Записки научных
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KNOWLEDGE BASE ON SPECIAL FUNCTIONS

1. Introduction

The books on special functions are common for physicists and many researches involve special functions as a key tool for the description the theoretical models. However, the majority of the existing sources expose knowledge only on the simplest special functions - namely that belonging to the Riemann (hypergeometric) class. On the other hand modern studies require the use of more sophisticated special functions (solutions of both linear and non-linear ODEs). In this latter case only restricted information is available at the moment and known formulas are too complicated and relations between them are not easy to comprehend within the traditional approach.

At first the problem of the "uniform" structural approach to the classification of special functions arises. Secondly, modern information technologies propose knowledge bases and data bases instead of books — there a problem of data navigation and processing can be solved by means of programming tools on the base of uniform principles.

The classification scheme for special functions of Heun class and hypergeometric functions has been elaborated in a series of articles by S. Slavyanov, W. Lay and A. Seeger [1]–[3]. Further the Painlevé transcendents were added to the classification scheme [4].

It allows to start the knowledge base project on this bases. As a prototype the program SFTools was created. Special functions presented in the system are hypergeometric-type functions, Heun functions, Painlevé transcendents, orthogonal polynomials. For these functions basic differential equations with emphasize to singularities, integral relations, series expansions, Riemann's P-symbols (Riemann scheme) and other important characteristics are exhibited. In order to cope with the information the processes of confluence and reduction of equations and their parameters are widely used.

For the present moment two versions of the system exist -

1 — a stand-alone dialogue system written in Borland Delphi 3 running under the MS Windows'95 or NT 4.0 or later environment;

2 — an Internet modification of the first system coded in Java. It can be reached at the following URL: www.cp.phys.spbu.ru

2. Necessary basic terms

The theory of special functions can be constructed either from the analytical theory of differential equations or by means of the group theory. For the project at hand the first approach was chosen.

A systematic classification of differential equations is carried out by distinguishing the equations according to the number and the character of their singular points. Consider the basic terms and principles used in the classification process.

All special functions of Heun and hypergeometric classes are specifically chosen solutions of ordinary second order linear differential equations with polynomial coefficients. The common form for such an equation is

$$L_z y(z) = P_0(z)y''(z) + P_1(z)y_1(z) + P_2(z)y(z) = 0 \qquad (1)$$

where P_0, P_1, P_2 are polynomials in z.

All points of the complex plane $z \in C$ in the vicinity of which the Cauchy problem (i.e., the initial value problem with arbitrary initial data at this point) can be solved are called *ordinary points* of equation (1). Those points where the Cauchy problem can not be posed are singular points (or singularities) of equation (1). These are the zeros of the polynomial P_0 and possibly the point at infinity. There are two types of singular points. Around *regular singularities* solutions of (1) can — in general — be expressed in terms of two Laurent series with a finite number of negative power terms times a power x^ρ, ρ complex. Laurent series approximating the solution near irregular singularities are infinite in both directions. As it was shown, for instance, by Ince [5] (see also [2]), any irregular singular point of the differential equation can be regarded as a result of a so-called *confluence process* (or a finite number of confluence processes) of regular singularities (a limiting process involving coefficients of the equation where singularities of the equation coalesce). This fact is the basis of a uniform approach to the classification of special functions.

The second process that should be used in parallel to confluence processes is the *reduction process* of the equation which means the decrease of the number of coefficients accompanied by decrease of the s-rank of a singularity (see below).

The hypergeometric class of equations comprises those ODEs that arise by confluence and reduction from the hypergeometric equation. The Heun class of equation comprises those equations that arise from the Heun one – equation with four regular singularities.

For each singular point z_i of ODE there is a vicinity in which functions

$$P(z) = \frac{P_1(z)}{P_0(z)}, \quad Q(z) = \frac{P_2(z)}{P_0(z)}$$

are expanded in the convergent Laurent series of the form

$$P(z) = \sum_{j=-K_1}^{\infty} E_{ij}(z - z_i)^j, \ E_{i,-K_1} \neq 0,$$
$$Q(z) = \sum_{j=-K_2}^{\infty} D_{ij}(z - z_i)^j, \ D_{i,-K_1} \neq 0.$$

The *s-rank* of a finite irregular singular point of ODE is defined as the number

$$R(z_i) = \max\{K_2(z_i)/2, K_1(z_i)\}.$$

The s-rank for the singular point at infinity is defined with the help of the Möbius transformation putting it to a finite point.

It follows from the definition that the s-rank of an irregular singularity either is an integer or is a halfinteger positive number. Singularities characterized by halfinteger s-ranks are called *ramified singularities*. Formal asymptotic series in their vicinity (so-called subnormal solutions) are constructed in terms of halfinteger powers. For irregular singularity possessing integer s-rank the formal asymptotic solution is so-called normal solution. Equations possessing ramified singularities are called *reduced confluent* equations. Among the regular singularities (with the corresponding Fröbenius solution) we distinguish between elementary ones (for which the difference between two characteristic exponents is equal to 1/2) and regular singularities which are not elementary. The first are characterized by the s-rank $R(z_i) = 1/2$ the second by the s-rank $R(z_i) = 1$.

One of the main notions in our classification scheme is the *s-rank multisymbol* of the ODE. This notion represents all qualitative characteristics of the corresponding equation. The s-rank multisymbol is defined as a set of the s-ranks for all singular points of the studied ODE. For example, the Heun equation from which the Heun class originates has the s-rank multisymbol $\{1, 1, 1; 1\}$ and the equation the solutions of which are the Bessel functions has the s-rank multisymbol $\{1; 3/2\}$. The semicolon in the multisymbol detaches the singularity at infinity from finite singular points.

A more refined description of the considered ODE is exposed by the *generalized Riemann scheme* (GRS) [3]. It comprises not only the s-ranks and locations of the singularities but also *characteristic exponents* of Fröbenius, normal, and subnormal solutions at the singularity [5].

It is needed to stress that the Möbius transforms of the independent variable and the appropriate gauge transforms of the dependent variable modify the equation to its various *forms*.

Above, only linear equations were discussed. The so-called Painlevé equations can be regarded as classical analogues of equations belonging to the Heun class regarded as quantum Schrödinger equations [4]. This definition is equivalent to the fact that Painlevé transcendents enjoy the Poincaré property [5].

3. Classification scheme

The classification scheme represented in our program "SFTools" covers specific solutions of two classes of linear differential equations (hypergeometric and Heun classes of equations), Painlevé transcendents related to nonlinear Painlevé equations and the classical orthogonal polynomials. All these four objects are connected with each other. Inside each class the equations are organized with respect to the characteristic features of their singularities and the way they have been derived from the original equation. The Heun equation with s-rank multisymbol $\{1, 1, 1; 1\}$ is the generating equation for the Heun class and the hypergeometric equation with s-rank multisymbol $\{1, 1; 1\}$ is the generating equation for the Riemann class. Each equation is presented on the scheme by its s-rank multisymbol. These s-rank multisymbols form a two-dimensional structure consisting of horizontally placed clusters. These clusters comprise all equations which could be obtained from the first (left) equation in the cluster by means of the sequence of reduction processes. The number of such clusters is defined by the number N of confluent cases of equations in one class and is equal to $N + 1$.

The scheme for the Painlevé class was built in order to reflect the correspondence between Painlevé equations and the equations of Heun class so that each equation on the scheme for the Painlevé class can be derived from the Heun equation located on the same place in the scheme for the Heun class. The s-rank multisymbols located in one and the same column represent differential equations having the same number of irreducible parameters.

Classical polynomials originate as particular solutions of specific differential equations with polynomials coefficients and therefore are also included in the scheme. However, we don't pronounce this feature in particular.

4. SFTools: Description of the program

4.1. Stand-alone version. The stand-alone dialogue version of the program is currently developed for running under Microsoft Windows'95 or Windows NT 4.0 or later versions. The main program as well as the help system are to work in 32-bit environment. All necessary files take a space of about 10 Mbytes. The required space is likely to slight changes when adding new information to the databases. A "Graphic User Interface" was created to work in 600 – 800 screen mode or higher. Minor modes are also supported. For to work with SFTools on a computer it should be installed data Borland Database Engine (BDE). This procedure is automatized by means of the installation program (see below).

We achieved to satisfy four main conditions to make SFTools powerful and flexible at the same time:

- Data should be stored in a way which can provide easy access from different instrument tools such as editors with mathematical support, symbolic algebra software etc.
- There should be an easy way for adding new data.
- Adding new and editing old data should as little as possible affect the development of other parts of the software product such as GUI, computational part etc.
- The process of installation of the program and necessary files (i.e., BDE) should be automatized and require least efforts from the user.

For these goals several external PARADOX databases were used for holding all scientific data. Access to the databases is provided by BDE. This certainly adds some requirements to the system. On the other hand the interaction between programmers and scientists becomes more convenient while a lot of data is expected to appear and change when the software is created. Our own experience shows that this aspect is very essential for the projects of such kind. The Graphic Interface of SFTools is to reflect as much of the properties of the classification scheme described

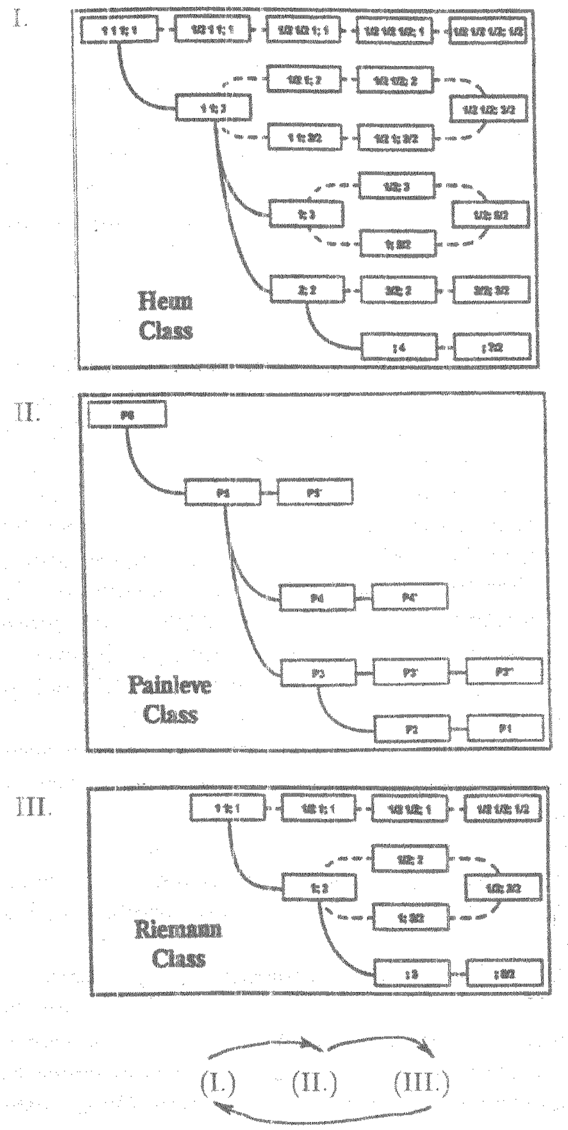

Fig. 1. Classification scheme for special functions of Heun, Riemann (hypergeometric) and Painleve classes
—– confluence process
..... reduction process
—→ correlation between classes.

above as possible. Technical details of creating this particular GUI aren't too different from any GUI created by Borland Delphi Developer. For this reason the details of it's realization are omitted here and only the end-user behavior is described.

The last mentioned condition is satisfied by using Install Shield Light 1.1 utility that generates all necessary codes and data for automatizing processes of program installation, necessary system files modifications and tuning of BDE.

SFTools session uses one and only one window which is organized as a tabset with four pages. Each page contains the information windows for one of three classes: Painlevé, Riemann (hypergeometric), Heun and Classical Polynomials. Classification schemes are constructed of coloured button-like elements. Each element carries the s-rank multisymbol (for the first three classes) or an abbreviation of the polynomial name (the latest one) corresponding to one particular equation. Being pushed, an element becomes active and all the information available for this equation appears on the screen in the information windows. Information which can be found in the current version of SFTools is the following:

for Painlevé class:	corresponding equation;
for Riemann class:	corresponding equation;
	Generalized Riemann Scheme (GRS);
	integral representation of solutions ;
	series solutions ;
for Heun class:	corresponding equation;
	Generalized Riemann Scheme (GRS);
for Classical	corresponding differential equation;
polynomials:	orthogonality condition
	recurrence relations
	corresponding difference equation.

Besides this, for each class there is one editable textbox where any comments can be inserted by the users during the program session. Active elements are highlighted by red on the electronic scheme.

As was mentioned above not all transitions between equations are known, yet. This concerns first of all the way of getting confluent equations. SFTools highlights in green those equations which can be derived from the one currently chosen. If the mouse pointer is moved upon such a green-coloured element new information a window appears which contains analytical formulas for the corresponding transformation. The basic mathematical theory which was presented briefly in the first part of this

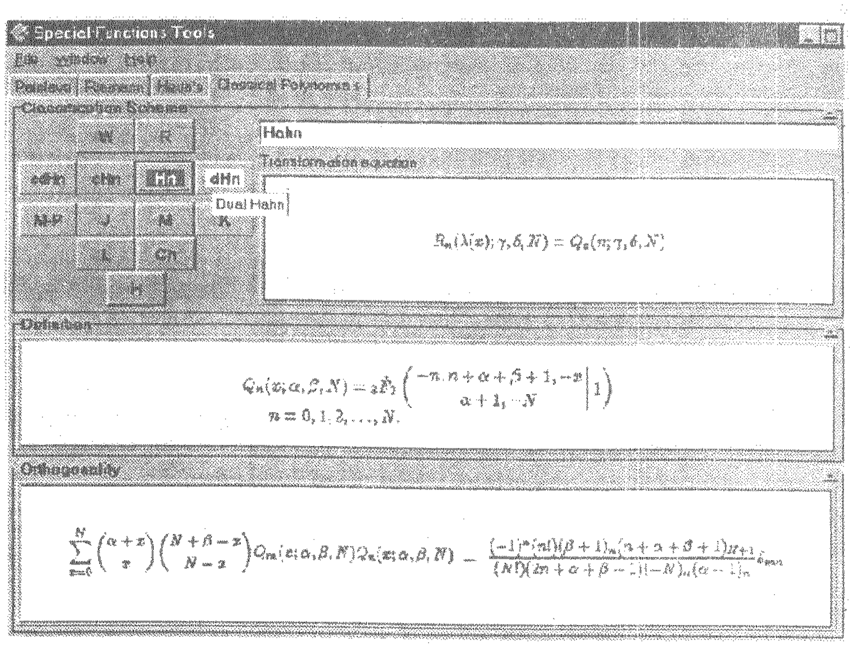

Fig. 2.

paper is exposed in the context help system of SFTools. Help information is stored in standard MS Windows 32-bit help file which can be accessed by direct opening as well as during SFTools session.

4.2. Internet version. The Internet version of SFTools program is developing on the base of stand-alone version bearing in mind modern trends in information technology. The program is supposed to work with any browser that supports HTML 2.0 and graphical user interface and frames. It supports nearly all the features of the stand-alone version and also has the possibility of a rapid modification and renewal of the information.

5. Possible future development

In the current version SFTools provides only static information. Even this information is significant since the large number of equations which were dispersed in many books is concentrated in one place and can be delivered in an easy interactive way. All relations between equations can

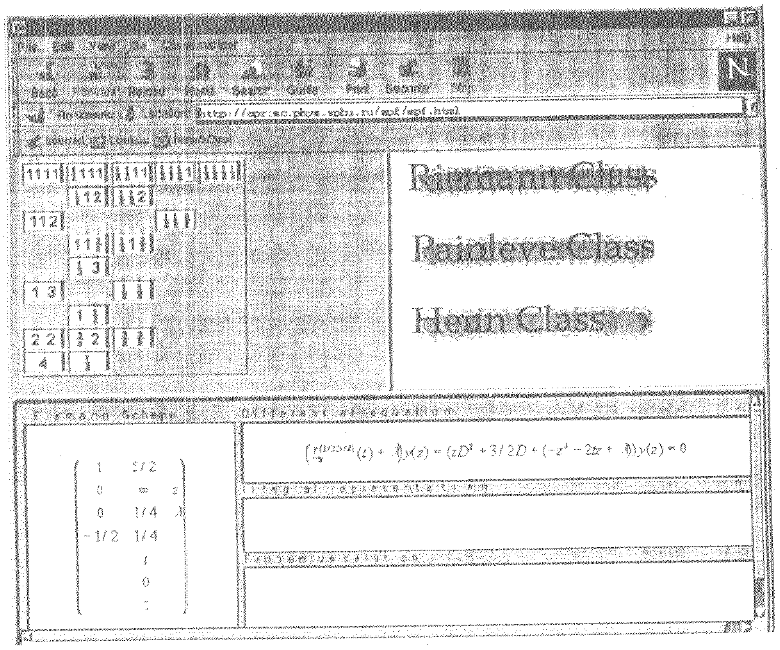

Fig. 3.

be easily seen as well. However, there is no way to use formulas from SFTools databases directly in popular scientific computer algebra systems. It is our next goal, to provide SFTools with this feature. The other direction of a program development is creating an interactive system that can accept user data as input, carry out all necessary transformations of data according to formulas in the database and then displays the results in user-specific terms. As soon as this step is achieved SFTools will be proceeded from an ordinary information system to a real scientific instrument.

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