Записки научных
семинаров ПОМИ
Том 271, 2000 г.

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REGULARITY TO STATIONARY WEAK SOLUTIONS IN THE THEORY OF GENERALIZED NEWTONIAN FLUIDS WITH ENERGY TRANSFER

To the memory of A. I. Ivanov

ABSTRACRT. In this work we prove regularity results for weak solutions to some stationary problems arising in the theory of generalized Newtonian fluids with energy transfer. Namely, we prove that these solutions are strong. In two-dimensional case we prove the Hölder continuity of the first gradient of solutions.

1. Introduction

In fluids mechanics, there exists various types of non-Newtonian fluids: fluids of the diferential type, of grade n, of rate type and integral type. Many considerations about these fluids can be seen in the Rajagopal's paper [23], like, the ability to shear thin or shear thicken, the ability to creep, the ability to relax stresses, the presence of normal stress diferences in simple shear flows or the presence of yield stress. One example can be the generalized Newtonian fluids.

A recent worry of many authors is the study of flow with energy transfer. Mathematically, it results in the study of a coupled system of nonlinear partial differential equations. We can mention, for example, Duvaut and Lions [7] that studied the evolution case to the Bingham fluid, Rodrigues [24, 25] that studied an unidirectional flow and a stationary free boundary problem for thermoconvective flows of Boussinesq–Stefan type, [1] and Baranger and Mikelic that proved existence of a least one weak solution for the system describing the stationary flow of a quasi-Newtonian fluid, with temperature dependent viscosity and with creation of heat by viscous dissipation. Unfortunately, all these works only give weak solutions.

We may also refer the works [4] and [21] which give existence and uniqueness of a new kind of solution: weak-renormalized solution to solve

This work is partially supported by the Russian Foundation for Basic Research, grant No. 99-01-00108.

problems motivated by turbulence modelling. For example, (k, ε)-model or models for the compled Atmosphere and Ocean (see references in [21]).

In this paper, we develop the study of regularity for solutions to the problem described above. We would like to mention here the paper [16] of Kagei, M. Ružička, and G. Thäter who proved existence of a strong solutions in non-stationary case for small data, and the papers [26, 27] of G. Seregin who investigated questions of regularity in isothermal non-stationary 2D- and 3D-cases.

New regularity criterious for steady isothermal equations when $n \geq 5$ are done in [8] (see also the references therein).

The new idea of this work is how standard techniques can be useful to obtain a new result of regularity. Assuming the existence of weak solutions to the coupled system constitued by motion and energy equations, first we prove higher summability to the gradient of the velocity, using the reverse Hölder inequality. Second, we prove that this higher summability is true up to the boundary. Next we prove higher summability to the gradient of the energy, using again the reverse Hölder inequality.

We estimate the second derivatives for the solution to a regularized problem of the motion equation, with help of finite differences. Finally, interior regularity for the gradient of the velocity is obtained.

There are papers devoted even to $L\ \log L$ growth of functionals but all of them consider problems without convective term.

At two-dimensional case, regularity of the gradient of the energy follows from the general theory. Regularity of the gradient of the velocity is not trivial. On the contrary, it is very complicated. But, if we have a $C^{1+\alpha}$-coefficient, then the problem of Hölder continuity for the gradient of the velocity is well studied by Seregin and others.

2. Statement of the problem

Let us consider the boundary-value problem of the following form:

$$\begin{aligned} -\operatorname{div}\sigma + (\nabla u)u &= f - \nabla\pi \quad &\text{in} \quad \Omega;\\ \operatorname{div} u &= 0 \quad &\text{in} \quad \Omega; \end{aligned} \tag{2.1}$$

$$-\operatorname{div}\mathcal{G} + u\cdot\nabla e = \sigma : \varepsilon(u) + g \quad \text{in} \quad \Omega; \tag{2.2}$$

$$u\big|_{\partial\Omega} = 0, \qquad e\big|_{\partial\Omega} = 0. \tag{2.3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ with a smooth boundary $\partial\Omega$, unknown functions are the velocity field $u : \Omega \to \mathbb{R}^n$, pressure field $\pi : \Omega \to \mathbb{R}$ and internal energy $e : \Omega \to \mathbb{R}$. We denote by $\varepsilon(u) = \frac{1}{2}\big(\nabla u +$

$(\nabla u)^T)$ the strain velocity tensor, $f : \Omega \to \mathbb{R}^n$ a given external force and $g : \Omega \to \mathbb{R}^n$ a given function describing heat sources or heat flow.

We use following notations: $\mathbb{M}^{n\times n}$ is the space of matrixes $n \times n$, and $\mathbb{M}^{n\times n}_{sym}$ is the subspace of symmetric matrixes,

$$A : B = A_{ij}B_{ij}, \quad \forall A, B \in \mathbb{M}^{n\times n} \quad a \cdot b = a_i b_i \quad \forall a, b \in \mathbb{R}^n$$

$$|A| = (A : A)^{1/2}, \qquad |a| = (a \cdot a)^{1/2}$$

$$a \otimes b = (a_i b_j),\ i, j = 1, \dots, n \quad a \odot b = (a \otimes b + b \otimes a)/2.$$

(here and everywhere repeated indices imply summation from 1 to n).

The problem (2.1)–(2.5) describes the steady-state motion of one class of generalized Newtonian fluid which has a constitutive relation between stress tensor σ and strain velocity tensor $\varepsilon(u)$ of the following form:

$$\sigma = \beta(\theta(e))a(\varepsilon(u)). \tag{2.4}$$

The viscosity β in this relation depends on the temperature θ where the temperature is a smooth function of the internal energy e. So we denote by $\mu \equiv \beta \circ \theta$. For the physical cases of the dependence σ on $\varepsilon(u)$, we may give as examples: the "power law" of Ostwald and de Waele, the Carreau, Prandtl–Eyring, Williamson, Cross, Ellis models (see [2]) or the Ladyzenskaya model (see [19]).

If there is a function W such that $a(\tau) = \frac{\partial W}{\partial \tau}(\tau)$ then we call it *a dissipative potential* of the fluid.

The relation between *heat flux* $\mathcal{G}$ and internal energy e is given by the formula

$$\mathcal{G} = k(e)b(\nabla e). \tag{2.5}$$

The particular case of $b(\xi) = \xi$, $k(s) = \text{const}$ is known as *Fourier law* of heat conductivity.

3. Assumptions and statement of the main result

We assume that the data of our problem satisfy the following structure conditions:

$a : \mathbb{M}^{n\times n}_{sym} \to \mathbb{M}^{n\times n}_{sym}$ is a function of class C^1 which satisfies the conditions

$$\begin{gathered} \exists \nu_0, \nu_1 > 0 : \nu_0(1 + |\tau|)^{p-2}|\varkappa|^2 \leqslant \Big(\frac{\partial a}{\partial \tau}(\tau)\varkappa\Big) : \varkappa, \\ \Big|\frac{\partial a}{\partial \tau}(\tau)\Big| \leqslant \nu_1(1 + |\tau|)^{p-2} \end{gathered} \tag{3.1}$$

for all τ and $\varkappa \in \mathbb{M}^{n\times n}_{sym}$.

$b : \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 which satisfies the conditions

$$\begin{gathered}\exists \varkappa_0, \varkappa_1 > 0 : \varkappa_0(1+|\xi|)^{q-2}|\eta|^2 \leqslant \Big(\frac{\partial b}{\partial \xi}(\xi)\eta\Big)\cdot \eta,\\ \Big|\frac{\partial b}{\partial \xi}(\xi)\Big| \leqslant \varkappa_1(1+|\xi|)^{q-2}\end{gathered} \tag{3.2}$$

for all ξ and $\eta \in \mathbb{R}^n$.

$\mu : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that

$$0 < \mu_0 \leqslant \mu(s) \leqslant \mu_1, \quad |\mu'(s)| \leqslant \mu_2; \tag{3.3}$$

$k : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that

$$0 < k_0 \leqslant k(s) \leqslant k_1, \quad |k'(s)| \leqslant k_2. \tag{3.4}$$

We shall use the following functional spaces:

$L^p(\Omega, \mathbb{R}^n)$, $W^{1,p}(\Omega, \mathbb{R}^n)$, $W^{1,p}_0(\Omega, \mathbb{R}^n)$ are the usual Lebegue and Sobolev spaces of vector functions with values in $\mathbb{R}^n$; $L^p(\Omega) = L^p(\Omega, \mathbb{R})$, e.t.c.; the norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$ we denote by $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{(1),p,\Omega}$, respectively;

$(W^{1,p}_0(\Omega))^*$ is the conjugate to $W^{1,p}_0(\Omega)$;

$J^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega; \mathbb{R}^n) : \mathrm{div} v = 0\}$;

$J^{\infty}_0(\Omega) = \{v \in C^{\infty}_0(\Omega; \mathbb{R}^n) : \mathrm{div} v = 0\}$;

$J^{1,p}_0(\Omega)$ is the closure of $J^{\infty}_0(\Omega)$ in $W^{1,p}(\Omega; \mathbb{R}^n)$.

For domains Ω with sufficiently smooth boundaries $J^{1,p}_0(\Omega) = J^{1,p}(\Omega) \cap W^{1,p}_0(\Omega)$.

We use also notations $|\omega| = \mathrm{meas}\,\omega$, $(f)_\omega = \frac{1}{|\omega|}\int_\omega f dx$. Unless otherwise stated constants C (that can be different from line to line) can depend only on ν_i, $\varkappa_i$, μ_i, k_i, δ_i, p, q, and n.

We assume the following conditions on the data of the problem:

$$f \in L^{p'}(\Omega, \mathbb{R}^n), \quad g \in L^1(\Omega). \tag{3.5}$$

In this paper we consider the problem (2.1)–(2.5) under the following restrictions on p and q:

$$p > \frac{3n}{n+2}, \tag{3.6}$$

$$q \geqslant n. \tag{3.7}$$

Definition. *We say that (u, π, e) is a weak solution to a system (2.1)–(2.5) iff the functions u, π and e belong to the spaces*

$$u \in J_0^{1,p}(\Omega), \qquad \pi \in L^{p'}(\Omega),$$

$$e \in W_0^{1,r}(\Omega) \qquad \forall\, r < \frac{n}{n-1}(q-1)$$

and satisfy the integral identities

$$\int_\Omega \Big(\mu(e)a(\varepsilon(u)) : \varepsilon(w) - u \otimes u : \nabla w\Big) dx =$$

$$= \int_\Omega (f \cdot w + \pi \operatorname{div} w) dx \quad \forall\, w \in W_0^{1,p}(\Omega; \mathbb{R}^n), \tag{3.8}$$

$$\int_\Omega \Big(k(e)b(\nabla e) \cdot \nabla \eta + u \cdot \nabla e \eta\Big) dx =$$

$$= \int_\Omega \Big(\mu(e)a(\varepsilon(u)) : \varepsilon(u)\eta + g\eta\Big) dx \quad \forall\, \eta \in W_0^{1,s'}(\Omega), \quad \forall\, s < \frac{n}{n-1}. \tag{3.9}$$

Remark 3.1. Note that all the terms in the integral identity (3.9) make sense froviding (3.5) holds. Indeed, to the first one we have $b(\nabla e) \in L^{\frac{r}{q-1}}(\Omega, \mathbb{R}^n)$ and $s' = \frac{r}{r-q+1}$; to the second one, for $p \leqslant n$ we have an imbedding $u \in J^{1,p}(\Omega) \hookrightarrow L^{\frac{pn}{n-p}}(\Omega, \mathbb{R}^n)$ where $\frac{pn}{n-p} > n$ if $n = 2, 3$ and (3.5) hold; while for the terms on the right hand side it is sufficient to remind that $W_0^{1,s'}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Existence of a weak solution to system (2.1)–(2.5) in stationary as well as in nonstationary cases was proved in [5–6]. In the present paper we investigate regularity of these weak solutions. Our model problem is a two-dimensional problem describing the motion of generalized Newtonian fluids under Fourier law of heat conductivity (case $n = 2$, $q = 2$, $p > \frac{3}{2}$). Our main result is the following

Theorem 3.1. *Assume that conditions (3.1)–(3.5) hold and suppose additionaly*

$$f \in L^{p_1}(\Omega, \mathbb{R}^n), \quad \textit{for some} \quad p_1 > p_0 \equiv \max\{2, p'\}; \tag{3.10}$$

$$g \in L^{p_2}(\Omega), \quad \textit{for some} \quad p_2 > 1. \tag{3.11}$$

Then there is at least one weak solution (u, π, e) *to the system* (2.1)–(2.5) *such that*

$$(1 + |\varepsilon(u)|)^{\frac{p-2}{2}} |\nabla \varepsilon(u)| \in L^2_{\mathrm{loc}}(\Omega). \tag{3.12}$$

Theorem 3.2. *Let all conditions of Theorem* 3.1 *be satisfied. Assume additionaly that* $n = 2$ *and*

$$g \in L^{2+\delta_1}(\Omega) \quad \textit{for some} \quad \delta_1 > 0. \tag{3.13}$$

Then for every weak solution (u, π, e) *which satisfies* (3.12) *the following assertions hold*

$$\nabla u \in C^{\delta}_{\mathrm{loc}}(\Omega, \mathbb{M}^{n\times n}), \qquad \nabla e \in C^{\delta}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$$

with some $\delta > 0$.

To prove Theorem 3.1 we introduce the following lemma:

Lemma 3.1. *Suppose conditions* (3.1), (3.3), and (3.6) *be satisfied. Suppose also that*

$$f \in L^{p_0}(\Omega; \mathbb{R}^n),$$

and that

$$\nabla e \in L^{n+\delta_2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n), \quad \textit{for some} \quad \delta_2 > 0. \tag{3.14}$$

Let the functions $u \in J^{1,p}(\Omega)$, $\pi \in L^{p'}(\Omega)$ *satisfy the integral identity* (3.8)*. Then* (3.12) *is true.*

Note that in case of growth condition $q > n$ by our definition of a weak solution the assertion (3.14) is satisfied automaticaly for any weak solution (u, π, e) to the system (2.1)–(2.5). But in case of $q = n$ additional considerations are necessary. From our point of view the case of $q = n$ is the most interesting one because for two-dimensional problem it includes Fourier law of heat conductivity.

In the case of $q = n$ we prove the following lemmas:

Lemma 3.2. *Suppose conditions* (3.1), (3.3), and (3.6) *be satisfied. Assuming that*

$$f \in L^{p'+\delta_3}(\Omega, \mathbb{R}^n), \quad \textit{for some} \quad \delta_3 > 0. \tag{3.15}$$

Suppose also that e *be an arbitrary measurable function and let functions* $u \in J^{1,p}(\Omega)$, $\pi \in L^{p'}(\Omega)$ *satisfy the integral identity* (3.8) *for the given* e*. Then*

$$\nabla u \in L^{p+\delta_4}_{\mathrm{loc}}(\Omega, \mathbb{M}^{n\times n}) \tag{3.16}$$

for some $\delta_4 \in (0, \delta_3)$. *Moreover if* $\partial\Omega$ *is of class* C^2 *and* $u \in \overset{\circ}{J}{}^{1,p}_0(\Omega)$ *then*

$$\nabla u \in L^{p+\delta_4}(\Omega, \mathbb{M}^{n\times n}). \tag{3.17}$$

Lemma 3.3. *Assume that* $q = n$, (3.1)–(3.4) *and* (3.17) *hold. Suppose also that*

$$g \in L^{1+\delta_5}(\Omega), \quad \text{for some} \quad \delta_5 > 0. \tag{3.18}$$

Then there exists a function $e \in W^{1,n}_0(\Omega)$ *satisfying* (3.9).

Lemma 3.4. *Suppose that* $q = n$, *conditions* (3.1)–(3.4) *be satisfied and also assertions* (3.16) *and* (3.18) *be true. Then for any function* $e \in W^{1,n}(\Omega)$ *satisfying* (3.9) *we have*

$$\nabla e \in L^{n+\delta_6}_{\text{loc}}(\Omega, \mathbb{R}^n) \tag{3.19}$$

for some $\delta_6 > 0$.

It remains to prove these four lemmas to conclude Theorem 3.1.

The outline of the paper is the following: in the Sec. 4 we prove Lemma 3.1. For this purpose we consider an approximation of e by smooth functions. Then we construct an approximation of u by functions u^m that have summable second derivatives. Using some multiplicative inequalities we get for u^m an estimate of type (3.12) which is uniform with respect to m. The last step in the proof of Lemma 3.1 is to pass to the limit in this estimate as $m \to \infty$. In the Sec. 5 we prove Lemmas 3.2–3.4. To prove Lemmas 3.2, 3.4 we derive a reverse Hölder inequality for functions ∇u, ∇e, from which their higher summability follows via theorems developed in [14, 15]. In the Sec. 6 we give a sketch and some references for the proof of Theorem 3.2.

4. Proof of the Lemma 3.1

Through out this section constants C besides ν_i, $\varkappa_i$, μ_i, k_i, p, q, n can depend also on Ω and the radius R that shall be chosen below.

Let $u \in \overset{\circ}{J}{}^{1,p}_0(\Omega)$, $\pi \in L^{p'}(\Omega)$, $e \in W^{1,n+\delta_2}(\Omega)$ satisfy the integral identity (3.8). Consider approximations of (3.8). Let $e^m \in C^1(\bar\Omega)$ such that $e^m \to e$ in $W^{1,n+\delta_2}(\Omega)$. For any m there are functions $u^m \in \overset{\circ}{J}{}^{1,p}_0(\Omega)$, $\pi^m \in L^{p'}(\Omega)$, such that

$$\begin{aligned}&\int\limits_\Omega \mu(e^m)a(\varepsilon(u^m)) : \varepsilon(w)dx = \\ &= \int\limits_\Omega (u^m \otimes u^m : \varepsilon(w) + f\cdot w + \pi^m \operatorname{div} w)dx, \quad \forall\, w \in W^{1,p}_0(\Omega).\end{aligned} \tag{4.1}$$

In this section we shall denote by K constants (which can be different from line to line) that can depend on f but do not depend on m. Taking in (4.1) a test function $w = u^m$ we get the estimate

$$\left\|\nabla u^m\right\|_{p,\Omega}^{p} \leqslant C\left\|f\right\|_{p',\Omega}^{p'}. \tag{4.2}$$

We remind here the following result of [3] (see also [13]).

Lemma 4.1. *Let $f \in (W_0^{1,p}(\Omega, \mathbb{R}^n))^*$ such that*

$$\langle f, w\rangle = 0, \quad \forall\, w \in J_0^{1,p}(\Omega).$$

Then there is a unique function $\pi \in L^{p'}(\Omega)$, $\int_\Omega \pi dx = 0$ such that

$$\langle f, w\rangle = \int\limits_\Omega \pi \operatorname{div} w dx, \quad \forall w \in W_0^{1,p}(\Omega; \mathbb{R}^n)$$

and the estimate

$$\left\|\pi\right\|_{p',\Omega} \leqslant C(\Omega)\left\|f\right\|_{(W_0^{1,p}(\Omega,\mathbb{R}^n))^*}$$

is true with constant C depending only on Ω.

This lemma together with (4.2) and (3.3), and the inequality

$$\left|a(\varepsilon(u^m))\right| \leqslant C(1 + |\varepsilon(u^m)|)^{p-1} \tag{4.3}$$

(which follows from (3.1)) imply the estimate

$$\left\|\pi^m\right\|_{p',\Omega} \leqslant C\Big(\left\|f\right\|_{p',\Omega} + 1\Big). \tag{4.4}$$

Using finite-difference technique it can be shown that for any $m \in \mathbb{N}$

$$\begin{gathered}(1 + |\varepsilon(u^m)|)^{\frac{p-2}{2}}|\nabla\varepsilon(u^m)| \in L^2_{\mathrm{loc}}(\Omega),\\ (1 + |\varepsilon(u^m)|)^{\frac{2-p}{2}}|\nabla\pi^m| \in L^2_{\mathrm{loc}}(\Omega).\end{gathered} \tag{4.5}$$

Now we claim to the estimate

$$\sup_m \int\limits_{\Omega'} (1 + |\varepsilon(u^m)|)^{p-2}|\nabla\varepsilon(u^m)|^2 dx < +\infty, \tag{4.6}$$

which holds for any $\Omega' \Subset \Omega$. We split the prove of (4.6) into two parts.

Proof of (4.6). Part 1: $p \geq 2$. In this case we assume $f \in L^2(\Omega)$. From (4.1) and (4.5) the identity follows

$$\int_\Omega \mu(e^m)\frac{\partial a}{\partial \tau}(\varepsilon(u^m))\varepsilon(u^m_{,k}) : \varepsilon(w)dx =$$

$$= -\int_\Omega \mu'(e^m)e^m_{,k}a(\varepsilon(u^m)) : \varepsilon(w)dx + \int_\Omega \Big((u^m \otimes u^m)_{,k} : \varepsilon(w) - \tag{4.7}$$

$$-f \cdot w_{,k} + \pi^m_{,k} \ \operatorname{div} w\Big)dx, \quad \forall w \in C_0^\infty(\Omega, \mathbb{R}^n).$$

Consider $B_{2R}(x_0) \Subset \Omega$ and $\varphi \in C_0^\infty(B_{2R})$, $\varphi \equiv 1$ on B_R, $0 \leqslant \varphi \leqslant 1$.

Choosing in (4.7) $w = \varphi^2 u^m_{,k}$ (taking into account (4.5) this substitution can be easily justified) we get

$$I_0 \equiv \int_\Omega \varphi^2\mu(e^m)\frac{\partial a}{\partial \tau}(\varepsilon(u^m))\varepsilon(u^m_{,k}) : \varepsilon(u^m_{,k})dx = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = -\int_\Omega \varphi^2\mu'(e^m)e^m_{,k}a(\varepsilon(u^m)) : \varepsilon(u^m_{,k}))dx,$$

$$I_2 = -\int_\Omega \Big(\mu(e^m)a(\varepsilon(u^m))\Big)_{,k} : u^m_{,k} \otimes \nabla\varphi^2 dx,$$

$$I_3 = -\int_\Omega f \cdot (\varphi^2 \Delta u^m + u^m_{,k}\varphi^2_{,k})dx,$$

$$I_4 = \int_\Omega (u^m \otimes u^m)_{,k} : (\varphi^2\varepsilon(u^m_{,k}) + u^m_{,k} \odot \nabla\varphi^2)dx,$$

$$I_5 = \int_\Omega \pi^m_{,k}u^m_{,k} \cdot \nabla\varphi^2 dx.$$

Let us start with the estimate for I_1. By the assumption (3.14) we can choose $s \in (1, +\infty)$ in such a way that $2s' = n + \delta_2$ and note that $2s < n^*$, where $n^* = \frac{2n}{n-2}$ if $n \geqslant 3$ and $n^* = +\infty$ if $n = 2$. Applying Hölder inequality we get

$$|I_1| \leqslant CI_0^{1/2}\Big\|\varphi\nabla e^m\Big(1 + \Big|\varepsilon(u^m)\Big|\Big)^{\frac{2-p}{2}} a(\varepsilon(u^m))\Big\|_{2,B_{2R}} \tag{4.8}$$

and from (4.3) we get

$$|I_1| \leqslant C I_0^{1/2} \left\| \nabla e^m \right\|_{2s', B_{2R}} \Big(1 + \left\| \varphi \varepsilon(u^m) \right\|_{ps, B_{2R}}^{\frac{p}{2}} \Big).$$

Choose $\theta \in (0,1)$ in such a way that $2s = 2(1-\theta) + \widehat{n}\theta$ where $\widehat{n} < n^*$. With help of Hölder inequality we obtain

$$\left\| \varphi \varepsilon(u^m) \right\|_{ps, B_{2R}}^{\frac{p}{2}} = \left\| \varphi^{\frac{p}{2}} |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{2s, B_{2R}} \leqslant$$

$$\leqslant \left\| |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{2, B_{2R}}^{\frac{1-\theta}{s}} \left\| \varphi^{\frac{p}{2}} |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{\widehat{n}, B_{2R}}^{\frac{\widehat{n}\theta}{2s}}.$$

Using Sobolev imbedding theorem we get the estimate

$$\left\| \varphi^{\frac{p}{2}} |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{\widehat{n}, B_{2R}}^{\frac{\widehat{n}\theta}{2s}} \leqslant C \left\| \nabla (\varphi^{\frac{p}{2}} |\varepsilon(u^m)|^{\frac{p}{2}}) \right\|_{2, B_{2R}}^{\frac{\widehat{n}\theta}{2s}} \leqslant$$

$$\leqslant C \Big(\left\| \varphi^{\frac{p}{2}} \nabla |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{2, B_{2R}}^{\frac{\widehat{n}\theta}{2s}} + \left\| |\varepsilon(u^m)|^{\frac{p}{2}} \right\|_{2, B_{2R}}^{\frac{\widehat{n}\theta}{2s}} \Big) \leqslant$$

$$\leqslant C \Big(I_0^{\frac{1}{2}\frac{\widehat{n}\theta}{2s}} + \left\| \varepsilon(u^m) \right\|_{p, B_{2R}}^{\frac{p}{2}\frac{\widehat{n}\theta}{2s}} \Big).$$

Hence we get for I_1 the estimate

$$|I_1| \leqslant C I_0^{\frac{1}{2}} (1 + I_0^{\frac{\widehat{n}\theta}{4s}}) \quad \text{with} \quad \frac{1}{2} + \frac{\widehat{n}\theta}{4s} < 1 \quad \text{as} \quad \widehat{n}\theta < 2s.$$

To study I_2 we split

$$I_2 \equiv - \int\limits_{\Omega} \Big(\mu(e^m) a(\varepsilon(u^m)) \Big)_{,k} : u^m_{,k} \otimes \nabla \varphi^2 dx = I_2' + I_2'',$$

where

$$I_2' = - \int\limits_{\Omega} \mu'(e^m) e^m_{,k} a(\varepsilon(u^m)) : u^m_{,k} \otimes \nabla \varphi^2 dx$$

$$I_2'' = - \int\limits_{\Omega} \mu(e^m) \frac{\partial a}{\partial \tau} (\varepsilon(u^m)) \varepsilon(u^m_{,k}) : u^m_{,k} \otimes \nabla \varphi^2 dx.$$

I_2' can be estimated as follows:

$$|I_2'| \leqslant$$
$$C\Big\|\varphi \nabla e^m \Big(1+\Big|\varepsilon(u^m)\Big|\Big)^{\frac{2-p}{2}} a(\varepsilon(u^m))\Big\|_{2,B_{2R}} \Big\|\Big(1+|\varepsilon(u^m)|\Big)^{\frac{p-2}{2}} \nabla u^m\Big\|_{2,B_{2R}}.$$

The first multiplier in the right hand side of the last inequality is exactly the term given in (4.8). So it is already estimated. Second multiplier less or equal to $C(1+\|\nabla u^m\|^{\frac{p}{2}}_{p,B_{2R}})$.

We estimate I_2'' using inequality (3.1):

$$|I_2''| \leqslant C\Big\|\varphi\Big(1+\Big|\varepsilon(u^m)\Big|\Big)^{\frac{p-2}{2}} \nabla\varepsilon(u^m)\Big\|_{2,B_{2R}} \Big\|(1+|\varepsilon(u^m)|)^{\frac{p-2}{2}} \nabla u^m\Big\|_{2,B_{2R}} \leqslant$$
$$\leqslant C I_0^{1/2}\Big(1+\Big\|\nabla u^m\Big\|^{\frac{p}{2}}_{p,B_{2R}}\Big).$$

For I_3 the following estimate is true

$$|I_3| \leqslant C\Big\|f\Big\|_{2,B_{2R}}\Big(I_0^{1/2}+\Big\|\nabla u^m\Big\|_{2,B_{2R}}\Big).$$

Let us consider now I_4. This estimate requires the restriction on n ($n=2,3$). By Hölder inequality we obtain

$$|I_4| \leqslant C\Big\|\varphi\nabla(u^m\otimes u^m)\Big\|_{2,B_{2R}}\Big(\Big\|\varphi\nabla\varepsilon(u^m)\Big\|_{2,B_{2R}}+\Big\|\nabla u^m\Big\|_{2,B_{2R}}\Big),$$

and then

$$\Big\|\varphi\nabla(u^m\otimes u^m)\Big\|_{2,B_{2R}} \leqslant C\Big\|u^m\Big\|_{4,B_{2R}}\Big\|\varphi\nabla u^m\Big\|_{4,B_{2R}}.$$

By Korn inequality we get

$$\Big\|\varphi\nabla u^m\Big\|_{4,B_{2R}} \leqslant C\Big(\Big\|\varphi\varepsilon(u^m)\Big\|_{4,B_{2R}}+\Big\|u^m\Big\|_{4,B_{2R}}\Big).$$

Reminding that in case of $p\geqslant 2$ and $n=2,3$ by Sobolev theorem we have the imbedding $W^{1,p}(B_{2R})\hookrightarrow L^4(B_{2R})$ we obtain the estimate

$$\Big\|u^m\Big\|_{4,B_{2R}} \leqslant C\Big(\Big\|\nabla u^m\Big\|_{p,B_{2R}}+\Big\|u^m\Big\|_{p,B_{2R}}\Big)$$

and hence (taking into account (4.2)) we have the estimate

$$|I_4| \leqslant K\Big(\Big\|\varphi\varepsilon(u^m)\Big\|_{4,B_{2R}}+1\Big)\big(I_0^{1/2}+1\big).$$

To estimate $\Big\|\varphi\varepsilon(u^m)\Big\|_{4,B_{2R}}$ we use the following multiplicative inequalities (see, for example, [18])

$$\Big\|\varphi\varepsilon(u^m)\Big\|_{4,B_{2R}} \leqslant \begin{cases} C\Big\|\nabla u^m\Big\|^{\frac{1}{2}}_{2,B_{2R}}\Big\|\nabla(\varphi\varepsilon(u^m))\Big\|^{\frac{1}{2}}_{2,B_{2R}} \\ C\Big\|\nabla u^m\Big\|^{\frac{1}{4}}_{2,B_{2R}}\Big\|\nabla(\varphi\varepsilon(u^m))\Big\|^{\frac{3}{4}}_{2,B_{2R}}. \end{cases}$$

Hence $|I_4| \leqslant K(I_0^\alpha+1)(I_0^{1/2}+1)$ with $\alpha=\frac{1}{4}$ if $n=2$ and $\alpha=\frac{3}{8}$ if $n=3$.

To estimate I_5 we find $\nabla\pi^m$ from the equations

$$\nabla\pi^m = \operatorname{div}\Big(\mu(e^m)a(\varepsilon(u^m))\Big) - f - (\nabla u^m)u^m \quad \text{a. e. in} \quad \Omega$$

and then we continue in the same way to I_2–I_4.

Gathering all the estimates for I_1–I_5 together we get an inequality

$$I_0 \leqslant K(I_0^\alpha+1) \quad \text{with some} \quad \alpha<1$$

from which the statement (4.6) follows.

Proof of (4.6) Part 2. : $\frac{3n}{n+2} < p < 2$ $(n=2,3)$. In this case we assume that $f\in L^{p'}(\Omega)$.

Taking in (4.7) the same test function $w=\varphi^2 u^m_{,k}$ we obtain

$$I_0 \equiv \int\limits_\Omega \varphi^2\mu(e^m)\frac{\partial a}{\partial\tau}(\varepsilon(u^m))\varepsilon(u^m_{,k}) : \varepsilon(u^m_{,k})dx = I_1+I_2+I_3+I_4+I_5,$$

where

$$\begin{aligned} I_1 &= -\int\limits_\Omega \varphi^2\mu'(e^m)e^m_{,k}a(\varepsilon(u^m)) : \varepsilon(u^m_{,k}))dx, \\ I_2 &= \int\limits_\Omega \mu(e^m)a(\varepsilon(u^m)) : (u^m_{,k}\otimes\nabla\varphi^2)_{,k}dx, \\ I_3 &= -\int\limits_\Omega (\varphi^2 f\cdot\Delta u^m + u^m_{,k}\varphi^2_{,k})dx, \\ I_4 &= \int\limits_\Omega (u^m\otimes u^m)_{,k} : (\varphi^2\varepsilon(u^m_{,k}) + u^m_{,k}\odot\nabla\varphi^2)dx, \\ I_5 &= -\int\limits_\Omega \pi^m(u^m_{,k}\cdot\nabla\varphi^2)_{,k}dx. \end{aligned}$$

I_1 can be estimated in the same way as in case of $p \geqslant 2$.

For I_2 the following estimate is true:

$$|I_2| \leqslant C\Big\|a(\varepsilon(u^m))\Big\|_{p',B_{2R}}\Big\|(u^m_{,k}\otimes\nabla\varphi^2)_{,k}\Big\|_{p,B_{2R}}. \tag{4.9}$$

Using Korn inequality one we can show that

$$\Big\|(u^m_{,k}\otimes\nabla\varphi^2)_{,k}\Big\|_{p,B_{2R}} \leqslant C\Big(\Big\|\varphi\nabla\varepsilon(u^m)\Big\|_{p,B_{2R}} + \Big\|\nabla u^m\Big\|_{p,B_{2R}}\Big). \tag{4.10}$$

Note that in case of $p<2$ due to Hölder inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ the following estimate holds for any measurable functions h and S

$$\Big(\int\limits_\omega |S|^p\,dx\Big)^{\frac{1}{p}} \leqslant \Big(\int\limits_\omega (1+|h|)^p\,dx\Big)^{\frac{1}{p}-\frac{1}{2}}\Big(\int\limits_\omega \frac{|S|^2}{(1+|h|)^{2-p}}dx\Big)^{\frac{1}{2}} \tag{4.11}$$

providing the right-hand side is well defined.

Taking in (4.11) $h=\varepsilon(u^m)$ and $S=\nabla\varepsilon(u^m)$ we obtain the estimate

$$\Big\|\varphi\nabla\varepsilon(u^m)\Big\|_{p,B_{2R}} \leqslant CI_0^{1/2}\Big(1+\Big\|\nabla u^m\Big\|^{1-\frac{p}{2}}_{p,B_{2R}}\Big). \tag{4.12}$$

Taking into account (4.2) and (4.3), from (4.9), (4.10), and (4.12) we obtain the inequality

$$|I_2| \leqslant K(I_0^{1/2}+1).$$

Let us consider I_3:

$$|I_3| \leqslant C\Big\|f\Big\|_{p',B_{2R}}(\Big\|\varphi\Delta u^m\Big\|_{p,B_{2R}} + \Big\|\nabla u^m\Big\|_{p,B_{2R}}).$$

Using again the inequality (4.12) we get

$$\Big\|\varphi\Delta u^m\Big\|_{p,B_{2R}} \leqslant CI_0^{\frac{1}{2}}\Big(1+\Big\|\nabla u^m\Big\|^{1-\frac{p}{2}}_{p,B_{2R}}\Big).$$

For I_4 we have

$$|I_4| \leqslant C\Big\|\varphi\nabla(u^m\otimes u^m)\Big\|_{p',B_{2R}}\Big(\Big\|\varphi\nabla\varepsilon(u^m)\Big\|_{p,B_{2R}} + \Big\|\varphi\nabla u^m\Big\|_{p,B_{2R}}\Big). \tag{4.13}$$

Applying Hölder inequality we obtain

$$\Big\|\varphi\nabla(u^m\otimes u^m)\Big\|_{p',B_{2R}} \leqslant C\Big\|u^m\Big\|_{2p',B_{2R}}\Big\|\varphi\nabla u^m\Big\|_{2p',B_{2R}}.$$

Using Korn inequality it follows

$$\Big\|\varphi\nabla u^m\Big\|_{2p',B_{2R}} \leqslant C\Big(\Big\|\varphi\varepsilon(u^m)\Big\|_{2p',B_{2R}} + \Big\|u^m\Big\|_{2p',B_{2R}}\Big).$$

To estimate $\Big\|u^m\Big\|_{2p',B_{2R}}$ we use the imbedding $W^{1,p}(B_{2R}) \hookrightarrow L^{2p'}(B_{2R})$ which is valid providing that (3.6) holds. So we get the estimate

$$\Big\|u^m\Big\|_{2p',B_{2R}} \leqslant C\Big(\Big\|\nabla u^m\Big\|_{p,B_{2R}} + \Big\|u^m\Big\|_{p,B_{2R}}\Big).$$

So from (4.13) taking into account (4.2), (4.12), and the last inequality we obtain the estimate $|I_4| \leqslant K\Big(\Big\|\varphi\varepsilon(u^m)\Big\|_{2p',B_{2R}} + 1)(I_0^{1/2}+1\Big)$.

To estimate $\Big\|\varphi\varepsilon(u^m)\Big\|_{2p',B_{2R}}$ we choose $\theta \in (0,1)$ in such a way that $2p' = p(1-\theta) + \frac{np}{n-p}\theta$. (It is possible due to the restriction (3.6).) Then by Hölder inequality we have

$$\Big\|\varphi\varepsilon(u^m)\Big\|_{2p',B_{2R}} \leqslant \Big\|\varepsilon(u^m)\Big\|_{p,B_{2R}}^{p\frac{1-\theta}{2p'}}\Big\|\varphi\varepsilon(u^m)\Big\|_{\frac{np}{n-p},B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}}.$$

Applying Sobolev imbedding theorem to the second multiplier in the right-hand side of the last relation we obtain

$$\Big\|\varphi\varepsilon(u^m)\Big\|_{\frac{np}{n-p},B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}} \leqslant C\Big\|\nabla(\varphi\varepsilon(u^m))\Big\|_{p,B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}} \leqslant$$

$$\leqslant C\Big(\Big\|\varphi\nabla\varepsilon(u^m)\Big\|_{p,B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}} + \Big\|\varepsilon(u^m)\Big\|_{p,B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}}\Big).$$

Using inequality (4.12) one more time we get the estimate

$$\Big\|\varphi\varepsilon(u^m)\Big\|_{2p',B_{2R}} \leqslant C\Big\|\nabla u^m\Big\|_{p,B_{2R}}^{p\frac{1-\theta}{2p'}}\Big(I_0^{\frac{1}{2}\frac{np}{n-p}\frac{\theta}{2p'}}\Big(1+\Big\|\nabla u^m\|_{p,B_{2R}}^{1-\frac{p}{2}}\Big)^{\frac{np}{n-p}\frac{\theta}{2p'}} +$$

$$+\Big\|\nabla u^m\Big\|_{p,B_{2R}}^{\frac{np}{n-p}\frac{\theta}{2p'}}\Big).$$

So for I_4 we have the estimate $|I_4| \leqslant K(I_0^\alpha+1)(I_0^{\frac{1}{2}}+1)$ with $\alpha = \frac{np}{n-p}\frac{\theta}{4p'}$ and $\frac{1}{2} + \frac{np}{n-p}\frac{\theta}{4p'} < 1$ if (3.6) is valid.

We are left to estimate I_5:

$$|I_5| \leqslant \|\pi^m\|_{p',B_{2R}}\|(u^m_{,k}\cdot\nabla\varphi^2)_{,k}\|_{p,B_{2R}},$$

and taking into account (4.4) we continue in the same way to I_2.

Gathering all the estimates for I_1–I_5 together for the case $\frac{3n}{n+2} < p < 2$ we again get the inequality

$$|I_0| \leqslant K(I_0^\alpha + 1) \quad \text{with some} \quad \alpha < 1.$$

The statement (4.6) is proved.

The last step in the proof of Lemma 3.1 is to pass to the limit in (4.6) as $m \to \infty$. From (4.2), (4.4), and (4.3) we have the estimate

$$\|\nabla u^m\|_{p,\Omega} + \|\sigma^m\|_{p',\Omega} + \|\pi^m\|_{p',\Omega} \leqslant K.$$

Note that we have the estimate

$$\Big\|\nabla \varepsilon(u^m)\Big\|_{p_1,\Omega'} \leqslant K, \quad \text{where } p_1 = \begin{cases} 2, & p \geqslant 2 \\ p, & p < 2. \end{cases} \tag{4.14}$$

Really this estimate is obvious in case of $p \geqslant 2$. In case of $p < 2$ it follows from (4.12).

Let us introduce the notations

$$G^m = a(\varepsilon(u^m)), \quad H^m = (1 + |\varepsilon(u^m)|^2)^{\frac{p}{4}}, \quad \sigma^m = \mu(e^m)G^m. \tag{4.15}$$

Similarly to (4.14) we have

$$\Big\|\nabla G^m\Big\|_{p_2,\Omega'} \leqslant C, \quad \text{where } p_2 = \begin{cases} p', & p \geqslant 2 \\ 2, & p < 2. \end{cases}$$

In case of $p \geqslant 2$ the last estimate follows from inequality (that is dual to (4.11))

$$\Big(\int_\omega |S|^{p'} dx\Big)^{\frac{1}{p'}} \leqslant \Big(\int_\omega (1+|h|)^p dx\Big)^{\frac{1}{p'} - \frac{1}{2}} \Big(\int_\omega \frac{|S|^2}{(1+|h|)^{p-2}} dx\Big)^{\frac{1}{2}}, \tag{4.16}$$

which holds for any h and S providing the right-hand side is well defined. We apply (4.16) to functions $h = \varepsilon(u^m)$ and $S = \nabla G^m$ and note that

$$\frac{|\nabla G^m|^2}{(1 + |\varepsilon(u^m)|)^{p-2}} \leqslant C(1 + |\varepsilon(u^m)|)^{p-2} |\nabla \varepsilon(u^m)|^2.$$

(The last relation follows from (3.1).) From the last inequality also follows

$$\|\nabla H^m\|_{2,\Omega'} \leqslant C.$$

From these estimates follow that there are subsequences of $\{u^m\}$, $\{\pi^m\}$ and $\{e^m\}$ (for which we keep the same notations) such that

$$\begin{gathered} u^m \rightharpoonup u \quad \text{in} \quad W^{1,p}(\Omega,\mathbb{R}^n), \quad \sigma^m \rightharpoonup \sigma \quad \text{in} \quad L^{p'}(\Omega,\mathbb{M}^{n\times n}), \\ \pi^m \rightharpoonup \pi \quad \text{in} \quad L^{p'}(\Omega), \quad e^m \to e \quad \text{in} \quad W^{1,n+\delta_2}(\Omega) \end{gathered} \tag{4.17}$$

$$\begin{gathered} \varepsilon(u^m) \rightharpoonup \varepsilon(u) \quad \text{in} \quad W^{1,p_1}(\Omega',\mathbb{M}^{n\times n}), \\ G^m \rightharpoonup G \text{ in } W^{1,p_2}(\Omega'), \quad H^m \rightharpoonup H \text{ in } W^{1,2}(\Omega') \end{gathered} \tag{4.18}$$

$$\begin{gathered} \varepsilon(u^m) \to \varepsilon(u) \quad \text{a.e. in} \quad \Omega', \quad G^m \to G \quad \text{a.e. in} \quad \Omega', \\ H^m \to H \quad \text{a.e. in} \quad \Omega', \quad e^m \to e \quad \text{a.e. in} \quad \Omega'. \end{gathered} \tag{4.19}$$

As $2p' < \frac{np}{n-p}$ providing (3.6) holds from Kondrashev–Relich–Sobolev theorem we get the strong convergence $u^m \to u$ in $L^{2p'}(\Omega',\mathbb{R}^n)$ and hence

$$u^m \otimes u^m \to u \otimes u \quad \text{in} \quad L^{p'}(\Omega',\mathbb{M}^{n\times n}). \tag{4.20}$$

Assertions (4.17) and (4.20) make possible to pass to the limit in the integral identity (4.1). So we get

$$\int_\Omega \sigma : \varepsilon(w)dx = \int_\Omega (u \otimes u : \varepsilon(w) + f\cdot w + \pi \operatorname{div}\ w)dx, \quad \forall w \in C_0^\infty(\Omega,\mathbb{R}^n).$$

From convergence (4.19), (4.15) and continuity of $a(\cdot)$, $\mu(\cdot)$ follows that

$$\sigma = \mu(e)G = \mu(e)a(\varepsilon(u)) \quad \text{a.e. in} \quad \Omega'$$

and also

$$H = (1+|\varepsilon(u)|^2)^{\frac{p}{2}} \quad \text{a.e. in} \quad \Omega'.$$

Hence (u,π,e) is a solution to the (3.8). As $H \in W^{1,2}(\Omega')$ we obtain

$$\int_{\Omega'} (1+|\varepsilon(u)|^2)^{\frac{p-2}{2}} |\nabla\varepsilon(u)|^2 dx < +\infty.$$

Lemma 3.1 is proved.

5. Proof of Lemmas 3.2–3.4

Throught out this section constants C can depend only on ν_i, $\varkappa_i$, μ_i, δ_i, p, q, n and do not depend on R and $x_0 \in \bar{\Omega}$.

Proof of Lemma 3.2. The idea of the prove is to get for $|\nabla u|$ a reverse Hölder inequality (see [15]), from which higher integrability of $|\nabla u|$ follows via Gehring's lemma ([14], see also [15]).

We start with the proof of (3.16). For every $x_0 \in \Omega$ we consider a ball $B_{2R}(x_0) \Subset \Omega$ and also a usual cut-off function $\varphi \in C_0^\infty(B_{2R}(x_0))$:

$$0 \leqslant \varphi \leqslant 1, \quad \varphi \equiv 1 \quad \text{on} \quad B_R, \quad |\nabla \varphi| \leqslant \frac{C}{R}. \tag{5.1}$$

Let d be an arbitrary vector and π_0 be an arbitrary number.

Choosing in (3.8) as a test function $v = \varphi^p(u-d)$ we obtain the identity

$$I_0 \equiv \int\limits_{B_{2R}(x_0)} \varphi^p \mu(e) a(\varepsilon(u)) : \varepsilon(u) dx = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = -\int\limits_{B_{2R}(x_0)} \mu(e) a(\varepsilon(u)) : (u-d) \otimes \nabla \varphi^p dx,$$

$$I_2 = \int\limits_{B_{2R}(x_0)} u \otimes u : (\varphi^p \nabla u + (u-d) \otimes \nabla \varphi^p) dx,$$

$$I_3 = \int\limits_{B_{2R}(x_0)} \varphi^p f \cdot (u-d) dx,$$

$$I_4 = \int\limits_{B_{2R}(x_0)} (\pi - \pi_0)(u-d) \cdot \nabla \varphi^p dx.$$

From condition (3.1) follows that

$$(a(\tau) - a(0)) : \tau \geqslant \frac{\nu_0}{p-1}[(1+|\tau|)^{p-1}|\tau| - 1].$$

Hence

$$a(\tau) : \tau \geqslant \frac{\nu_0}{2(p-1)}|\tau|^p - C$$

and

$$I_0 \geq \frac{\mu_0 \nu_0}{2(p-1)} \left\| \varphi \varepsilon(u) \right\|_{p, B_{2R}}^p - CR^n.$$

(We denoted by $B_{2R} = B_{2R}(x_0)$.) Taking in Korn inequality

$$\Big\|\nabla v\Big\|^p_{p,B_{2R}} \leqslant C(n,p)\Big\|\varepsilon(v)\Big\|^p_{p,B_{2R}}, \quad \forall\, v \in W^{1,p}_0(B_{2R},\mathbb{R}^n)$$

the test function $v = \varphi(u-d)$ we get the inequality

$$\Big\|\varphi\nabla u\Big\|^p_{p,B_{2R}} \leqslant C\Big(\Big\|\varphi\varepsilon(v)\Big\|^p_{p,B_{2R}} + \frac{1}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}}\Big).$$

For I_1 we have the estimate

$$|I_1| \le \frac{C}{R}\Big(\int\limits_{B_R} \varphi^p|a(\varepsilon(u))|^{p'}dx\Big)^{1/p'}\|u-d\|_{p,B_{2R}}.$$

Using (4.3) and Young inequality we obtain

$$|I_1| \le \theta\int\limits_{B_{2R}}(1+|\varepsilon(u)|)^p dx + \frac{C(\theta)}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}},$$

where the number $\theta > 0$ can be chosen arbitrary small.

Similarly to I_1 for I_2 and I_3 we get the estimates

$$|I_2| \underset{\varphi^p\le\varphi}{\le} \Big\|u\Big\|^2_{2p',B_{2R}}\Big\|\varphi\nabla u\Big\|_{p,B_{2R}} + \Big\|u\Big\|^2_{2p',B_{2R}}\frac{C}{R}\Big\|u-d\Big\|_{p,B_{2R}} \leqslant$$

$$\le \theta\Big\|\varphi\nabla u\Big\|^p_{p,B_{2R}} + C(p,\theta)\Big\|u\Big\|^{2p'}_{2p',B_{2R}} + \frac{C}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}}$$

and

$$|I_3| \le \Big\|f\Big\|^{p'}_{p',B_{2R}} + \frac{C}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}} \quad \text{if} \quad R<1.$$

Let us consider I_4:

$$|I_4| \le \theta_1\Big\|\pi-\pi_0\Big\|^{p'}_{p',B_{2R}} + \frac{C(\theta_1)}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}},$$

where the number $\theta_1 > 0$ will be chosen later.

Gathering estimates of I_1–I_4 together and choosing θ small enough we get the inequality

$$\begin{aligned}\Big\|\nabla u\Big\|^p_{p,B_R} \le \frac{C(\theta_1)}{R^p}\Big\|u-d\Big\|^p_{p,B_{2R}} + C\Big\|u\Big\|^{2p'}_{2p',B_{2R}} + \\ +C\Big\|f\Big\|^{p'}_{p',B_{2R}} + \theta_1\Big\|\pi-\pi_0\Big\|^{p'}_{p',B_{2R}}.\end{aligned} \tag{5.2}$$

Next we shall exclude the pressure from the right-hand side of (5.2). Fix now $\pi_0 = (\pi)_{B_{2R}(x_0)}$. Unfortunatly, the constant $C(\Omega)$ given in Lemma 4.1 depends on Ω. So we shall use the following modification of Lemma 4.1 that can be derived from Lemma 4.1 via simple rescaling arguments:

Lemma 5.1. *Assume that for* $\pi \in L^{p'}(\Omega)$, $F \in L^{p'}(\Omega, \mathbb{M}^{n\times n})$, $G \in L^{p'}(\Omega, \mathbb{R}^n)$ *the following identity holds:*

$$\int\limits_{B_R} (\pi - (\pi)_{B_R}) \operatorname{div} w dx = \int\limits_{B_R} (F : \nabla w + G \cdot w) dx, \quad \forall\, w \in W_0^{1,p}(B_R).$$

Then there is a constant C *depending only on* n, p *such that the estimate*

$$\|\pi - (\pi)_{B_R}\|_{p',B_R} \leqslant C(n,p)(\|F\|_{p',B_R} + R\|G\|_{p',B_R})$$

holds.

From Lemma 5.1 the estimate follows

$$\Big\|\pi - \pi_0\Big\|_{p',B_{2R}} \le C_1\Big(\mu_1 \Big\|a(\varepsilon(u))\Big\|_{p',B_{2R}} + \Big\|u\Big\|^2_{2p',B_{2R}} + R\Big\|f\Big\|_{p',B_{2R}}\Big),$$

where $C_1 = C_1(n,p)$. Introducing this estimate in (5.2), taking $d = (u)_{B_{2R}(x_0)}$, and recalling the Gagliardo inequality

$$\Big\|u - (u)_{B_{2R}}\Big\|_{p,B_{2R}} \leqslant C(n,p)\Big\|\nabla u\Big\|_{\frac{np}{n+p},B_{2R}}$$

(in which constant C depends only on n and p), we get the estimate

$$\Big\|\nabla u\Big\|^p_{p,B_R} \le \frac{C}{R^p}\Big\|\nabla u\Big\|^p_{\frac{pn}{n+p},B_{2R}} + C(\theta_1)\Big\{\Big\|u\Big\|^{2p'}_{2p',B_{2R}} + \Big\|f\Big\|^{p'}_{p',B_{2R}} + R^n\Big\} +$$
$$+\theta_1 C_1 \mu_1 \nu_1 \Big\|\nabla u\Big\|^p_{p,B_{2R}}.$$

In order to apply the global version on higher summability, we recognize

$$\frac{1}{R^n}\int\limits_{B_R} |\nabla u|^p \le C\Big(\frac{1}{R^n}\int\limits_{B_{2R}} |\nabla u|^{pn/(n+p)} dx\Big)^{\frac{n+p}{n}} +$$
$$+C(\theta_1)\Big(\frac{1}{R^n}\int\limits_{B_{2R}} |u|^{2p'} dx + \frac{1}{R^n}\int\limits_{B_{2R}} |f|^{p'} dx\Big) + \tag{5.3}$$

$$+\theta_1 C_1 \mu_1 \nu_1 \frac{1}{R^n} \int\limits_{B_{2R}} |\nabla u|^p dx + C.$$

Let us choose now the number $\theta_1 > 0$ in such a way that $\theta_1 C_1 \mu_1 \nu_1 < 1$. Now we can apply the following generalization of Ghering lemma (see [15], p. 122):

Lemma 5.2. *Let be $\delta_7 > 0$ and positive functions $\Phi \in L^p(\Omega)$ and $\Psi \in L^{p+\delta_7}(\Omega)$ satisfy the estimate*

$$\Big(\frac{1}{R^n} \int\limits_{B_{\alpha R}(x_0)} \Phi^p dx\Big)^{1/p} \leqslant B_1 \Big(\frac{1}{R^n} \int\limits_{B_R(x_0)} \Phi^q dx\Big)^{1/q} +$$
$$+B_2 \Big(\frac{1}{R^n} \int\limits_{B_R(x_0)} \Psi^p dx\Big)^{1/p} + \theta \Big(\frac{1}{R^n} \int\limits_{B_R(x_0)} \Phi^p dx\Big)^{1/p}$$

for all $x_0 \in \Omega$, $R < \frac{1}{2}\mathrm{dist}\,(x_0, \partial\Omega)$ with some constant $\alpha \in (0,1)$, $\theta \in (0,1)$, $q < p$ and positive B_1, B_2. Then there is $\delta \in (0, \delta_7)$ such that $\Phi \in L^{p+\delta}_{\mathrm{loc}}(\Omega)$.

From (5.3) the desired result follows with help of Lemma 5.2 if we take $\Phi = |\nabla u|$ and $\Psi = |u|^{\frac{2p'}{p}} + |f|^{\frac{p'}{p}} + 1$. Due to (3.6) and (3.15) we have $\Psi \in L^{p+\delta_7}(\Omega)$ for $\delta_7 = \min\{\delta_3, \frac{np}{n-p} - 2p'\}$. The first statement of Lemma 3.2 is proved.

Let us turn to the proof of (3.17). Consider now a point $x_0 \in \partial\Omega$ and let φ be the same cut-off function as above. To prove higher integrability of u near the boundary we choose in (3.8) the test function $v = \varphi^p u$. Due to the first condition in (2.3) this function is admissible. Similarly to (5.2) we obtain

$$\begin{aligned} \Big\|\nabla u\Big\|^p_{p,\Omega_R(x_0)} \le \frac{C(\theta_1)}{R^p} \Big\|u\Big\|^p_{p,\Omega_{2R}(x_0)} + C\Big\|u\Big\|^{2p'}_{2p',\Omega_{2R}(x_0)} + \\ +C\Big\|f\Big\|^{p'}_{p',\Omega_{2R}(x_0)} + \theta_1 \Big\|\pi - \pi_0\Big\|^{p'}_{p',\Omega_{2R}(x_0)}, \end{aligned} \tag{5.5}$$

where π_0 is an arbitrary number and

$$\Omega_R(x_0) = B_R(x_0) \cap \Omega.$$

Instead of Lemma 5.2 we are going to use the following

Lemma 5.3. *Assume that $\partial\Omega$ is of class C^2. Let be $\delta_7 > 0$ and positive functions $\Phi \in L^p(\Omega)$ and $\Psi \in L^{p+\delta_7}(\Omega)$ satisfy the estimate*

$$\begin{aligned}\Big(\frac{1}{R^n}\int\limits_{\Omega_{\alpha R}(x_0)}\Phi^p\,dx\Big)^{1/p} &\leqslant B_1\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)}\Phi^q\,dx\Big)^{1/q}+\\ &+B_2\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)}\Psi^p\,dx\Big)^{1/p}+\theta\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)}\Phi^p\,dx\Big)^{1/p},\end{aligned} \tag{5.6}$$

for all $x_0 \in \bar\Omega$, $R > 0$ with some constant $\alpha \in (0,1)$, $\theta \in (0,1)$, $q < p$ and positive B_1, B_2. Then there is $\delta \in (0,\delta_7)$ such that $\Phi \in L^{p+\delta}(\Omega)$.

Proof of Lemma 5.3 consists on two steps.

Step 1. We prove higher summability of Φ near the plane part $\partial_0\Omega$ of the boundary $\partial\Omega$, which means the function Φ is given on a half a ball $B^+_{R_0} \equiv \{x \in \mathbb{R}^n : |x| < R_0, x^n > 0\}$ and it satisfies (5.5) for all $x_0 \in B^+_{R_0} \cup \Gamma_{R_0}$, $\Gamma_{R_0} = \{|x'| < R_0, x^n = 0\}$ and all $R < \operatorname{dist}\{x_0, \Gamma'_{R_0}\}$, $\Gamma'_{R_0} = \{|x| = R_0, x^n > 0\}$. (We have used the notation $x' = (x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$.) Consider an even continuation of Φ with respect to the plane part of Γ_{R_0}:

$$\bar\Phi(x', x^n) = \begin{cases}\Phi(x', x^n), & x^n > 0\\ \Phi(x', -x^n), & x^n < 0.\end{cases}$$

Then $\bar\Phi$ satisfies the estimate (5.4) for all $x_0 \in B_{R_0}$ and R small enough. Due to the Lemma 5.2 $\bar\Phi \in L^{p+\delta}(B_{R_0})$ for $R < R_0$ and hence

$$\Phi \in L^{p+\delta}(B^+_{R_0}), \quad R < R_0.$$

Step 2. To prove higher summability of Φ near non-plane part of the boundary $\partial\Omega$ we remind the following facts:

Let $\partial\Omega$ be of class C^2. Then there are constants $M > 0$, $R_0 > 0$ such that for any $x_0 \in \partial\Omega$ there are local coordinate system $\{y^i_{x_0}\}_{i=1}^n$ and function $\psi_{x_0} : \{|y'| < R_0\} \to \mathbb{R}$ of class C^2 such that

$$\psi_{x_0}(0) = 0, \quad \nabla\psi_{x_0}(0) = 0, \quad |\nabla^2\psi_{x_0}(y')| \leqslant M, \quad \forall |y'| < R_0 \tag{5.7}$$

and

$$\Omega_R(x_0) = B_R(x_0) \cap \Omega = \{y \in \mathbb{R}^n : |y'| < R, y^n > \psi_{x_0}(y')\}$$

for any $R < R_0$. For any point $x_0 \in \partial\Omega$ and $R < R_0$ we consider a diffeomorphism $\phi_{x_0} : B_R(x_0) \to \mathbb{R}^n$ of class C^2

$$\phi_{x_0}(y) = \begin{pmatrix} y' \\ y^n - \psi_{x_0}(y') \end{pmatrix}. \tag{5.8}$$

Note that Jacobian of the transformation ϕ_{x_0} is equal to 1. Denote

$$\omega_R^+(x_0) := \phi_{x_0}^{-1}(B_R^+).$$

The following fact is well-known:

Lemma 5.4. *Let $\partial\Omega$ be of class C^2. Then there constants $\lambda \in (0,1)$ and $R_1 > 0$ depending only on M such that*

$$\omega_{\lambda R}^+(x_0) \subset \Omega_R(x_0) \subset \omega_{\frac{1}{\lambda}R}^+(x_0)$$

for any $x_0 \in \partial\Omega$ and any $R < R_1$.

So from (5.6) and Lemma 5.4 the estimate

$$\left(\frac{1}{R^n}\int\limits_{\omega_{\beta R}^+(x_0)} \Phi^p\,dx\right)^{1/p} \leqslant B_1\left(\frac{1}{R^n}\int\limits_{\omega_R^+(x_0)} \Phi^q\,dx\right)^{1/q} +$$

$$+B_2\left(\frac{1}{R^n}\int\limits_{\omega_R^+(x_0)} \Psi^p\,dx\right)^{1/p} + \theta\left(\frac{1}{R^n}\int\limits_{\omega_R^+(x_0)} \Phi^p\,dx\right)^{1/p}$$

follows for all $x_0 \in \partial\Omega$, $R < \lambda R_1$ with $\beta = \lambda^2\alpha$.

From this estimate via change of variables $y = \phi_{x_0}(x)$ we get the inequality

$$\left(\frac{1}{R^n}\int\limits_{B_{\beta R}^+} \widetilde{\Phi}^p\,dy\right)^{1/p} \leqslant B_1\left(\frac{1}{R^n}\int\limits_{B_R^+)} \widetilde{\Phi}^q\,dy\right)^{1/q} +$$

$$+B_2\left(\frac{1}{R^n}\int\limits_{B_R^+} \widetilde{\Psi}^p\,dy\right)^{1/p} + \theta\left(\frac{1}{R^n}\int\limits_{B_R^+} \widetilde{\Phi}^p\,dy\right)^{1/p}$$

where $\widetilde{\Phi} = \Phi \circ \phi_{x_0}$. From the first step of the proof of Lemma 5.3 follows that $\widetilde{\Phi} \in L^{p+\delta}(B_R^+)$ for any $R < \lambda R_1$ and by the back change of variables $x = \phi_{x_0}^{-1}(y)$ we get $\Phi \in L^p(\omega_R^+(x_0))$ for any $R < R_0$. Covering now domain Ω with the finite number of subdomains $B_R(x_i)$ and $\omega^+_R(x_j)$ we obtain the second statement of the Lemma 5.3.

Let us turn back to the estimate (5.5). As before we take $\Phi = |\nabla u| \in L^p(\Omega)$ and $\Psi = |u|^{\frac{2p'}{p}} + |f|^{\frac{p'}{p}} + 1 \in L^{p+\delta_7}(\Omega)$ for $\delta_7 = \min\{\delta_3, \frac{np}{n-p} - 2p'\}$. So we just need to estimate the first and the last terms in the right-hand side of (5.5).

For the first one we have the inequality

$$\left\| u \right\|_{p,\Omega_{2R}(x_0)} \leqslant \left\| u \right\|_{p,\omega^+_{\frac{2}{\lambda}R}(x_0)} = \left\| \widetilde{u} \right\|_{p,B^+_{\frac{2}{\lambda}R}},$$

where $\widetilde{u} = u \circ \phi_{x_0}$. As $u = 0$ on $\partial\Omega$ we have

$$\widetilde{u}(y', 0) = 0, \quad |y'| < R_0,$$

and hence we can apply Gagliardo inequality

$$\left\| \widetilde{u} \right\|_{p,B^+_{\frac{2}{\lambda}R}} \leqslant C(n,p) \left\| \nabla \widetilde{u} \right\|_{\frac{np}{n+p},B^+_{\frac{2}{\lambda}R}}$$

(here constant $C(n,p)$ depends only on n and p). Making the back change of variables $x = \phi^{-1}_{x_0}(y)$ we get

$$\left\| \nabla \widetilde{u} \right\|_{\frac{np}{n+p},B^+_{\frac{2}{\lambda}R}} \leqslant C_2 \left\| \nabla u \right\|_{\frac{np}{n+p},\omega^+_{\frac{2}{\lambda}R}},$$

where C_2 depends only on M. Thus we have finally

$$\left\| u \right\|_{p,\Omega_{2R}(x_0)} \leqslant C_3 \left\| \nabla u \right\|_{\frac{np}{n+p},\Omega_{\frac{2}{\lambda^2}R}},$$

with C_3 depending only on n, p, and M. We are left to estimate the last term in the right-hand side of (5.5). To do this we introduce one more lemma:

Lemma 5.5. *Suppose that for functions $F \in L^{p'}(\Omega; \mathbb{M}^{n\times n})$, $G \in L^{p'}(\Omega; \mathbb{R}^n)$ and $\pi \in L^{p'}(\Omega)$ the following identity hold:*

$$\int\limits_\Omega (F : \nabla w + G \cdot w)dx = \int\limits_\Omega \pi \operatorname{div} w dx, \quad w \in W^{1,p}_0(\Omega; \mathbb{R}^n). \tag{5.9}$$

Then there exist C_4 such that for all $x_0 \in \partial\Omega$ and all $R < R_0$

$$\begin{aligned} \left\| \pi - (\pi) \right\|_{p',\omega^+_R(x_0)} \leqslant C_4 \Big(\left\| F \right\|_{p',\omega^+_R(x_0)} + \left\| G \right\|_{p',\omega^+_R(x_0)} + \\ + \sup_{|y'|<R} |\nabla \psi_{x_0}| \left\| \pi - (\pi)_{\omega^+_R(x_0)} \right\|_{p',\omega^+_R(x_0)} \Big). \end{aligned} \tag{5.10}$$

Constant C_4 depends only on n, p, M. Here R_0 and ψ_{x_0} are given by (5.7).

Proof. Consider the transformation (5.8). Then by change of variables $y = \phi_{x_0}(x)$ we get from (5.9) the relation

$$\int\limits_{B_R^+} (\widetilde{\pi} - (\widetilde{\pi})_{B_R^+}) \operatorname{div} w dy = \int\limits_{B_R^+} \widetilde{F}_{ij} \frac{\partial \widetilde{w}_i}{\partial y_j} dy +$$

$$+ \int\limits_{B_R^+} \left(\widetilde{F}_{in} + \widetilde{F}_{ij} \frac{\partial \psi}{\partial y_j} - (\widetilde{\pi} - (\widetilde{\pi})_{B_R^+}) \frac{\partial \psi}{\partial y_i}) \right) \frac{\partial w_i}{\partial y_n} dy +$$

$$+ \int\limits_{B_R^+} (\widetilde{F}_{nn} + \widetilde{F}_{ni} \frac{\partial \psi}{\partial y_n}) dy, \quad \forall\, \widetilde{w} \in W_0^{1,p}(B_R^+, \mathbb{R}^n).$$

Here $\widetilde{F} = F \circ \phi_{x_0}$, $\widetilde{\pi} = \pi \circ \phi_{x_0}$ and repeated indices for i and j means the sum from 1 to $n-1$. Note that $(\widetilde{\pi})_{B_R^+} = (\pi)_{\omega_R^+}$. From Lemma 5.1 the estimate

$$\left\| \widetilde{\pi} - (\widetilde{\pi})_{B_R^+} \right\|_{p', B_R^+} \leqslant C_5 \Big(\left\| \widetilde{F} \right\|_{p', B_R^+} + \| \widetilde{G} \|_{p', B_R^+} +$$

$$+ \sup_{|y'|<R} |\nabla \psi_{x_0}| \left\| \widetilde{\pi} - (\widetilde{\pi})_{B_R^+} \right\|_{p', B_R^+} \Big)$$

follows. Similarly to Lemma 5.1 by rescaling arguments it can be shown that constant C_5 depends only on n and p. Making back change of variables $x = \phi_{x_0}^{-1}(y)$ we get the estimate (5.10) and taking into account the identity $(\widetilde{\pi})_{B_R^+} = (\pi)_{\omega_R^+(x_0)}$ we get the estimate (5.10) with constant C_4 depending only on n, p, and M. Lemma 5.5 is proved.

Denote by $F = \mu(e) a(\varepsilon(u)) + u \otimes u$ and $G = f$, we apply Lemma 3.8 taking into account relation (3.8). We obtain

$$\left\| \pi - (\pi)_{\omega_R^+(x_0)} \right\|_{p', \omega_R^+(x_0)} \leqslant$$

$$\leqslant C_4 \Big(\mu_1 \left\| a(\varepsilon(u)) \right\|_{p', \omega_R^+(x_0)} + \left\| u \right\|^2_{2p', \omega_R^+(x_0)} + \left\| f \right\|_{p', \omega_R^+(x_0)} \Big) +$$

$$+ C_4 \sup_{|y'|<R} |\nabla \psi_{x_0}| \left\| \pi - (\pi)_{\omega_R^+(x_0)} \right\|_{p', \omega_R^+(x_0)},$$

which holds for any $R < R_0$ and $x_0 \in \partial\Omega$.

Due to the conditions (5.7) we have

$$\sup_{|y'|<R} |\nabla \psi_{x_0}(y')| \leqslant MR$$

and hence the estimate

$$\Big\|\pi - (\pi)_{\omega^+_R(x_0)}\Big\|_{p',\omega^+_R(x_0)} \leqslant$$

$$\leqslant C_4\Big(\mu_1\Big\|a(\varepsilon(u))\Big\|_{p',\omega^+_R(x_0)} + \Big\|u\Big\|^2_{2p',\omega^+_R(x_0)} + \Big\|f\Big\|_{p',\omega^+_R(x_0)}\Big)$$

holds for all $x_0 \in \partial\Omega$ and R small enough. (To be precise, for any $R < R_2 \equiv \min\{R_0, \frac{1}{2C_4M}\}$. So R_2 depends only on n, p, and M.)

Combining the last estimate with the inequality (5.5) and using inclusion $\omega^+_{\frac{2}{\lambda}R}(x_0) \subset \Omega_{\frac{2}{\lambda^2}R}(x_0)$ from Lemma 5.4 we get finally the estimate

$$\Big(\frac{1}{R^n}\int\limits_{\Omega_{\alpha R}(x_0)} |\nabla u|^p dx\Big)^{1/p} \leqslant \theta\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)} |\nabla u|^p dx\Big)^{1/p} +$$

$$C\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)} |\nabla u|^{\frac{np}{n+p}} dx\Big)^{(n+p)/(np)} +$$

$$+C\Big(\frac{1}{R^n}\int\limits_{\Omega_R(x_0)} (|f|^{p'} + |u|^{2p'})dx\Big)^{1/p'},$$

for all $x_0 \in \bar{\Omega}$ and R small enough. Now higher integrability of $|\nabla u|$ on Ω follows from Lemma 5.3. The second statement of Lemma 3.2 is proved.

Proof of Lemma 3.3.

Since $g \in L^{1+\delta_5}(\Omega) \hookrightarrow (W^{1,n}_0(\Omega))^*$ and $u \in W^{1,p+\delta_3}(\Omega, \mathbb{R}^n)$ the right hand side of (3.9) gives a bounded linear functional on a space $W^{1,n}_0(\Omega)$, assuming (3.1) and (3.3). So we can apply an existence theorem (for example, see [22] or [30]) if we define the operator $A : W^{1,n}_0(\Omega) \to (W^{1,n}_0(\Omega))^*$ by

$$\langle Ae, w\rangle := \int\limits_\Omega (k(e)b(\nabla e) - eu) \cdot \nabla w dx, \quad \forall w \in W^{1,n}_0(\Omega).$$

Indeed, from the assumptions (3.2) and (3.4) this operator is coercive, bounded hemicontinuous and monotonous. Lemma 3.3 is proved.

Proof of Lemma 3.4. We follow the same arguments we used in the proof of Lemma 3.2.

From Lemma 3.3 follows that (3.9) holds for any test function $\eta \in W_0^{1,n}(\Omega)$. Taking in (3.9) test function $w = \varphi^n(e - (e)_{B_{2R}})$ where φ is a cut-off function satisfying (5.1) we obtain the identity

$$\int_\Omega \varphi^n k(e) b(\nabla e) \cdot \nabla e dx =$$

$$= -\int_\Omega (k(e)(e - (e)_{B_{2R}}) b(\nabla e) \cdot \nabla \varphi^n + \widetilde{g} \varphi^n (e - (e)_{B_{2R}})) dx, \qquad (5.11)$$

where

$$\widetilde{g} = u \cdot \nabla e + g + \sigma : \varepsilon(u).$$

Since $u \in L^{pn/(n-p)}(\Omega, \mathbb{R}^n)$ and $\nabla e \in L^n(\Omega, \mathbb{R}^n)$, it follows

$$\widetilde{g} \in L^{1+\delta_8}(\Omega),$$

with some $\delta_8 > 0$. Denote by $s < 1 + \delta_8$. Using Hölder's inequalities for each term in the right hand side of (5.11) and taking into account (3.2) we get

$$\|\varphi \nabla e\|^n_{n,B_{2R}} \le$$
$$\leqslant \frac{C}{R} \|\varphi \nabla e\|^{n-1}_{n,B_{2R}} \|e - (e)_{B_{2R}}\|_{n,B_{2R}} + C(\|\widetilde{g}\|_{s,B_{2R}} \|e\|_{s',B_{2R}} + R^n).$$

By Young's inequality we conclude that

$$\|\nabla e\|^n_{n,B_R} \le \frac{C}{R^n} \|e - (e)_{B_{2R}}\|^n_{n,B_{2R}} + C(\|\widetilde{g}\|^s_{s,B_{2R}} + \|e\|^{s'}_{s',B_{2R}} + R^n).$$

Remembering the Gagliardo inequality

$$\|e - (e)_{B_{2R}}\|_{n,B_{2R}} \le C(n) \|\nabla e\|_{n/2,B_{2R}}$$

and dividing by R^n we obtain

$$\frac{1}{R^n} \int_{B_R} |\nabla e|^n dx \le C\left(\frac{1}{R^n} \int_{B_{2R}} |\nabla e|^{n/2} dx\right)^2 +$$
$$+ \frac{C}{R^n}\left(\int_{B_{2R}} |\widetilde{g}|^s dx + \int_{B_{2R}} |e|^{s'} dx\right) + C.$$

Applying Lemma 5.2 with $p = n$, $q = \frac{n}{2}$ and functions

$$\Phi = |\nabla e|, \quad \Psi = |\widetilde{g}|^{\frac{s}{n}} + |e|^{\frac{s'}{n}} + 1, \quad \Psi \in L^{n+\delta_9}(\Omega)$$

with some $\delta_9 > 0$, we obtain the desired result. Lemma 3.3 is proved.

6. Proof of the Theorem 3.2

From Theorem 3.1 we get $(1 + |\varepsilon(u)|^2)^{\frac{p}{4}} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and hence in 2D case we get $\varepsilon(u) \in L^s_{\mathrm{loc}}(\Omega, \mathbb{M}^{n\times n})$, $\forall s < +\infty$. If condition (3.13) holds then the right-hand side of the second equation in system (2.1)–(2.3) belongs to $L^{2+\delta_1}_{\mathrm{loc}}(\Omega)$. Hölder continuity of ∇e follows now from the general theory of regularity for elliptic equations, see for instance the book [20].

To prove Hölder continuity of ∇u on a subdomain $\Omega' \Subset \Omega$ we can utilize with small technical changes technique developed in the serie of papers [29, 9]. So, in [29, 10] for problems of plasticity it is shown how to operate with nonlinear principal part of the equation in case of $p < 2$, the case $p \geqslant 2$ is treated in [10, 9] arguments of [11, 12] shows how to operate with divergence-free condition and convective term. In this way in 2D-case internal full regularity for solutions to systems of type

$$-\operatorname{div} a(\varepsilon(u)) + (\nabla u)u = f - \nabla\pi, \quad \operatorname{div} u = 0 \tag{6.1}$$

can be established as well as partial regularity in 3D-case. For 2D-case we can also refer to [17] were full regularity up to the boundary of solutions to Dirichlet problem for systems of type (6.1) was proved under growth condition on the dissipative potential of the fluid $p > \frac{3}{2}$ while for $p > \frac{6}{5}$ solutions with internal regularity was obtained.

Theorem 3.2 is proved.

In conclusion we would like to remark that for today theory of internal regularity for solutions to Modified Navier–Stokes system (6.1) in the isothermal case is developed much better than in case of reologic relations (2.4) including the temperature. So, in [26] nonstationary 2D-case was considered and Hölder continuity of the spacial gradient of the solution on any subdomain of the parabolic cylinder was obtained under the assumption $p = 2$. In [27, 28] nonstationary 3D-case was studied and partial regularity of the solutions was obtained under appropriate assumptions on p.

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Поступило 10 ноября 2000 г.

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