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ESTIMATES OF DEVIATIONS FROM EXACT SOLUTIONS OF ELLIPTIC VARIATIONAL INEQUALITIES

To the memory of A. I. Ivanov

ABSTRACT. In this paper, we present a method of deriving majorants of the difference between exact solutions of elliptic type variational inequalities and any given functions lying in the admissible functional class of the problem under consideration. We analyse three classical problems associated with stationary variational inequalities: a variational problem with two obstacles, elasto-plastic torsion problem and a problem with friction type boundary conditions. The majorants are obtained by a modification of duality technique earlier used for variational problems with uniformly convex functionals [9–11]. These majorants naturally reflects properties of exact solutions and possess necessary continuity conditions.

1. Introduction

Mathematical models related to variational inequalities often arise in applied sciences (see, e.g., [1]). Existence, uniqueness and differentiability properties of their solutions were analyzed by many authors (see [2, 3, 4] and the references therein). Much effort has been directed toward creating numerical methods for variational inequalities. At this point we can refer to [5–8] and to works of many other authors.

In the present paper, we are focused on the problem, which is important for verifying the accuracy of numerical approximations of variational inequalities. Let u be the exact solution of a problem and v be an admissible function from the "energy" functional space V of the considered problem. Our goal is to derive a functional defined by v and by the given data of our problem that measures the deviation of $v - u$ in the norm of V. This functional must be nonnegative and vanish if and only if v coincides with exact solution u. Besides, it must be explicitly computable

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and possess proper continuity properties.

Below we present a method of deriving majorants of the deviation, which can be regarded as a modification of duality technique earlier used for variational problems with uniformly convex functionals [9–11]. We consider three classical problems related to variational inequalities: problem with obstacles, elasto-plastic torsion problem and a problem with friction type boundary conditions. In this short note, we confine ourselves to the simplest case of stationary 2D elliptic variational inequalities. However, possible generalizations are rather transparent and will be considered in subsequent papers.

Let us begin with general remarks. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and $j : V \to \mathbb{R}$ be a given convex continuous functional. We consider the following

Problem $\mathcal{P}$. Find $u \in K$ such that

$$a(u, w-u) + j(w) - j(u) \geq \langle f, w-u \rangle \qquad \forall w \in K, \tag{1.1}$$

where K is a convex closed subset of V and $f \in V^*$.

The solution u of (1.1) is a minimizer of the following variational problem: Find $u \in K$ such that

$$J(u) = \inf_{w \in K} J(w), \quad J(w) = \frac{1}{2}a(w,w) + j(w) - \langle f, w \rangle . \tag{1.2}$$

By (1.1) we easily obtain the inequality

$$\begin{aligned} J(v) - J(u) = {} & \frac{1}{2}a(v-u, v-u) + \\ & + a(u, v-u) - \langle f, v-u \rangle + j(v) - j(u) \geq \\ \geq {} & \frac{1}{2}a(v-u, v-u) , \end{aligned} \tag{1.3}$$

which implies the basic "deviation" estimate

$$\frac{1}{2}\| v-u \|_a^2 \leq J(v) - \inf \mathcal{P} \quad \forall v \in K , \tag{1.4}$$

where $\inf \mathcal{P}$ denotes the exact lower bound of the functional J and $\| v \|_a := (a(v,v))^{1/2}$. However, in general, the quantity $\inf \mathcal{P}$ is unknown. In this paper, we show how to circumvent this difficulty and obtain explicitly computable majorants of $\| v-u \|_a$ for the afore mentioned variational inequalities.

2. The obstacle problem

2.1 Variational formulations. We consider the bilinear form

$$a(v,w) := \int\limits_{\Omega} A\nabla v \cdot \nabla w \, dx \,, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$ and $A = \{a_{ij}\}$ is a symmetric matrix satisfying the usual ellipticity conditions

$$\nu_1 \, |\xi|^2 \; \leq \; A\xi \cdot \xi \; \leq \; \nu_2 \, |\xi|^2 \,, \qquad \forall \xi \in \mathbb{R}^n \,. \tag{2.2}$$

By V_0 we denote the subset of $W_2^1(\Omega)$ formed by functions vanishing on $\partial\Omega$. Let $\phi, \psi \in W_2^2(\Omega)$ be two given functions such that $\phi(x) \leq \psi(x)$ for all $x \in \Omega$, $j \equiv 0$, $\langle f, v \rangle = \int_\Omega f v \, dx$, and

$$K = K_{\phi\psi} := \big\{ v \in V_0 \;\; \| \;\; \phi(x) \; \leq v(x) \; \leq \psi(x) \text{ a.e. in } \Omega \big\}.$$

Then $\mathcal{P}$ is the classical obstacle problem (see, e.g., [2, 4]). A solution u minimizes the functional

$$J(v) \; = \; \int\limits_{\Omega} A\nabla v \cdot \nabla v \, dx \; - \; \int\limits_{\Omega} f v \, dx$$

on the set $K_{\phi\psi}$. In general, it divides Ω into three sets:

$$\begin{aligned}
\Omega^u_{\oplus} :=& \big\{ x \in \Omega \;\; \| \qquad u(x) = \psi(x) \big\} \,, \\
\Omega^u_{\ominus} :=& \big\{ x \in \Omega \;\; \| \qquad u(x) = \phi(x) \big\} \,, \\
\Omega^u_{0} :=& \big\{ x \in \Omega \;\; \| \;\; \phi(x) < u(x) < \psi(x) \big\} \,.
\end{aligned}$$

The sets $\Omega^u_{\oplus}$ and $\Omega^u_{\ominus}$ are called *upper* and *lower coincidence sets* and the open set Ω^u_0 is the complementary set, where a solution satisfies the differential equation. Thus, we see that this problem involves free boundaries, which are unknown a priori.

Differentiability properties of solutions to linear and quasiliniear problems with obstacles were investigated by many authors (see, e.g., [2–4, 12, 13]). In particular, it was proved that under natural assumptions on external data, $u \in W_2^2(\Omega)$ and even for very smooth data u may have limit regularity (which is W^2_∞). Hereafter, we assume that these assumptions are fulfilled and, thus, a solution possesses W_2^2 – regularity.

2.2 Perturbed problem. Let us introduce the perturbed functional

$$J_\lambda(v) := J(v) - \int\limits_\Omega \lambda \cdot (\mathrm{v} - \Phi)\, dx,$$

where $\Phi = (\phi, -\psi)$ and $\mathrm{v} = (\, v,\, -v)$.

$$\lambda \in \Lambda_+ := \big\{ (\lambda_1, \lambda_2) \;\|\; \lambda_i \in L_2(\Omega), \lambda_i(x) \geq \; 0 \,\text{a.e. in}\, \Omega,\; i = 1, 2 \big\}.$$

It is easy to see that

$$\begin{aligned} \sup_{\lambda \in \Lambda_+} J_\lambda(v) \;&=\; J(v) - \inf_{\lambda \in \Lambda_+} \int\limits_\Omega \lambda \cdot (\mathrm{v} - \Phi)\, dx = \\ &= \begin{cases} J(v) & \text{if } v \in K_{\phi\psi} \\ -\infty & \text{if } v \notin K_{\phi\psi}. \end{cases} \end{aligned} \tag{2.3}$$

With the functional J_λ we associate a perturbed variational problem.

Problem $\mathcal{P}_\lambda$. Find $u_\lambda \in V_0$ such that

$$J_\lambda(u_\lambda) = \inf_{v \in V_0} J_\lambda(v) := inf \mathcal{P}_\lambda. \tag{2.4}$$

Since

$$\inf_{v \in V_0} J_\lambda(v) \leq \inf_{v \in K_{\phi\psi}} J_\lambda(v) = \inf_{v \in K_{\phi\psi}} J(v) = inf\, \mathcal{P},$$

we see that

$$\inf \mathcal{P}_\lambda \;\leq\; \inf \mathcal{P} \quad \forall\; \lambda \in \Lambda_+. \tag{2.5}$$

It follows from (1.4) and (2.5) that

$$\frac{1}{2} \|\, v - u\,\|_a^2 \;\leq\; J(v) \,-\, \inf \mathcal{P}_\lambda \qquad \lambda \in \Lambda_+. \tag{2.6}$$

To estimate the right hand side of (2.5) we introduce a dual counterpart for Problem $\mathcal{P}_\lambda$.

2.3. Dual perturbed problem. By using the Lagrangian

$$L(v, \tau^*, \lambda) := \int\limits_\Omega \left(\tau^* \cdot \nabla v - \frac{1}{2} \widehat{A} \tau^* \cdot \tau^* - f v - \lambda \cdot (\mathrm{v} - \Phi) \right)\, dx,$$

where $\hat{A}$ is the matrix inverse to A, we can define the perturbed functional as follows

$$J_\lambda(v) = \sup_{\tau^* \in Y^*} L(v, \tau^*, \lambda), \qquad Y^* := L_2(\Omega, \mathbb{R}^2).$$

Known minimax theorems of the calculus of variations lead to a new variational problem, which is dual to Problem $\mathcal{P}_\lambda$.

Problem $\mathcal{P}^*_\lambda$. Find τ^*_λ such that

$$J^*_\lambda(\tau^*_\lambda) = \sup_{q^* \in Q^*_{f\lambda}} J^*_\lambda(q^*),$$

where

$$J^*_\lambda(q^*) = \int_\Omega \left(-\frac{1}{2}\widehat{A}q^* \cdot q^* + \lambda \cdot \Phi \right) dx,$$

$$Q^*_{f\lambda} := \left\{ q^* \in Y^* \ \| \ \int_\Omega q^* \cdot \nabla v \, dx = \int_\Omega (fv + \lambda \cdot \mathrm{v}) \, dx \ \forall v \in V_0 \right\}.$$

We note that the set $Q^*_{f\lambda}$ is a closed affine manifold in Y^* and the functional $-J^*_\lambda$ is convex and continuous on Y^*. Therefore, Problem $\mathcal{P}^*_\lambda$ has a solution and

$$\inf \mathcal{P}_\lambda = \sup \mathcal{P}^*_\lambda. \tag{2.7}$$

2.4. Estimates of the deviation. Relations (1.3) and (2.7) yield the estimate

$$\frac{1}{2}\| v - u \|^2_a \leq J(v) - \sup \mathcal{P}^*_\lambda \leq J(v) - J^*_\lambda(q^*), \tag{2.8}$$

which holds for any $v \in K_{\phi\psi}$, $q^* \in Q^*_{f\lambda}$ and $\lambda \in \Lambda_+$.

By rewriting the right-hand side of (2.8), we obtain

$$\begin{aligned} \frac{1}{2}\| v - u \|^2_a \leq & \int_\Omega \left(\frac{1}{2} A\nabla v \cdot \nabla v - fv \right) dx + \int_\Omega \frac{1}{2}\widehat{A}\tau^* \cdot \tau^* \, dx - \\ & - \int_\Omega \lambda \cdot \Phi \, dx + \frac{1}{2} \int_\Omega \left(\widehat{A}q^* \cdot q^* - \widehat{A}\tau^* \cdot \tau^* \right) dx \end{aligned} \tag{2.9}$$

By using the identity $\widehat{A}a \cdot a - \widehat{A}b \cdot b = \widehat{A}(a-b) \cdot (a-b) + 2\widehat{A}b \cdot (a-b)$ and the integral identity

$$\int_\Omega fv \, dx = \int_\Omega q^* \cdot \nabla v \, dx - \int_\Omega \lambda \cdot \mathrm{v} \, dx \quad \forall q^* \in Q^*_{f\lambda},$$

we obtain

$$\begin{aligned}\frac{1}{2}\| v-u \|_a^2 \leq \tfrac{1}{2}&\int\limits_\Omega (A\nabla v-\tau^*)\cdot(\nabla v-\widehat{A}\tau^*)\,dx+\\ &+\int\limits_\Omega \lambda\cdot(\mathrm{v}-\Phi)\,dx+\frac{1}{2}\int\limits_\Omega \widehat{A}(q^*-\tau^*)\cdot(q^*-\tau^*)+\\ &+\int\limits_\Omega (\nabla v-\widehat{A}\tau^*)\cdot(\tau^*-q^*)\,dx.\end{aligned}\tag{2.10}$$

The last integral can be estimated as follows

$$\begin{aligned}\int\limits_\Omega (\nabla v-\widehat{A}\tau^*)\cdot(\tau^*-q^*)dx \leq \tfrac{\beta}{2}&\int\limits_\Omega A(\nabla v-\widehat{A}\tau^*)\cdot(\nabla v-\widehat{A}\tau^*)\,dx+\\ &+\tfrac{1}{2\beta}\int\limits_\Omega \widehat{A}(q^*-\tau^*)\cdot(q^*-\tau^*)\,dx,\end{aligned}\tag{2.11}$$

where β is any positive number.

Let us introduce the quantity

$$d^2(\tau^*,Q^*_{f\lambda}) := \inf_{q^*\in Q^*_{f\lambda}}\int\limits_\Omega \widehat{A}(q^*-\tau^*)\cdot(q^*-\tau^*)\,dx\,,\tag{2.12}$$

which can be considered as a distance between τ^* and the set $Q^*_{f\lambda}$.

Then the estimate (2.11) comes in the form

$$\begin{aligned}\| v-u \|_a^2 \leq (1+\beta)d^2(\tau^*,Q^*_{f\lambda})+(1+\beta^{-1})\|\nabla v-\widehat{A}\tau^*\|_a^2+\\ +\,2\int\limits_\Omega \lambda\cdot(\mathrm{v}-\Phi)\,dx.\end{aligned}\tag{2.13}$$

If $\tau^*\in Q^*:=\left\{\tau^*\in Y^* \;\;\|\;\; \operatorname{div}\tau^*\in L_2(\Omega)\right\}$ then $d(\tau^*,Q^*_{f\lambda})$ can be estimated by the duality procedure (see [9–11]). As a result of this, we obtain

$$d(\tau^*,Q^*_\lambda) \leq C_{\Omega,A}\,\|\operatorname{div}\tau^*+f+\lambda_1-\lambda_2\|\;,\tag{2.14}$$

where

$$C_{\Omega,A} \;=\; \inf_{w\in V_0}\frac{\| w \|_a}{\|w\|}$$

and $\|\cdot\|$ is the usual norm in $L_2(\Omega)$.

Then (2.13) comes in the form

$$\| v-u \|_a^2 \leq C^2_{\Omega,A}(1+\beta)\, \|\operatorname{div}\tau^* + f + \lambda_1 - \lambda_2\|^2 + \\ +(1+\beta^{-1})\| \nabla v - \widehat{A}\tau^* \|_a)^2 + 2\int\limits_\Omega \lambda\cdot(\mathrm{v}-\Phi)\,dx. \quad (2.15)$$

To obtain more a rigorous upper bound of the deviation $\| v-u\|_a$ one should find a suitable function λ minimizing the right-hand side of (2.15).

For any continuous function v one can define three sets (some of which may be empty)

$$\begin{aligned} \Omega^v_0 :=& \{x\in\Omega \;\|\; \phi(x) < v(x) < \psi(x)\},\\ \Omega^v_\ominus :=& \{x\in\Omega \;\|\; v(x)=\phi(x)\},\\ \Omega^v_\oplus :=& \{x\in\Omega \;\|\; v(x)=\psi(x)\}. \end{aligned}$$

In (2.15), we can set the functions λ_1 and λ_2 as follows:

$$\begin{aligned} \lambda_1 = \lambda_2 &= 0 && \text{a.e. in } \Omega^v_0,\\ \lambda_1 = -\langle \operatorname{div}\tau^*+f\rangle_\ominus,\ \lambda_2 &= 0 && \text{a.e. in } \Omega^v_\ominus,\\ \lambda_1 = 0,\ \lambda_2 &= \langle \operatorname{div}\tau^*+f\rangle_\oplus, && \text{a.e. in } \Omega^v_\oplus. \end{aligned}$$

This choice leads to the estimate

$$\| v-u\|_a^2 \ \leq\ \mathcal{M}_1(v,\tau^*,\beta) := \left(1+\frac{1}{\beta}\right)\|\nabla v-\widehat{A}\tau^*\|_a^2 + C^2_{\Omega,A}(1+\beta)\times \\ \times\Big[\int\limits_{\Omega^v_0} |r(\tau^*)|^2\,dx + \int\limits_{\Omega^v_\oplus}\langle r(\tau^*)\rangle^2_\ominus\,dx + \int\limits_{\Omega^v_\ominus}\langle r(\tau^*)\rangle^2_\oplus\,dx\Big], \quad (2.16)$$

where $r(\tau^*) = \operatorname{div}\tau^* + f$ and $\langle\ \rangle_\ominus$ and $\langle\ \rangle_\oplus$ denote negative and positive parts of a quantity.

The majorant $\mathcal{M}_1$ consists of four terms. The first term penalizes an error in the duality relation $\nabla v = \widehat{A}\tau^*$ and the other three terms penalize "improper" behavior of $r(\tau^*)$ on the sets Ω^v_0, $\Omega^v_\oplus$, and $\Omega^v_\ominus$, respectively. It is easy to see that the majorant $\mathcal{M}_1$ is a nonnegative function of its arguments, which vanishes if and only if

$$\nabla v(x) = \widehat{A}\tau^*(x) \qquad \text{for a.e. } x\in\Omega, \qquad (2.17)$$

$$\operatorname{div}\tau^*(x) + f(x) \leq 0, \qquad \text{for a.e. } x\in\Omega^v_\ominus, \qquad (2.18)$$

$$\operatorname{div}\tau^*(x) + f(x) = 0, \qquad \text{for a.e. } x\in\Omega^v_0, \qquad (2.19)$$

$$\operatorname{div}\tau^*(x) + f(x) \geq 0 \qquad \text{for a.e. } x\in\Omega^v_\oplus. \qquad (2.20)$$

In this case, for any $w \in K_{\phi\psi}$, we have

$$\int_\Omega A\nabla v \cdot \nabla(w-v)\,dx - \int_\Omega f(w-v)\,dx =$$
$$= \int_\Omega (\operatorname{div}\tau^* + f)(v-w)\,dx = \int_{\Omega^v_\ominus} (\operatorname{div}\tau^* + f)(\phi - w)\,dx +$$
$$+ \int_{\Omega^v_0} (\operatorname{div}\tau^* + f)(v-w)\,dx + \int_{\Omega^v_\oplus} (\operatorname{div}\tau^* + f)(\psi - w)\,dx \ge 0\,. \quad (2.21)$$

This inequality means that v is a solution of the considered problem and, thus, we arrive at the following statement:

Proposition 1. *For any $\beta > 0$, the functional $\mathcal{M}_1(v, \tau^*, \beta)$ is nonnegative and vanishes if and only if $v = u$ and $\tau^* = A\nabla u$, where u is a solution of the variational inequality*

$$a(u, w-u) \ge \int_\Omega f(w-u)\,dx \qquad \forall w \in K_{\phi\psi}\,. \quad (2.22)$$

To obtain more a rigorous upper bound of the deviation $\| v - u \|_a$ one should find λ by minimizing the right-hand side of the estimate (2.15). This procedure can be done exactly what results in the estimate

$$\| v - u \|_a^2 \le \mathcal{M}_2(v, \tau^*, \beta) := (1 + \frac{1}{\beta})\| \nabla v - \widehat{A}\tau^* \|_a^2 +$$
$$+ \int_\Omega R(v, \operatorname{div}\tau^* + f, \beta)\,dx\,, \quad (2.23)$$

where

$$R(v, r, \beta) = \begin{cases} -\frac{(\phi - v)^2}{c_\beta} + 2r(\phi - v) & \text{if } \; c_\beta r + v \le \phi \\ \quad c_\beta r^2 & \text{if } \; \phi < c_\beta r + v < \psi \\ -\frac{(\psi - v)^2}{c_\beta} + 2r(\psi - v) & \text{if } \; c_\beta r + v \ge \psi \end{cases} \quad (2.24)$$

and $c_\beta = C^2_{\Omega,A}(1+\beta)$ The functional $\mathcal{M}_2$ is defined for any $v \in K_{\phi\psi}$, $\tau^* \in Q^*$, and $\beta > 0$. By the method of its construction, we see that $\mathcal{M}_2(v, \tau^*, \beta) \le \mathcal{M}_1(v, \tau^*, \beta)$. Thus, the following assertion is evident.

Proposition 2. *For any $\beta > 0$, the functional $\mathcal{M}_2(v, \tau^*, \beta)$ is nonnegative and vanishes if and only if $v = u$ and $\tau^* = A\nabla u$, where u is a solution of the variational inequality* (2.22).

The first term in (2.24) has the same structure as in (2.16), while the second one replaces three last terms in (2.16) and, in a sense, inherits all their properties. To make this fact more transparent, we present the shape of the integrand R, considered as a function of its two arguments for the case, in which $\phi = 0$ and $\psi = 1$ (so that $0 \leq v \leq 1$).

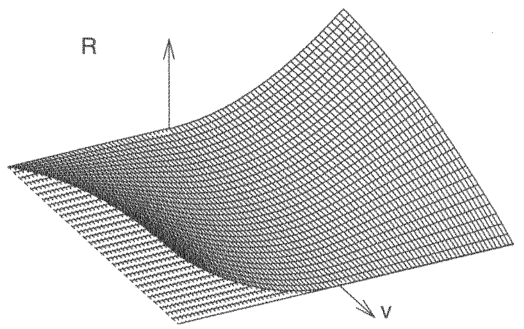

Fig. 1.

We see that $R = 0$ in three cases:
(a) $r = 0$,
(b) $v = \phi = 0$ and $r < 0$,
(c) $v = \psi = 1$ and $r > 0$.

Remark 1. One can prove that for any positive β the majorant $\mathcal{M}_2(v, \tau^*, \beta)$ possesses necessary continuity properties with respect to the first and second arguments. Namely, $\mathcal{M}_2(v_k, \tau_k^*, \beta) \to 0$ if $v_k \to u$ in V_0 and $\tau_k^* \to A\nabla u$ in Q^*.

Remark 2. If $\psi = +\infty$ (i.e., if only one obstacle exists) then the function R has a more compact form:

$$R(v, \tau^*, \beta) = c_\beta \left[|r|^2 - \langle \frac{v - \phi}{c_\beta} + r \rangle_\ominus^2 \right]$$

For the paradigm of the membrane problem, this case was analyzed in [14], where one can also find some numerical examples.

Remark 3. Right-hand sides of estimates (2.16) and (2.24) contain free parameters: they are valid for any $\beta > 0$ and $\tau^* \in Q^*$. Therefore, optimal majorant of the deviation requires a minimization over β and τ^*.

3. Elasto-plastic torsion problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. We consider the classical problem on torsion of a "long" elasto-plastic bar, whose cross-section is Ω. If the bar is made of an isotropic material then this problem can be reduced to the following variational inequality (see, e.g., [1]): find $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,dx \geq \mu \int_\Omega (v-u)\,dx \quad \forall v \in K, \tag{3.1}$$

where μ is a positive parameter,

$$K := \left\{ v \in V_0 \;\|\; |\nabla v| \leq 1 \text{ a.e. in } \Omega \right\},$$

and V_0 is defined as in the previous section.

Mathematical properties of this problem has been investigated by many authors. This problem has a unique solution, which has a free boundary separating the sets

$$\Omega_e := \left\{ x \in \Omega \;\|\; |\nabla u| < 1 \right\} \text{ and } \Omega_p := \left\{ x \in \Omega \;\|\; |\nabla u| = 1 \right\}.$$

These sets are called the *elastic* and *plastic* sets, respectively. It is known that under natural assumptions on external data a solution u has second generalized derivatives summable with any power (see, e.g., [2]).

If Ω is 1-connected then the solution of problem (3.1) coincides with the solution of the following obstacle problem (see, e.g., [2, 7, 15])

Problem $\mathcal{P}$. Find $u \in K_d$ such that

$$J(u) = \inf_{v \in K_d} J(v), \quad J(v) = \frac{1}{2} \int_\Omega \left(|\nabla v|^2 - \mu v \right) dx, \tag{3.2}$$

where

$$K_d := \left\{ v \in V_0 \;\|\; |v| \leq d(x, \partial\Omega) \text{ for a.e. } x \in \Omega \right\}$$

and $d(x, \partial\Omega)$ denotes the distance to $\partial\Omega$.

Problem (3.2) is, in fact, an obstacle problem with two obstacles considered in Sec. 2. Therefore, we can apply above estimates (2.16) or (2.24) with $\phi = -d(x, \partial\Omega)$ and $\psi = d(x, \partial\Omega)$. In particular, if v has a fixed sign in Ω (e.g., v is nonnegative in Ω) then (2.16) gives the estimate

$$\| v - u \|_a^2 \leq \left(1 + \frac{1}{\beta}\right) \| \nabla v - \tau^* \|_a^2 + \\ + C_\Omega^2 (1+\beta) \Big[\int\limits_{\Omega_e(v)} (\mathrm{div}\tau^* + \mu)^2 \, dx + \int\limits_{\Omega_p(v)} \langle \, \mathrm{div}\tau^* + \mu \, \rangle_\ominus^2 \, dx \Big] , \quad (3.3)$$

where $\Omega_e(v)$ and $\Omega_p(v)$ are elastic and plastic sets associated with function $v \in K$ and C_Ω is a constant in the Friederihs–Poincaré inequality.

In the case of a multi-connected domain, the elasto-plastic torsion problem can also be represented as an obstacle problem (see [2]). Therefore, estimates (2.16) and (2.24) give a way to estimate $v - u$ for such problems.

4. Model problem with friction

4.1. Basic problem. We analyse the inequality (1.1) provided that $\partial\Omega$ consists of two measurable nonintersecting parts Γ_0 and Γ_1,

$$K = V_0(\Gamma_0) := \left\{ v \in W_2^1(\Omega) \ \| \ v = 0 \text{ on } \Gamma_0 \right\},$$

and

$$j(v) = \gamma \int\limits_{\Gamma_1} |v| \, dx \, .$$

In this case, Problem $\mathcal{P}$ can be considered as a model problem involving friction type boundary condition on the boundary Γ_1 and the positive parameter γ has a meaning of the coefficient of friction. By n we shall denote the unit outward normal to Γ_1.

Our problem has a variational form: find $u \in V_0(\Gamma_0)$ such that

$$J(u) \, = \inf_{v \in V_0(\Gamma_0)} J(v) \, = \, \inf \mathcal{P} \, .$$

This problem has a unique solution. Moreover, if $\Gamma_1 = \partial\Omega$ is smooth and $f \in L_2(\Omega)$, then $u \in W_2^2(\Omega)$. Hereafter, we assume that external data are such that this solution has square summable second derivatives.

Then, directly from the variational inequality, it follows that u satisfies the relations

$$\begin{aligned} \Delta u + f &= 0 && \text{in } \Omega && (4.1)\\ u &= 0 && \text{on } \Gamma_0 && (4.2)\\ |u_{,n}| < \gamma \;\Rightarrow\; u &= 0 && \text{on } \Gamma_1 && (4.3)\\ |u_{,n}| = \gamma \;\Rightarrow\; u &\le 0 && \text{on } \Gamma_1 && (4.4)\\ |u_{,n}| = -\gamma \;\Rightarrow\; u &\ge 0 && \text{on } \Gamma_1, && (4.5) \end{aligned}$$

where $u_{,n}$ denotes the normal component of ∇u. It is not difficult to verify that conditions (4.3)–(4.5) are equivalent to the relations

$$\begin{aligned} |u_{,n}| &\le \gamma && \text{on } \Gamma_1\,, && (4.6)\\ u\,u_{,n} + \gamma\,|u| &= 0 && \text{on } \Gamma_1\,, && (4.7) \end{aligned}$$

which present "friction" conditions on Γ_1.

4.2. Perturbed problems. Let us consider the following perturbed functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega \left(|v|^2 - fv\right)\,dx + \gamma\int_{\Gamma_1} \lambda v\,d\Gamma\,, \tag{4.8}$$

where

$$\lambda \in \Lambda_0 := \left\{\lambda \in L_\infty(\Omega) \;\|\; |\lambda| \le 1 \text{ a.e. on } \Gamma_1\right\}.$$

The perturbed Problem $\mathcal{P}_\lambda$ is to find $u_\lambda \in V_0(\Gamma_0)$ such that

$$J_\lambda(u_\lambda) = \inf_{v\in V_0(\Gamma_0)} J_\lambda(v) = \inf \mathcal{P}_\lambda\,.$$

This problem is a simple quadratic problem, which has a unique solution for any $\lambda \in \Lambda_0$.

It is easy to see that

$$J(v) = \sup_{\lambda\in\Lambda_0} J_\lambda(v)$$

and, consequently,

$$\inf_{v\in V_0(\Gamma_0)} J_\lambda(v) \le \inf_{v\in V_0(\Gamma_0)} J(v) = \inf \mathcal{P}. \tag{4.9}$$

The perturbed problem that stands in the left-hand side of (4.9) has a dual counterpart. The respective problem $\mathcal{P}^*_\lambda$ is to find $\tau^*_\lambda \in Q^*_{\gamma\lambda}$ such that

$$I^*_\lambda(\tau^*_\lambda) = \sup_{q^* \in Q^*_{\gamma\lambda}} I^*_\lambda(q^*)\,,$$

where

$$I^*_\lambda(q^*) = -\frac{1}{2}\int\limits_\Omega |q^*|^2\, dx$$

and

$$Q^*_{\gamma\lambda} := \left\{ q^* \in Y^* \;\|\; \int\limits_\Omega q^* \cdot \nabla v\, dx = \int\limits_\Omega f v\, dx - \gamma \int\limits_{\Gamma_1} \lambda v\, d\Gamma \;\; \forall v \in V_0(\Gamma_0) \right\}.$$

This problem also has a unique solution. Moreover,

$$\inf \mathcal{P}_\lambda = \sup \mathcal{P}^*_\lambda\,.$$

4.3. Estimates of the deviation. In view of the above connection between lower and upper bounds in Problems $\mathcal{P}_\lambda$ and $\mathcal{P}^*_\lambda$, we obtain

$$\frac{1}{2}\|\nabla(v-u)\|^2 \le J(v) - \sup \mathcal{P}^*_\lambda \le J(v) - I^*_\lambda(q^*) \quad \forall q^* \in Q^*_{\gamma\lambda}\,. \tag{4.10}$$

By arguments similar to those used in Sec. 2, we rearrange the right-hand side of (4.12) and find that for any positive β and any $\tau^* \in Y^*$ holds the estimate

$$\|\nabla(v-u)\|^2 \le \left(1+\frac{1}{\beta}\right)\|\nabla v - \tau^*\|^2 + \\ + (1+\beta) \inf_{q^* \in Q^*_{\gamma\lambda}} \|q^* - \tau^*\|^2 + \gamma \int\limits_{\Gamma_1} (\lambda v + |v|)\, dx\,. \tag{4.11}$$

Let $\tau^* \in Q^*$, then

$$\inf_{q^* \in Q^*_{\gamma\lambda}} \|q^* - \tau^*\|^2 \le \\ \le C^2_{\Omega\Gamma}\left(\int\limits_\Omega |\mathrm{div}\tau^* + f|^2\, dx + \int\limits_{\Gamma_1} |\gamma\lambda - \tau^*_n|^2\, d\Gamma \right), \tag{4.12}$$

where $\tau_n^* = \tau^* \cdot n$, and $C_{\Omega\Gamma}$ a constant in the inequality

$$\int\limits_\Omega |w|^2 \, dx + \int\limits_{\Gamma_1} |w|^2 \, d\Gamma \le C_{\Omega\Gamma}^2 \int\limits_\Omega |\nabla w|^2 \, dx \quad \forall w \in V_0(\Gamma_0) . \tag{4.13}$$

Combining (4.11) and (4.12), we obtain

$$\begin{aligned} \|\nabla(v-u)\|^2 \le & \left(1+\frac{1}{\beta}\right) \|\nabla v - \tau^*\|^2 + \\ & + (1+\beta) C_{\Omega\Gamma}^2 \left(\int\limits_\Omega |\mathrm{div}\tau^* + f|^2 \, dx + \int\limits_{\Gamma_1} |\gamma\lambda - \tau_n^*|^2 \, d\Gamma \right) + \\ & + \gamma \int\limits_{\Gamma_1} (\lambda v + |v|) \, dx . \end{aligned} \tag{4.14}$$

Let us choose here λ as follows:

$$\begin{aligned} \lambda(x) &= \tau_n^*(x)/\gamma && \text{if } |\tau_n^*(x)| \leqslant \gamma \\ \lambda(x) &= -1 && \text{if } \tau_n^*(x) < -\gamma \\ \lambda(x) &= +1 && \text{if } \tau_n^*(x) > \gamma \end{aligned}$$

In this case the resulting estimate has the form

$$\begin{aligned} \|\nabla(v-u)\|^2 \le \mathcal{M}_3(v,\tau^*,\beta) = & \left(1+\frac{1}{\beta}\right) \|\nabla v - \tau^*\|^2 + \\ & + (1+\beta) \, C_{\Omega\Gamma}^2 \int\limits_\Omega |\mathrm{div}\tau^* + f|^2 \, dx + \int\limits_{\Gamma_1} \Theta_1(v,\tau_n^*,\beta) \, dx , \end{aligned} \tag{4.15}$$

where

$$\Theta_1(v,\tau_n^*,\beta) = \begin{cases} (1+\beta) C_{\Omega\Gamma}^2 \, |\tau_n^* + \gamma|^2 + \gamma(|v| - v) & \text{if } \tau_n^* < -\gamma \\ \tau_n^* v + \gamma \, |v| & \text{if } |\tau_n^*| \le \gamma \\ (1+\beta) C_{\Omega\Gamma}^2 \, |\tau_n^* - \gamma|^2 + \gamma(|v| + v) & \text{if } \tau_n^* > +\gamma \end{cases}$$

It is easy to see that the first and the third branches of the above function give positive values and, consequently, zero values may arise only in the second branch. Thus, for any $\beta > 0$, the functional $\mathcal{M}_3(v,\tau^*,\beta)$ is nonnegative and vanishes if and only if

$$\nabla v = \tau^*, \qquad \text{a.e. in } \Omega \tag{4.16}$$

$$\mathrm{div}\, \tau^* + f = 0 \qquad \text{a.e. in } \Omega \tag{4.17}$$

$$|\tau_n^*| \le \gamma \quad \text{and} \quad \tau_n^* v + \gamma \, |v| = 0 \qquad \text{a.e. on } \Gamma_1 \tag{4.18}$$

Since $\nabla v \cdot n = v_{,n}$, the above relations holds if and only if the pair of functions (v, τ^*) meets the relations (4.16), (4.17), and the conditions (4.6) and (4.7) on the contact boundary Γ_1.

The above is summarized in the following assertion:

Proposition 3. *For any $\beta > 0$, the functional $\mathcal{M}_3(v, \tau^*, \beta)$ is nonnegative and vanishes if and only if $v = u$ and $\tau^* = \nabla u$.*

To obtain a sharper estimate, the right-hand side of (4.14) should be minimized over $\lambda \in [-1, 1]$. Then, we obtain

$$\|\nabla(v-u)\|^2 \leq \left(1 + \frac{1}{\beta}\right) \|\nabla v - \tau^*\|^2 + \\ + (1+\beta)\, C^2_{\Omega\Gamma} \int\limits_{\Omega} |\mathrm{div}\tau^* + f|^2 \, dx + \int\limits_{\Gamma_1} \Theta_2(v, \tau^*_n, \beta)\, dx\,,$$

where

$$\Theta_2(v, \tau^*_n, \beta) = \gamma\,|v| + \inf_{\xi \in [-1,1]} \left\{ (1+\beta) C^2_{\Omega\Gamma} (\gamma\xi - \tau^*_n)^2 + \gamma v \xi \right\}\,.$$

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