, let us look at the proof of Proposition 2, replacing Q^+ by $Q^+(7/8)$, s by m_1, m by m_0. Clearly, that arguments of Step 1 and Step 2 in the proof of Proposition 3 are valid under conditions

$$v \in W^{2,1}_{m_0,n}(Q^+(7/8);\mathbb{R}^3), \quad p \in W^{1,0}_{m_0,n}(Q^+(7/8)),$$
$$f \in L_{m_1,n}(Q^+(7/8);\mathbb{R}^3).$$

But, by (41), (49), and (57), these conditions are fulfilled. As a result of Step 1 and Step 2 in the proof of Proposition 2, we obtain the fact that

$$v \in W^{2,1}_{m_1,n}(Q^+(13/16);\mathbb{R}^3), \quad p \in W^{1,0}_{m_1,n}(Q^+(13/16)),$$

and the estimate

$$\|\nabla^2 v\|_{m_1,n,Q^+(13/16)} + \|\nabla p\|_{m_1,n,Q^+(13/16)} \leqslant c_{40}$$

with the constant c_{40}, depending on the same arguments as the constant c_{36}.

Step 2. By (51), there is a natural number $i > 1$ with the following properties

$$m_0 < m_1 < m_2 < \ldots < m_i < 3, \quad m_{i+1} \geqslant 3,$$

$$m_{k+1} = \frac{l_k q}{l_k + q}, \quad l_k = \frac{3m_k}{3 - m_k}, \quad k = 0, 1, \ldots, i.$$

Repeating arguments of Step 1 2 $i+1$ times, we observe that

$$v \in W^{2,1}_{m_{i+1},n}(Q^+(3/4); \mathbb{R}^3), \quad p \in W^{1,0}_{m_{i+1},n}(Q^+(3/4)),$$

and the estimate

$$\|\nabla^2 v\|_{m_{i+1},n,Q^+(3/4)} + \|\nabla p\|_{m_{i+1},n,Q^+(3/4)} \leqslant c_{41}$$

is valid. Here, the constant c_{41} depends on the same arguments as the constant c_{36}.

Now, we consider two cases. In the first of them

$$m_{i+1} > 3.$$

Then, by the embedding theorem, we have

$$\|\nabla v\|_{\infty,n,Q^+(3/4)} \leqslant c_{42}(n,q)\Big(\|\nabla^2 v\|_{m_{i+1},n,Q^+(3/4)} + \|\nabla v\|_{m_{i+1},n,Q^+(3/4)}\Big)$$

and, therefore,

$$\|g_1\|_{q,n,Q^+(3/4)} \leqslant \|\nabla v\|_{\infty,n,Q^+(3/4)} \|v\|_{q,\infty,Q^+}.$$

The same arguments, that were used in Step 1, show us that

$$v \in W^{2,1}_{q,n}(Q^+(5/8); \mathbb{R}^3), \quad p \in W^{1,0}_{q,n}(Q^+(5/8)), \tag{58}$$

and the estimate

$$\|\nabla^2 v\|_{q,n,Q^+(5/8)} + \|\nabla p\|_{q,n,Q^+(5/8)} \leqslant c_{43} \tag{59}$$

is valid for the constant c_{43}, depending on the same arguments as the constant c_{36}.

Now, we assume that

$$m_{i+1} = 3.$$

Let us fix a number $r > \frac{3q}{q-3}$ and define $\overline{m} = \frac{rq}{r+q} > 3$. Then we observe that

$$\|g_1\|_{\overline{m},n,Q^+(3/4)} \leqslant \|\nabla v\|_{r,n,Q^+(3/4)} \|v\|_{q,\infty,Q^+(3/4)} \leqslant$$

$$\leqslant c_{44}(r,n,q)\Big[\big\|\nabla^2 v\big\|_{3,n,Q^+(3/4)} + \big\|\nabla v\big\|_{3,n,Q^+(3/4)}\Big]\,\big\|v\big\|_{q,\infty,Q^+}.$$

The latter estimate shows that the second case can be reduced to the first one but in a smaller cylinder, say, in $Q^+\left(\frac{11}{16}\right)\left(\frac{5}{8} < \frac{11}{16} < \frac{3}{4}\right)$.

Step 3. The result of Step 1 and Step 2 is statement (58) and estimate (59). Arguring as in the proof of Lemma 1, we derive estimate (48) from (59). Proposition 3 is proved.

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