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PARTIAL REGULARITY UP TO THE BOUNDARY OF WEAK SOLUTIONS OF ELLIPTIC SYSTEMS WITH NONLINEARITY Q GREATER THAN TWO

To the memory of A. I. Ivanov

ABSTRACT. Nonlinear elliptic systems with q-growth are considered. It is assumed that additional nonlinear terms of the systems have q-growth in the gradient, $q > 2$. For Dirichlet and Neumann boundary-value problems we study the regularity of weak bounded solutions in the vicinity of the boundary.

In the case of small dimensions ($n \le q+2$), the Hölder continuity or partial Hölder continuity of the solutions up to the boundary is proved. In a previous article the author studied the same problem for $q = 2$.

1. Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$, with sufficiently smooth boundary $\partial\Omega$.

Consider $u\colon \Omega \to \mathbb{R}^N$, $N > 1$, that is a solution of the problem

$$\mathcal{L}^{(k)} u \equiv -\frac{d}{dx_\alpha}\, a^k_\alpha(x, u, u_x) + b^k(x, u, u_x) = 0, \quad k \le N, \quad x \in \Omega; \tag{1}$$

$$u\big|_{\partial\Omega} = \varphi(x) \tag{2_1}$$

or

$$\frac{\partial u}{\partial n_a}\bigg|_{\partial\Omega} = \psi(x, u), \tag{2_2}$$

where $\frac{\partial u^{(k)}}{\partial n_a} = a^k_\alpha(x, u, u_x)\cos(\mathbf{n}, x_\alpha)$, $k \le N$, $\mathbf{n} = \mathbf{n}(x)$ is the outward normal (with respect to Ω) at a point $x \in \partial\Omega$, $\alpha \le n$.

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We denote by $u_x = \{u^k_{x_\beta}\}^{k\le N}_{\beta\le n}$ the gradient of $u = (u^1, \dots, u^N\}$ and fix a number $q > 2$. On the set $\mathcal{M} = \overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{Nn}$, the functions $a = \{a^k_\alpha\}$ and $b = \{b^k\}$ are assumed to satisfy the following conditions:

(a)

$$|a(x, u, p)| \le \mu(1 + |p|^{q-1}). \tag{3}$$

(b) The functions $a = \{a^k_\alpha(x, u, p)\}$ are uniformly continuous on $\mathcal{M}$ with respect to x and u in the following sense:

$$|a(x, u, p) - a(y, v, p)| \le (1 + |p|^{q-1})\omega(|u|;\ |x - y|^q + |u - v|^q), \tag{4}$$

$$x, y \in \overline{\Omega}, \quad u, v \in \mathbb{R}^N, \quad p \in \mathbb{R}^{Nn}.$$

Here, $\omega(s, t)$ is a bounded nondecreasing function continuous in $s, t \ge 0$: it is concave in t for any fixed $s > 0$.

(c) The matrix a is uniformly elliptic, i.e.,

$$\left|\frac{\partial a(x, u, p)}{\partial p}\right| \le \mu(1 + |p|^{q-2}), \tag{5}$$

$$\left\langle \frac{\partial a(x, u, p)}{\partial p}\xi, \xi \right\rangle \ge (\varkappa + \nu|p|^{q-2})|\xi|^2, \quad \xi \in \mathbb{R}^{Nn}, \tag{6}$$

for some positive constants μ, ν, χ; $(x, u, p) \in \mathcal{M}$.

(d) The function b is the Caratheodory function on $\mathcal{M}$ and for a fixed $M > 0$

$$|b(x, u, p)| \le a_0|p|^q + b \quad \text{for all} \quad (x, u, p) \in \mathcal{M}, \quad |u| \le M, \tag{7}$$

$$2(q - 1)a_0 M < \nu. \tag{8}$$

Remark 1. From (6) it follows that

$$\big(a(x, u, p_1) - a(x, u, p_2)\big)(p_1 - p_2) \ge (\varkappa + \nu_0|p_1 - p_2|^{q-2})|p_1 - p_2|^2 \tag{9}$$

with a positive constant $\nu_0 = \nu_0(\nu, q)$.

Moreover, from (3) and (6)

$$a(x, u, p) \cdot p \ge \left(\varkappa + \frac{\nu}{q - 1}\,|p|^{q-2}\right)|p|^2 - \mu|p|. \tag{10}$$

In this paper, we study the regularity of weak solutions $u \in W^1_q(\Omega) \cap L_\infty(\Omega)$ of problem (1), (2_1) and (1), (2_2) in the vicinity of $\partial\Omega \in \mathbb{C}^2$. For simplicity, we assume here that φ and ψ are equal to zero.

For the case of $q = 2$, the boundary regularity of weak solutions of (1), (2_i), $i = 1, 2$, was studied by the author in [1, 6].

In [2], the local interior regularity of weak solutions u of (1) was stated for $q \geq 2$. There it was proved that u_x is a Hölder function on an open set $\Omega_0 \subset\subset \Omega$, and $\operatorname{meas}_n(\Omega \setminus \Omega_0) = 0$.

When the regularity of a solution *up to the boundary* is of interest for us, it is natural to look for sharper estimates of the dimension of the singular set. As is known, even if we study interior regularity and $b = 0$, we are able to sharp the estimate of the singular set only in some particular cases: the matrix a has some special structure or $\dim \Omega = n \leq q + 2$.

Here we consider boundary-value problems (1), (2_i), $i = 1, 2$, for the case of small dimensions ($n \leqslant q + 2$).

To study regularity near the boundary,we apply the so-called "direct" method. In the frame of the method we straight a part of $\partial\Omega$ and invoke the model nonlinear problem in a half of a ball. If a solution of the model problem is known to be *bounded*, then one may exploit a well-developed method (see [8–10]) to prove the partial Hölder continuity of the solution under consideration. Nowadays, for model nonlinear systems in nonsmooth domains and, moreover, under mixed boundary conditions, maximum principle-type estimates have yet not been obtained. Here we continue to work out a modification of the direct method, which is not based on the "maximum principle" for the model problem (see [1, 6]).

We adopt the following notation:

$$u_x = \{u^k_{x_\alpha}\}^{k \leq N}_{\alpha \leq n}, \quad |u_x|^2 = \sum_{\substack{l \leq N \\ \alpha \leq n}} (u^k_{x_\alpha})^2,$$

$$u_{xx} = \{u^k_{x_\alpha x_\beta}\}^{k \leq N}_{\alpha, \beta \leq n}, \quad |u_{xx}|^2 = \sum_{\substack{k \leq N \\ \alpha, \beta \leq n}} (u^k_{x_\alpha x_\beta})^2.$$

$$B_R(x^0) = \{x \in \mathbb{R}^n \colon |x - x^0| < R\},$$

$$S_R(x^0) = \{x \in \mathbb{R}^n \colon |x - x^0| = R\},$$

$$B^+_R(x^0) = B_R(x^0) \cap \{x_n > x^0_n\},$$

$$\Omega_R(x^0) = B_R(x^0) \cap \Omega, \quad \gamma_R(x^0) = B_R(x^0) \cap \{x_n = 0\};$$

We write B_1, B^+_1, $\Gamma_1, \ldots$ in place of $B_1(0)$, $B^+_1(0)$, $\Gamma_1(0), \ldots$, and put

$$v^0_R = \fint_{\Omega_R(x^0)} v \, dx = \frac{1}{\operatorname{meas}_n \Omega_R} \int_{\Omega_R(x^0)} v \, dx,$$

$$\fint_{\Omega_R(x^0)} |v|^q dx = \frac{1}{R^{n-q}} \int_{\Omega_R(x^0)} |v|^q dx.$$

We denote by c and c_i positive constants depending on the parameters of the problem under consideration. For $p \in [1, \infty]$, we write $\|\cdot\|_{p,A}$ in place of $\|\cdot\|_{L_p(A)}$.

$\mathcal{H}(A)$ is the k-dimensional Hausdorff measure of a set A; $\dim_{\mathcal{H}} A$ is the Hausdorff dimension of A.

$\overset{\circ}{W}{}^1_q(B_R^+ \cup \Gamma_R)$ is the function space obtained by taking the closure in $W^1_q(B_R^+)$ of the space of smooth functions with support in $B_R^+ \cup \Gamma_R$.

Let $L^{p,\lambda}(\Omega)$ and $\mathcal{L}^{p,\lambda}(\Omega)$ be the Morrey and Campanato spaces, respectively (see [7]).

Finally, $t_+ = \max\{t, 0\}$.

2. Preliminaries and statements.

We investigate problem (1) and (2_2). All propositions proved here are also valid for problem (1) and (2_1), and can be derived in a simpler way. To save the place, we put $\psi = 0$ in (2_2). As was pointed in [1], problem (1) and (2_2) is invariant under a nondegenerate transformation of spatial variables in a small neiborhood of a fixed point of $\partial\Omega \in \mathbb{C}^2$. (We mean that all the assumptions (a)–(d) preserve the form and " smallness" condition (8) holds with respect to the new parameters.) It allows us to "straight" a part of $\partial\Omega$ and to study the problem in a local setting. Using the initial notation for the independent variables and functions, we study the problem

$$\begin{cases} -(a^k_\alpha(x, u, u_x))_{x_\alpha} + b^k(x, u, u_x) = 0, \quad x \in B_1^+, \\ a^k_n(x, u, u_x)\big|_{\Gamma_1} = 0, \quad k = 1, \dots, N. \end{cases} \tag{11}$$

A weak solution $u \in W^1_q(B_1^+) \cap L_\infty(B_1^+)$ of (11) satisfies the identity

$$\int_{B_1^+} [a^k_\alpha(x, u, u_x) h^k_{x_\alpha} + b^k(x, u, u_x) h^k] \, dx = 0,$$

$$h \in \overset{\circ}{W}{}^1_q(B_1^+ \cup \Gamma_1) \cap L_\infty(B_1^+). \tag{12}$$

Henceforth we assume that $M = \|u\|_{\infty,\Omega}$ in (7) and (8).

Formulate the main results of the present paper.

Theorem 1. *Let $u \in W_q^1(B_1^+) \cap L_\infty(B_1^+)$ be a weak solution of problem (11). If functions a_k^α, b^k satisfy conditions (a), (b), (c), and (d) with $q \geq 2$ on $\mathcal{M}^+ = \overline{B_1^+} \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ then*

(1) *$u \in \mathbb{C}^\alpha(B_1^+ \cup \Gamma_1)$ for any $\alpha < 1$ if $n = 2$;*

(2) *there exists $\lambda > 2$ such that*

if $n \in (2, q]$, then $u \in \mathbb{C}^\alpha(B_1^+ \cup \Gamma_1)$ with $\alpha < 1 - \frac{n-\lambda}{q}$;

if $n \in (q, q+\lambda)$, then $u \in \mathbb{C}^\alpha((B_1^+ \cup \Gamma_1) \setminus \Sigma)$ for $\alpha < 1 - \frac{n-\lambda}{q}$ and

$\mathcal{H}_{n-q-\delta}(\Sigma) = 0$ for some $\delta > 0$.

Theorem 1 is proved in Sec. 4.

Obviously, if $q > n$, then the embedding theorem guarantees that $u \in \mathbb{C}^\beta(\overline{B_1^+})$, where $\beta = 1 - n/q > 0$. In this case, the Hölder exponent α in Theorem 1 is greater than β. A more interesting case is $n \in (q, q+\lambda)$.

In the case where $n = 2$, $q \geq 2$, Theorem 1 ensures the Hölder continuity of the solution in $\overline{B_{1-\sigma}^+}$, $\sigma < 1$, with *any* $\alpha < 1$.

Theorem 2. *Let $n = 2$ and let the assumptions of Theorem 1 hold. Suppose in addition that the functions $a_\alpha^k(x, u, p)$ are Hölder continuous on $\overline{B_{1-\sigma}^+}$ with respect to x and u. More precisely, we assume that for a fixed $\theta \in (0, 1]$, any $x, y \in \overline{B_{1-\sigma}^+}$, u, $v \in \mathbb{R}^N$, $|u|$, $|v| \leq M$, and any $p \in \mathbb{R}^{Nn}$ we have*

$$|a(x, u, p) - a(y, v, p)| \leq L(|x - y|^\theta + |u - v|^\theta)(1 + |p|^{q-1}), \tag{13}$$

where $L = \mathrm{const} > 0$.
Then there exists $\theta_0 \in (0, 1)$ such that $u_x \in C^{\theta_0}(\overline{B_{1-2\sigma}})$.

This theorem is proved in Sec. 5.

3. Fundamental estimates for the nonlinear model problem.

Here we consider the following nonlinear problem:

$$\frac{d}{dx_\alpha} \mathcal{A}_\alpha^k(v_x) = 0, \quad x \in B_1^+, \tag{14}$$

$$\mathcal{A}_n^k(v_x)\big|_{\Gamma_1} = 0. \tag{15}$$

The functions $\mathcal{A}_\alpha^k$ are assumed continuously differentiable in $p \in \mathbb{R}^{Nn}$ and

$$\begin{gathered}\Big(\sum_{k,l\in N}\ \sum_{\alpha,\beta\le n}\Big(\frac{\partial\mathcal{A}_\alpha^k(p)}{\partial p_\beta^l}\Big)^2\Big)^{1/2} \le \hat\mu(1+|p|^{q-2}),\\ \frac{\partial\mathcal{A}_\alpha^k(p)}{\partial p_\beta^l}\xi_\alpha^k\xi_\beta^l \ge (\hat\chi+\hat\nu|p|^{q-2})|\xi|^2, \quad \xi\in\mathbb{R}^{Nn}.\end{gathered} \tag{16}$$

We follow [3], denoting

$$V(p) = (1+|p|^2)^{1/2}, \quad W(p) = V^{\frac{q-2}{2}}(p)p, \quad p \in \mathbb{R}^{Nn}.$$

(One can find useful technical relations for $W(p)$ in [12], §2.)

Proposition 1. *If assumptions* (16) *hold and* $v \in W_q^1(B_1^+)$ *is a weak solution of* (14), (15), *then*

(1) *v is twice differentiable in B_σ^+, $\sigma < 1$, $W(v_x) \in W_2^1(B_\sigma^+)$ and for every $x^0 \in \overline{B_\sigma^+}$, $2\rho < 1-\sigma$,*

$$\int\limits_{\Omega_\rho(x^0)} |(W(v_x))_x|^2 dx \le \frac{c_1}{\rho^2}\int\limits_{\Omega_{2\rho}(x^0)} V^{q-2}(v_x)|v_x - l|^2 dx \tag{17}$$

with any $l \in \mathbb{R}^{Nn}$;

(2) *there exists a number $p_0 > 2$ such that $W(v_x) \in W_{p_0}^1(B_\sigma^+)$, $\sigma < 1$, and*

$$\begin{aligned}\Big(\fint\limits_{\Omega_\rho(x^0)} |(W(v_x))_x|^{p_0} dx\Big)^{2/p_0} &\le\\ \le c_2 \fint\limits_{\Omega_{2\rho}(x^0)} |(W(v_x))_x|^2 dx, \quad & x^0\in\overline{B_\sigma^+},\ 2\rho<1-\sigma.\end{aligned} \tag{18}$$

The constants c_1, c_2, and p_0 depend on $\hat\nu$, $\hat\chi$, and $\hat\mu$ from (16).

Estimate (17) with $l = 0$ and estimate (18) were derived by S.Campanato for system (14) under the Dirichlet boundary condition $v\big|_{\Gamma_1} = 0$ (see [3]). We also observe that for the case $q = 2$ Proposition 1 was stated in [11] under a more general nonlinear boundary condition than (15). Since under boundary condition (15) both assertions of Proposition 1 can be proved in the same way as was done in [3], we omit the proof of them here.

As a consequence of this statement, we obtain the next proposition. (See the proof of Theorem 3.1 in [3].)

Proposition 2. *For any point $y^0 \in B_1^+ \cup \Gamma_1$ and $R < \operatorname{dist}(y^0, S_1^+)$ the following estimates are valid:*

$$\int\limits_{\Omega_\rho(y^0)} |(W(v_x))_x|^2 dx \le c_3 \left(\frac{\rho}{R}\right)^\gamma \int\limits_{\Omega_R(y^0)} |(W(v_x))_x|^2 dx, \quad \rho \le R, \tag{19}$$

where $\gamma = n(1 - \frac{2}{p_0}) > 0$;

$$\int\limits_{\Omega_\rho(y^0)} |W(v_x)|^2 dx \le c_4 \left(\frac{\rho}{R}\right)^\lambda \int\limits_{\Omega_R(y^0)} |W(v_x)|^2 dx, \quad \rho \le R, \tag{20}$$

where $\lambda = \min\{n, 2+\gamma\}$;
if $2 \le n < q + \lambda$, then

$$\int\limits_{\Omega_\rho(y^0)} |v|^q dx \le$$

$$\le c_5 \left(\frac{\rho}{r}\right)^n \left\{ \int\limits_{\Omega_r(y^0)} |v|^q dx + r^q \int\limits_{\Omega_r(y^0)} |v_x|^q dx \right\}, \quad \rho \le r \le R. \tag{21}$$

The constants c_3, c_4, and c_5 depend only on the parameters $\hat{\nu}$, $\hat{\mu}$, and $\hat{\chi}$ from conditions (16).

4. Proof of Theorem 1. Assume that the assumptions of Theorem 1 hold.

Lemma 1. *There exist numbers $s > 1$, $c > 0$ and $l \ge 2$ such that $u_x \in L_{qs}(B_r^+)$, $r < 1$, and for every $x^0 \in B_1^+ \cup \Gamma_1$ and $R \le \frac{1}{l} \operatorname{dist}\{x^0, S_1^+\}$ we have*

$$\left(\fint\limits_{\Omega_R(x^0)} |u_x|^{qs} dx \right) \le c \left(\fint\limits_{\Omega_{lR}(x^0)} (1 + |u_x|^q) dx \right)^s. \tag{22}$$

The constants s and c depend on $\left(\frac{\nu}{q-1} - 2a_0 M\right)^{-1}$, μ, $\varkappa$, and b; l is a universal constant.

The proof of this lemma reduces to the standard application of an appropriate modification of the Gehring Lemma (see [7, Chap. 5] or [5]) and is omitted.

Lemma 2. *Let $n \geq q$. Then for a fixed $\sigma \in (0,1)$ we have $u_x \in L^{q,n-q}(B^+_{1-\sigma})$ and*

$$\|u_x\|^q_{q,n-q;B^+_\sigma} \leq c_\sigma(M^q+1), \tag{23}$$

where the constant c_σ depends on σ^{-1} and on the same parameteres as the constant c in (22).

Estimate (23) is easily derived from identity (12) with a test function $h = u\xi^q$, where ξ is a cut-off function for any $B_{2\rho}(x^0)$, $x^0 \in B^+_\sigma$, and $\rho < \frac{1-\sigma}{2}$ (see Lemma 3.2 [1] for $q = 2$).

To prove Theorem 1, we fix $\sigma \in (0,1)$, $x^0 \in \overline{B^+_{1-\sigma}}$ and $R < \frac{\sigma}{2l}$, where $l \geq 2$ is the number occurring in Lemma 1. On the set $\Omega_R(x^0) = B_R(x^0) \cap B^+_1$ we consider the model problem

$$\begin{aligned} &\frac{d}{dx_\alpha}\overset{\circ}{a}{}^k_\alpha(v_x) = 0, \quad x \in \Omega_R(x^0),\ k \leq N, \\ &\overset{\circ}{a}{}^k_n(v_x)\big|_{\gamma_R(x^0)} = 0, \quad v\big|_{S'_R(x^0)} = u, \end{aligned} \tag{24}$$

where $\gamma_R(x^0) = B_R(x^0) \cap \Gamma_1$ (we consider the more interesting case of $\gamma_R(x^0) \neq \emptyset$), $S'_R(x^0) = \partial\Omega_R(x^0) \setminus \gamma_R(x^0)$, $\overset{\circ}{a}{}^k_\alpha(p) = a^k_\alpha(x^0,u^0,p) - a^k_\alpha(x^0,u^0,0)$, and $u^0 = \displaystyle\fint_{\Omega_R(x^0)} u\,dx$, u being the solution under study.

There exists a unique solution $v \in W^1_q(\Omega_R(x^0))$ of (24) satisfying the identity

$$\int_{\Omega_R(x^0)} \overset{\circ}{a}{}^k_\alpha(v_x)h^k_{x_\alpha}\,dx = 0, \quad h \in \overset{\circ}{W}{}^1_q(\Omega_R(x^0) \cup \gamma_R(x^0)). \tag{25}$$

From (25) with $h = v - u$, we easily deduce that

$$\int_{\Omega_R(x^0)} |v_x|^q dx \leq c(\nu,\mu,q) \int_{\Omega_R(x^0)} (|u_x|^q + 1)dx. \tag{26}$$

We may consider v as a solution of problem (14) and (15) where we take $\Omega_R(x^0)$ instead of B^+_1. The functions $\overset{\circ}{a}{}^k_\alpha(p)$ satisfy all the assumptions from (16), and Propositions 1 and 2 are valid for v. Taking $y^0 \in \Omega_{\frac{R}{2}}(x^0)$ and putting $r = \frac{R}{2}$ in (21), we obtain

$$\int_{\Omega_\rho(y^0)} |v|^q dx \leq c\rho^n \left\{ \frac{1}{R^n} \int_{\Omega_{R/2}(y^0)} |v|^q dx + R^{q-n} \int_{\Omega_{R/2}(y^0)} |v_x|^q dx \right\}.$$

Since $\Omega_{R/2}(y^0) \subset \Omega_R(x^0)$, from the previous inequality it follows that

$$\sup_{y^0\in\Omega_{R/2}(x^0)} \frac{1}{\rho^n} \int\limits_{\Omega_\rho(y^0)} |v|^q dx \le \frac{c}{R^n} \int\limits_{\Omega_R(x^0)} |v-u|^q dx + cM^q +$$

$$+ R^{q-n} \int\limits_{\Omega_R(x^0)} |v_x|^q dx \underset{(26)}{\le} \frac{c}{R^{n-q}} \int\limits_{\Omega_R} |(v-u)_x|^q dx +$$

$$\frac{c}{R^{n-q}} \int\limits_{\Omega_R(x^0)} |u_x|^q dx + cR^q \underset{(23),(26)}{\le} c(\sigma^{-1})(M^q+1).$$

This yields

$$\|v\|^q_{\infty,\Omega_{R/2}(x^0)} \le c(M^q+1) = m_0^q, \tag{27}$$

where the constant m_0 depends on σ^{-1}, M, $\big(\frac{\nu}{q-1} - 2a_0M\big)^{-1}$, μ, $\varkappa$, and b.

From (12) and (25) we derive the identity

$$\int\limits_{\Omega_R(x^0)} [a^k_\alpha(x^0,u^0,u_x) - a^k_\alpha(x^0,u^0,v_x)]h^k_{x_\alpha} dx =$$

$$= \int\limits_{\Omega_R(x^0)} [\Delta a^k_\alpha \cdot h^k_\alpha - a^k_\alpha(x^0,u^0,0)h^k_{x_\alpha} - b^k(x,u,u_x)h^k]\, dx,$$

$$\forall\, h \in \overset{\circ}{W}{}^1_q(\Omega_R(x^0)\cup\gamma_R(x^0)) \cap L_\infty(\Omega_R(x^0)), \tag{28}$$

where $\Delta a^k_\alpha = a^k_\alpha(x^0,u^0,u_x) - a^k_\alpha(x,u,u_x)$.

Moreover, inequality (27) implies the estimate

$$\|w\|_{\infty,\Omega_{R/2}(x^0)} \le M + m_0 = m \tag{29}$$

for the function $w = u - v$, $w \in \overset{\circ}{W}{}^1_q(\Omega_R(x^0)\cup\gamma_R(x^0))$.

We put $h = w(T^s - (|w|+m)^s)_+$ in (28). Here, m is as in (29) and the number $T > 0$ is determined by the condition

$$T^s - (2m)^s = \frac{T^s}{2}. \tag{30}$$

Note that $T = 2^{1+1/s}m \underset{s\to 0}{\to} \infty$; the value of the small parameter s will be specified in the sequel.

Now we make use of assumptions (3)–(7) and estimate (26) to deduce the inequality

$$\int_{\Omega_R} |w_x|^q (T^s - (|w|+m)^s)_+ dx \leq$$

$$\leq c_1 s T^s \int_{\Omega_R} (1+|u_x|^q) dx + c_2 \int_{\Omega_R} \omega(M; R^q + |u-u^0|^q)(1+|u_x|^{q-1}) \times$$

$$\times \{ |w_x| (T^s - (|w|+m)^s)_+ + s T^s \theta_+ |w_x| \} +$$

$$+ c_3 \int_{\Omega_R} (1+|u_x|^q)|w|(T^s - (|w|+m)^s)_+ dx + c_4 T^s R^n, \qquad (31)$$

where $\theta_+ = \{x \in \Omega_R |\ |w| < T - m\}$ and $c_i = c_i(\nu, \mu, q, a_0, b)$, $i \leq 4$.

We estimate the right-hand side of (31) by the Cauchy inequality and obtain

$$\int_{\Omega_R} |w_x|^q (T^s - (|w|+m)^s)_+ dx \leq$$

$$\leq c_5 s T^s \int_{\Omega_R} |u_x|^q dx + c_6 H(s) T^s R^n + c_7 T^s J_R^{(1)} + c_8 T^s J_R^{(2)}, \qquad (32)$$

where

$$J_R^{(1)} = \int_{\Omega_R} |u_x|^q \omega^{q/q-1}(M; R^q + |u-u^0|^q)\, dx$$

and

$$J_R^{(2)} = \int_{\Omega_R} |u_x|^q |w| \theta_+ dx.$$

The constant $H(s)$ may increase as s tends to 0; c_5, c_6, and c_7 do not depend on s and T.

Relations (29) and (30) imply that

$$\int_{\Omega_R} |w_x|^q (T^s - (|w|+m)^s)_+ dx \geq \frac{T^s}{2} \int_{\Omega_{R/2}} |w_x|^2 dx.$$

Now we apply this inequality to (32) and divide the result by T^s:

$$\int_{\Omega_{R/2}} |w_x|^q dx \leq c_9 \left[s \int_{\Omega_R} |u_x|^q dx + H(s) R^n + J_R^{(1)} + J_R^{(2)} \right]. \qquad (33)$$

As usual, $J_R^{(1)}$ is estimated with the help of the Gehring Lemma; more precisely:

$$
\begin{aligned}
J_R^{(1)} &\le \left(\fint_{\Omega_R} |u_x|^{qs}\,dx\right)^{1/s}\left(\int_{\Omega_R} \omega^{\frac{q}{q-1}\cdot s'}(\dots)\,dx\right)^{1/s'}\cdot|\Omega_R|^{1/s} \underset{(22)}{\le}\\
&\le c\int_{\Omega_{lR}}(1+|u_x|^q)\,dx\left(\fint_{\Omega_R}\omega(M;R^q+|u-u^0|^q)dx\right)^{1/s'}\le\\
&\le c\int_{\Omega_{lR}}(1+|u_x|^q)dx\cdot\omega\left(M;R^q+c_0\fint_{\Omega_R}|u_x|^q dx\right)\le\\
&\le\omega_0\left(M;R^q+\fint_{\Omega_R}|u_x|^q dx\right)\int_{\Omega_{lR}}|u_x|^q dx+cR^n,
\end{aligned}
$$

where the function $\omega_0(t;\tau)$ has the same properties as ω.

To estimate $J_R^{(2)}$, we note that $|w|\theta_+\le T$ in $\Omega_R(x_0)$ and apply estimate (22) once more:

$$
\begin{aligned}
J_R^{(2)} &\le\left(\fint_{\Omega_R}|u_x|^{sq}dx\right)^{1/s}\left(\int_{\Omega_R}(|w|\theta_+)^{s'}dx\right)^{1/s'}|\Omega_R|^{1/s}\le\\
&\le c\int_{\Omega_{lR}}(1+|u_x|^q)\,dx\left(\fint_{\Omega_R}(|w|\theta_+)^{s'}dx\right)^{1/s'}\le\\
&\le c\int_{\Omega_{lR}}(1+|u_x|^q)\,dx\left(\fint_{\Omega_R}|w|^q dx\right)^{1/s'}T^{1-q/s'}\le\\
&\le c\int_{\Omega_{lR}}(1+|u_x|^q)dx\left(\fint_{\Omega_R}|w_x|^q dx\right)^{1/s'}T^{1-q/s'}\le\\
&\le cT^{1-q/s'}\int_{\Omega_{lR}}(1+|u_x|^q)\,dx\left(\fint_{\Omega_R}(1+|u_x|^q)dx\right)^{1/s'}\le\\
&\le cT^{1-q/s'}\left(R^q+\fint_{\Omega_R}|u_x|^q\right)^{1/s'}\int_{\Omega_{lR}}|u_x|^q\,dx+cT^{1-q/s'}R^n.
\end{aligned}
$$

Now from (33) we deduce that

$$\int\limits_{\Omega_{R/2}} |w_x|^q dx \le \Big[c_9 s + \omega_0\Big(M; R^q + \fint\limits_{\Omega_R} |u_x|^q dx\Big) +$$

$$+ c_{10} T^{1-q/s'} \Big(R^q + \fint\limits_{\Omega_R} |u_x|^q dx\Big)^{1/s'}\Big] \int\limits_{\Omega_{lR}} |u_x|^q dx + c_{11} H(s) R^n.$$

Hence,

$$\int\limits_{\Omega_{R/2}} |w_x|^2 dx \le$$

$$\le \Big(\hat{c}s + G\Big(lR, \fint\limits_{\Omega_{lR}} |u_x|^q dx; T\Big)\Big) \int\limits_{\Omega_{lR}} |u_x|^q dx + cH(s)R^n, \quad (34)$$

where $G(r,\xi;T) = \omega_0(M; c(r^q+\xi)) + cT^{1-q/s'}(r^q+\xi)^{1/s'}$; $G(r,\xi,T) \to 0$ if T is fixed and $r,\xi \to 0$.

From (20) and (34) it follows that

$$\int\limits_{\Omega_\rho(x^0)} |u_x|^q dx \le$$

$$\le c\Big[\Big(\frac{\rho}{R}\Big)^\lambda + s + G\Big(lR, \fint\limits_{\Omega_{lR}(x^0)} |u_x|^q dx, T\Big)\Big] \int\limits_{\Omega_{lR}} |u_x|^q dx + H_0(s)(lR)^n,$$

$$\rho \le \frac{R}{2}, \quad H_0(s) \underset{s\to 0}{\to} +\infty.$$

Obviously, a similar inequality is also valid for $\rho \in \big(\frac{R}{2}, lR\big]$. In what follows,we write R in place of lR; $R \le \frac{\sigma}{2}$. In a result, we have

$$\int\limits_{\Omega_\rho(x^0)} |u_x|^q dx \le c_0\Big[\Big(\frac{\rho}{R}\Big)^\lambda + s + G\Big(R, \fint\limits_{\Omega_R(x^0)} |u_x|^q dx, T\Big)\Big] \times$$

$$\times \int\limits_{\Omega_R} |u_x|^q dx + H_0(s)R^n, \quad \rho \le R \le \frac{\sigma}{2}. \quad (35)$$

Now we put $\varphi(\rho, x^0) = \displaystyle\fint_{\Omega_\rho(x^0)} |u_x|^q dx$. By (35), $\varphi(\rho) = \varphi(\rho, x^0)$ satisfies the inequality

$$\varphi(\rho) \leq c_0 \left[\left(\frac{\rho}{R}\right)^{\lambda-n+q} + s\left(\frac{R}{\rho}\right)^{n-q} + \left(\frac{R}{\rho}\right)^{n-q} G(R, \varphi(R), T) \right] \varphi(R) + \\ + \frac{H_0(s) R^n}{\rho^{n-q}}.$$

For $\rho = \tau R$ with an arbitrary $\tau \in (0,1)$, we obtain

$$\varphi(\tau R) \leq c_0 \left[\tau^{\lambda-n+q} + \frac{s}{\tau^{n-q}} + \frac{G(R, \varphi(R), T)}{\tau^{n-q}} \right] \varphi(R) + \frac{H_0(s)}{\tau^{n-q}} R^q. \quad (36)$$

If $n \in (2, q+\lambda)$, then $\lambda = 2+\gamma$ and $\lambda - n + q > 0$. We fix $0 < \alpha < \frac{\lambda-n+q}{q}$ and put $\beta \in \left(\alpha, \frac{\lambda-n+q}{q}\right)$.

First we choose $\tau \in (0,1)$ in such a way that

$$c_0 \tau^{\lambda-n+q} = \frac{\tau^{q\beta}}{3}. \quad (37)$$

For this τ we choose $s \in (0,1)$ so that

$$\frac{c_0 s}{\tau^{n-q}} \leq \frac{\tau^{q\beta}}{3}. \quad (38)$$

The above τ and s are determined by the data of the problem (via c_0, λ, q, and β). Now the numbers $T = 2^{1+1/s}$ and $H_0 = H_0(s)$ are fixed in (36), and we obtain

$$\varphi(\tau R) \leq \left\{ \frac{2}{3}\, \tau^{q\beta} + G_0(R, \varphi(R)) \right\} \varphi(R) + H_1 R^q, \quad (39)$$

where

$$G_0(R, \varphi(R)) = \frac{c_0 G(R, \varphi(R), T)}{\tau^{n-q}}, \quad H_1 = \frac{H_0(s)}{\tau^{n-q}};$$

$G_0(r, \xi) \to 0$ as $r, \xi \to 0$.

Now we suppose that the point x_0 chosen satisfies the condition

$$\varliminf_{r \to 0} \varphi(r, x^0) = 0. \quad (40)$$

(Note that (40) is valid for any $x_0 \in \overline{B^+_{1-\sigma}}$ if $n \leq q$.) There exists a number $R_1 < \frac{\sigma}{2}$ such that $G_0(R_1, \varphi(R_1, x^0)) < \frac{\tau^{q\beta}}{3}$, and in the usual way, by iterations, we deduce that

$$\varphi(\rho, x^0) \leq c\left(\frac{\rho}{R_1}\right)^{q\alpha}, \quad \rho \leq R_1. \tag{41}$$

Moreover, for R_1 fixed, inequality (41) remains valid in a neighborhood of x_0, i.e., there is an $r_0 \in \left(0, \frac{\sigma}{2}\right)$ such that the function $\varphi(\rho, x)$ satisfies (41) for all $x \in B_{r_0}(x^0) \cap (B_1^+ \cup \Gamma_1)$. This means that $u_x \in L^{q,n-q+q\alpha}(B_r(x^0) \cap B_1^+)$, $r < r_0$. From this it follows (see, e.g., [7,Chapter 3]) that u is Hölder continuous with the exponent α on the set $B_r(x^0) \cap (B_1^+ \cup \Gamma_1)$. The set of singular points (i.e., those where (40) is invalid) can be estimated in the standard way.

Now let $n = 2$. Under the restrictions $2 \leq q \leq n \leq q + 2$, we have only the case of $q = 2$. For $q = 2$ the problem was studied by the author in [1]. If we assume that $q > n = 2$, then $u \in C^{\alpha_1}$ with $\alpha_1 = 1 - \frac{2}{q} > 0$ by the embedding theorem. Now we show that in this case the Hölder exponent is an arbitrary number in the interval $(0, 1)$.

We denote $\varkappa(\rho) = \int\limits_{\Omega_\rho(x^0)} |u_x|^q dx$ for any fixed $x^0 \in \overline{B^+_{1-\sigma}}$ and write (35) in the form

$$\varkappa(\rho) \leq c\left[\left(\frac{\rho}{R}\right)^2 + s + G(R, R^{\alpha_1 q}\varkappa(R), T)\right]\varkappa(R) + H_0(s)R^2,$$
$$\rho \leq R \leq \frac{\sigma}{2}. \tag{42}$$

It is obvious that $\varkappa(\rho)$ is a bounded and nondecreasing function in $\rho \in [0, R]$; $\lambda = 2 = n$.

For a fixed $\tau \in (0, 1)$ we put $\rho = \tau R$ and rewrite (42) in the form

$$\varkappa(\tau R) \leq c[\tau^2 + s + G(R, R^{\alpha_1 q}\varkappa(R), T)]\varkappa(R) + H_0(s)R^2.$$

Now we fix arbitrary numbers $\beta_1 < \beta < 1$ and choose τ to have the inequality

$$c\tau^2 = \frac{\tau^{2\beta}}{3}.$$

Then we put $s = \frac{\tau^{2\beta}}{3}$ and obtain a fixed number $T = 2^{1+1/s}m$. After that we choose $R_1 < \frac{\sigma}{2}$ satisfying the condition

$$G(R_1, R_1^{\alpha_1 q}\varkappa(R_1), T) < \frac{\tau^{2\beta}}{3}$$

and recall that $G(\xi,\eta,T)$ is a nondecreasing function in ξ and η for a fixed T.

This allows us to make iterations in the inequality

$$\varkappa(\tau R) \leq \tau^{2\beta}\varkappa(R) + HR^2$$

with $R \leq R_1$ and $H = H_0(s)$. In a result, we obtain

$$\varkappa(\rho) \leq A\left[\left(\frac{\rho}{R}\right)^{2\beta_1}\varkappa(R) + H\rho^{2\beta_1}\right], \quad \rho \leq R \leq R_1,$$

where $\beta_1 < \beta$, and the constant A depends only on the data.

Since

$$\int\limits_{\Omega_\rho(x^0)} |u_x|^q dx \leq c\rho^{2\beta_1}, \rho \leq R_1,$$

we have

$$\int\limits_{\Omega_\rho(x^0)} |u - u_{x^0,\rho}|^q dx \leq c\rho^{q+2\beta_1}, \rho \leq R_1.$$

The latter inequality holds for every point $x^0 \in \overline{B^+_{1-\sigma}}$. This means that $u \in \mathcal{L}^{q,q+2\beta_1}(B^+_{1-\sigma})$ and, as a consequence, $u \in \mathbb{C}^\alpha(\overline{B^+_{1-\sigma}})$, where $\alpha = 1 - \frac{2(1-\beta_1)}{q}$. We may claim that α is an arbitrary exponent in $(0,1)$, because $\beta_1 \in (0,1)$ was fixed arbitrarily.

Theorem 1 is proved.

5. Proof of Theorem 2.

The assumptions of Theorem 2 imply those of Theorem 1, and any weak solution u of (11), $n = 2$, is of class $\mathbb{C}^\alpha$ where $\alpha < 1$ is arbitrary. If $x^0 \in \overline{B^+_{1-2\sigma}}$ and $R < \sigma$ are fixed, then we have the estimates

$$\operatorname*{osc}_{\Omega_R(x^0)} u \leq cR^\alpha, \qquad \int\limits_{\Omega_R(x^0)} |u_x|^q dx \leq cR^{2-q+q\alpha}, \tag{42}$$

where the constant $c > 0$ depends on the fixed σ.

We consider the model problem

$$\begin{cases} \dfrac{d}{dx_\alpha} a^k_\alpha(x^0, u^0, v_x) = 0 \qquad \text{in} \quad \Omega_R(x^0) \\ a^k_n(x^0, u^0, v_x)\big|_{\gamma_R(x^0)} = 0, \qquad v\big|_{S'_R(x^0)} = u. \end{cases} \tag{43}$$

It does not coincide with (24), because $a^k_\alpha(x^0,u^0,0)$ may not be equal to zero. But estimates (17) and (19) are valid for a solution of system (43) under the given condition on $\gamma_R(x^0)$.

We have the following chain of inequalities for $\rho \le \frac{R}{2}$:

$$\int\limits_{\Omega_\rho(x^0)} |v_x - (v_x)_{x^0,\rho}|^q dx \le$$
$$\le c\int\limits_{\Omega_\rho} (|v_x|^2 + |(v_x)_{x^0,\rho}|^2)^{\frac{q-2}{2}} |v_x - (v_x)_{x^0,\rho}|^2 dx \le$$
$$\underset{\{1\}}{\le} c\int\limits_{\Omega_\rho} |W(v_x) - W((v_x)_{x^0,\rho})|^2 dx \underset{\{2\}}{\le} c\int\limits_{\Omega_\rho} |W(v_x) - (W(v_x))_{x^0,\rho}|^2 dx \le$$
$$\underset{\{3\}}{\le} c\rho^2 \int\limits_{\Omega_\rho} |(W(v_x))_x|^2 dx \underset{\{4\}}{\le} c\left(\frac{\rho}{R}\right)^\gamma \rho^2 \int\limits_{\Omega_{R/2}} |(W(v_x))_x|^2 dx \le$$
$$\underset{\{5\}}{\le} c\left(\frac{\rho}{R}\right)^{2+\gamma} \int\limits_{\Omega_R} (1+|v_x|^2)^{\frac{q-2}{2}} |v_x|^2 dx.$$

Here, $\{1\}$ and $\{2\}$ are some known technical inequalities (see, for example, (2.4) and (2.6) in [12]). At step $\{3\}$ we applied the Poincare inequality; at steps $\{4\}$ and $\{5\}$ we made use of estimates (19) and (17), respectively.

As a result, we arrive at the inequality

$$\int\limits_{\Omega_\rho(x^0)} |v_x - (v_x)_{x^0,\rho}|^q dx \le c\left(\frac{\rho}{R}\right)^{2+\gamma} \int\limits_{\Omega_R(x^0)} |v_x|^2 (1+|v_x|^{q-2})\, dx \tag{44}$$

with a constant c dependent on n, q, ν, μ, and $\varkappa$. Obviously, (44) holds for all $\rho \in (0,R]$, but not only $\rho \le \frac{R}{2}$. From the integral idendity for (43) we obtain

$$\int\limits_{\Omega_R} |w_x|^q dx \le c\int\limits_{\Omega_R} (1+|u_x|^q)\, dx,$$

where $w = u - v \in \overset{\circ}{W}{}^1_q(\Omega_R \cup \gamma_R)$.

Moreover,

$$\int\limits_{\Omega_R} [a^k_\alpha(x^0,u^0,u_x) - a^k_\alpha(x^0,u^0,v_x)] h^k_{x_\alpha} dx +$$

$$+\int\limits_{\Omega_R}(\Delta a_\alpha^k h^k_{x_\alpha}+b^k h^k)\,dx=0 \tag{45}$$

with $h\in\overset{\circ}{W}{}^1_q(\Omega_R\cup\gamma_R)\cap L_\infty(\Omega_R)$. Here, $\Delta a_\alpha^k = a_\alpha^k(x^0,u^0,u_x)-a_\alpha^k(x,u,u_x)$, $u^0=\fint\limits_{\Omega_R} u\,dx$, $\Omega_R=\Omega_R(x^0)$, and $\gamma_R=\gamma_R(x^0)$.

Substituting $h=w\min\left\{1,\frac{l}{|w|}\right\}$ with $l>>1$ into (45), we derive the inequality

$$\int\limits_{\Omega_R}(\varkappa+\nu_1|w_x|^{q-2})|w_x|^2\min\left\{1,\frac{l}{|w|}\right\}\le$$

$$\le c(\mu)\Bigg[\int\limits_{K_{l,R}}|w_x|(1+|u_x|^{q-1})dx\cdot R^{\alpha\theta}+$$

$$+\int\limits_{K_{l,R}}|w_x|^2\Bigg(\int\limits_0^1(1+|u_x+tw_x|^{q-2})\,dt\Bigg)dx+$$

$$+R^{\alpha\theta}\int\limits_{\Omega_R}|w_x|\min\left\{1,\frac{l}{|w|}\right\}(1+|u_x|^{q-1})\,dx+$$

$$+\int\limits_{\Omega_R}(a_0|u_x|^q+b)\min\{|w|,l\}\,dx\Bigg],$$

where $K_{l,R}=\{x\in\Omega_R|\ |w(x)|>l\}$.

From this inequality we deduce that

$$\int\limits_{\Omega_R}|w_x|^q\min\left\{1,\frac{l}{|w|}\right\}dx\le c_1\int\limits_{K_{l,R}}(|w_x|^q+|u_x|^q+1)\,dx+$$

$$+c_2R^{\frac{\alpha\theta q}{q-1}}\int\limits_{\Omega_R}|u_x|^qdx+c_3I_{l,R}+c_4\delta\int\limits_{\Omega_R}|w_x|^qdx+c(\delta)R^{2+\frac{q}{q-1}},$$

where $I_{l,R}=\int\limits_{\Omega_R}|u_x|^q\min\{|w|,l\}\,dx$, and any $\delta\in(0,1)$.

We observe that

$$\int\limits_{\Omega_R}|w_x|^q\min\left\{1,\frac{l}{|w|}\right\}dx\ge\int\limits_{\{\Omega_R:|w|\le l\}}|w_x|^qdx=$$

$$= \int\limits_{\Omega_R} |w_x|^q dx - \int\limits_{K_{l,R}} |w_x|^q dx.$$

Putting $\delta = \frac{1}{2c_4}$ and estimating $\int\limits_{\Omega_R} |u_x|^q dx$ by (42), we arrive at the inequality

$$\int\limits_{\Omega_R} |w_x|^q dx \leq c_5 \int\limits_{K_{l,R}} (|w_x|^q + |u_x|^q + 1)\, dx + c_6 R^{2-q+q\alpha+\frac{\alpha\theta q}{q-1}} + c_7 I_{l,R}. \quad (46)$$

To estimate $I_{l,R}$, we consider the identity

$$\int\limits_{\Omega_R} (a^k_\alpha(x,u,u_x)h^k_\alpha + b^k(x,u,u_x)h^k)\, dx = 0$$

with a function $h = (u - u(x^0)) \min\{|w|, l\} \in \overset{\circ}{W}{}^1_q(\Omega_R \cup \gamma_R) \cap L_\infty(\Omega_R)$. From this relation we derive the inequality

$$\begin{aligned} I_{l,R} \frac{\nu}{q-1} &\leq \mu \int\limits_{\Omega_R} |u_x| \min\{|w|, l\}\, dx + \\ &+ c_8 R^\alpha \int\limits_{\Omega_R} (1 + |u_x|^{q-1})|w_x|\, dx + c_9 R^\alpha \Big(I_{l,R} + \int\limits_{\Omega_R} |w|\, dx \Big). \end{aligned} \quad (47)$$

Now we fix R_0 in order to have the inequality

$$c_9 R_0^\alpha \leq \frac{\nu}{2(q-1)},$$

and from (47) with $R \leq R_0$ we deduce that

$$I_{l,R} \leq c(\epsilon) R^{\alpha \frac{q}{q-1} + 2 - q + q\alpha} + \varepsilon \int\limits_{\Omega_R} |w_x|^q dx, \quad \varepsilon > 0. \quad (48)$$

Now we put $\varepsilon = \frac{1}{2c_7}$ and conclude from (46) and (48) that

$$\int\limits_{\Omega_R} |w_x|^q dx \leq c_{10} \int\limits_{K_{l,R}} (|w_x|^q + |u_x|^q)\, dx + c_{11} R^{2-q+q\alpha+\frac{\alpha\theta q}{q-1}}. \quad (49)$$

All the constants $c_1, \ldots, c_{11}$ depend on ν, μ, $\varkappa$, a_0, and b, but not on $l \gg 1$.

Turning l to infinity, from (49) we obtain the inequality

$$\int\limits_{\Omega_R} |w_x|^q dx \le cR^{2-q+q\alpha+\frac{\alpha\theta q}{q-1}}. \tag{50}$$

From (42), (44), and (50) it follows that

$$\begin{aligned}\int\limits_{\Omega_\rho(x^0)} & |u_x - (u_x)_{x^0,\rho}|^q dx \le \\ & \le c_{12}\left(\frac{\rho}{R}\right)^{2+\gamma} R^{2-q+q\alpha} + c_{13}R^{2-q+q\alpha+\frac{\alpha\theta q}{q_1}},\end{aligned} \tag{51}$$

for any $x^0 \in \overline{B^+_{1-2\sigma}}$, $\rho \le R \le R_0$.

For a number $s > 1$ we put $\rho = R^s$ and rewrite (51) in the form

$$\int\limits_{\Omega_\rho(x^0)} |u_x - (u_x)_{x^0,\rho}|^q dx \le c(\rho^{l_1} + \rho^{l_2}), \tag{52}$$

where $l_1 = 2 + \gamma - \frac{q+\gamma-q\alpha}{s}$ and $l_2 = \frac{2-q+q\alpha+\frac{\alpha\theta q}{q-1}}{s}$.

We choose $\alpha \in (0,1)$ and $s > 1$ for the fixed θ and γ, where $\theta \in (0,1]$ is the exponent from (13), $\gamma = 2\big(1-\frac{2}{p_0}\big)$, and $p_0 > 2$ is the number from Proposition 1.

For $\eta = \frac{\theta}{q-1}$ we put $\alpha = \frac{2+\gamma(1+\eta/2)}{2+\gamma(1+\eta)}$ and $s = 1 + \frac{q\eta}{2+\gamma(1+\eta)}$, then

$$\begin{aligned} l_1 &= 2 + \alpha_1 q, \quad \alpha_1 = \frac{\gamma\eta}{2(2+\gamma(1+\eta)+q\eta)}, \\ l_2 &= 2 + \alpha_2 q, \quad \alpha_2 = \alpha_1(1+\eta). \end{aligned}$$

Now from (51) we obtain

$$\int\limits_{\Omega_\rho(x^0)} |u_x - (u_x)_{x^0,\rho}|^q dx \le c\rho^{2+q\alpha_1},$$

for $\rho \le \rho_0 = R_0^s$ and any $x^0 \in \overline{B^+_{1-2\sigma}}$.

Hence, u_x belongs to the Campanato space $\mathcal{L}^{q,2+q\alpha_1}(B^+_{1-2\sigma})$. This guarantees that $u_x \in \mathbb{C}^{\alpha_1}(\overline{B^+_{1-2\sigma}})$ if $n = 2$. Theorem 2 is proved.

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References

1. A. Arkhipova, *On the Neumann problem for nonlinear elliptic systems with quadratic nonlinearity.* — St. Petersburg Math. J. **8**, No. 5 (1997), 845–861.
2. Ch. Hamburger, *An elementary partial regularity proof for solutions of nonlinear elliptic systems.* — Manuscr. Math. **95**, No. 1 (1998), 11–31.
3. S. Campanato, *Elliptic systems with non-linearity $q \geq 2$. Regularity of the solution of the Dirichlet problem.* — Ann. Matem. Pure Appl. **147** (1987), 117–150.
4. S. Campanato, *Hölder continuity of the solutions of some nonlinear elliptic systems.* — Adv. Math. **48** (1983).
5. A. Arkhipova, *Reverse Holder Inequalities with boundary integrals and L^p-estimates for solutions of elliptic and parabolic nonlinear boundary-value problems.* — Am. Math. Soc. Transl. **164**(2) (1995).
6. A. Arkhipova, *On the regularity of solutions of boundary-value problems for quasilinear elliptic systems with quadratic nonlinearity.* J. Math. Sci.. **80** (1996), No. 6, 2208–2225; in Russian: Probl. Mat. Anal., No.15, S.-Ptersburg Gos. Univ., St.-Petersburg, 1995, 47–69.
7. M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems.* — Princeton, NJ (1983).
8. S. Campanato, *A maximum principle for nonlinear elliptic systems; boundary fundamental estimates.* — Adv. Math. **66** (1987), 291–317.
9. S. Campanato, *Nonlinear elliptic systems with quadratic growth.* — Confer. Sem. Mat. Univ. Bari **208** (1986), 19 pp.
10. M. Giaquinta and E. Ciusti, *Nonlinear elliptic systems with quadratic growth.* — Manuscr. Mat. **24** (1978), 323–349.
11. A. Arkhipova, *On the regularity of solutions of model nonlinear elliptic systems with boundary condition of the type of an oblique derivative.* — Zap. Nauchn. Semin. POMI **221** (1995), 30–57.
12. M. Giaquinta and G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals.* — Manuscr. Math., **57** (1986), No. 1, pp. 55–99.

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