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CORRELATORS OF THE LADDER SPIN MODEL IN THE STRONG COUPLING LIMIT

Abstract. Temperature and time dependent correlation functions of the spin–$\frac{1}{2}$ ladder model with infinitely strong coupling are calculated.

Introduction

Correlation functions give the most detailed information about a physical system. Their description in integrable models solvable by means of the Bethe Ansats [1] is based essentially on representations (in the thermodynamic limit) as the Fredholm determinants of linear integral operators. These operators being of a very special kind ("integrable" integral operators in the sense of paper [2], it is possible to derive a system of calssical integrable equations for quantum correlation functions and to calculate the large time and distance asymptotics [3–8], see also [9] for a review).

A representation of this kind was first obtained long ago ([10, 11]) for the simplest two-point equal-time correlator in the model of the one-component one-dimensional impenetrable bosons described by the quantum nonlinear Schrödinger equation with infinitely large coupling. The algebraic Bethe Ansatz [12–14] made it possible to obtain the determinant representations (containing in this case the auxiliary "dual" fields) also for the case of finite coupling. It was done first in [15] (see also [9], and paper [16] for the recent progress in this direction). In the case of infinite coupling (i.e., at the "free fermion points" of integrable models) there are no auxiliary fields, and the situation, though untrivial, is much simpler. The determinant representation for the time and temperature dependent correlation functions of the impenetrable bosons was obtained in [17], and for the case of the isotropic spin–1/2 $XX0$ Heisenberg chain it was given in papers [18, 19].

The multi-component one-dimensional Bose and Fermi gases can be solved by the multi-component ("nested") Bethe Ansatz [20, 21]. In particular, for the two-component gas with a finite coupling the nested

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Ansatz is also two-component, the second component being the Bethe Ansatz for the auxiliary XXX Heisenberg chain which cannot be solved explicitly. The approach for calculating correlation functions of two-component models at free fermion points suggested recently in papers [22, 23] (where two-component impenetrable Bose and Fermi gases were considered) is based on the fact that the auxiliary lattice problem of the nested Bethe Ansatz in the case of the infinitely strong coupling is reduced to the problem of finding eigenvectors of the isotropic spin–1/2 $XX0$ Heisenberg chain, i.e., it can be solved explicitly. This is the reason that the explicit formulae for the correlation functions are also obtained. The structure of the answers is, however, more bulky than in the one-component case, namely, the correlation functions in the thermodynamic limit are expressed now as integrals over some parameter of the Fredholm determinant of linear integral operators depending on this parameter. These linear integral operators are "integrable" operators. Moreover, they are practically the same as for the one-component impenetrable bosons. One can thus derive (just reproducing the corresponding procedure for the one-component case, see, e.g., [24]) the system of differential equations for the correlation functions in this case which also appears to be essentially the same as for the one-component impenetrable bosons. This allows to construct the asymptotics at large time and distance [25, 26]. This approach was applied in papers [27–29] to calculate the correlators of the one-dimensional Hubbard model [30] in the limit of infinitely strong repulsion. The Hubbard model is a lattice analogue of the two-component Fermi gas.

The aim of the present paper is to calculate temperature time-dependent correlation functions of the "ladder" spin model consisting of two Heisenberg $XX0$ spin–1/2 chains interacting via the third local spin components in the limit of infinitely strong coupling. By means of the Jordan–Wigner transformation, the Hamiltonian of the ladder model can be transformed to the Hamiltonian of the Hubbard model. So the relation between the ladder model and the Hubbard model is in this sense similar to the relation between the $XX0$ Heisenberg chain and free fermions on the lattice [31]. The expression of the local spin operators are, however, nonlocal in terms of the local fermion operators. So we calculate the matrix elements necessary for calculating correlator functions directly in terms of the local spin operators, similarly to the case of the simple $XX0$ chain [19].

1. The spin ladder model

Let us consider the spin "ladder" model with the Hamiltonian

$$H_U = -\sum_{\alpha=1}^{2}\sum_{m=1}^{L}(s^+_{m,\alpha}s^-_{m+1,\alpha} + s^-_{m+1,\alpha}s^+_{m,\alpha}) + \widetilde{U}\sum_{m=1}^{L} s^z_{m,1}s^z_{m,2}$$
$$- \widetilde{h}_1\sum_{m=1}^{L} s^z_{m,1} - \widetilde{h}_2\sum_{m=1}^{L} s^z_{m,1} - C \tag{1}$$

imposing the periodic boundary conditions,

$$s^a_{L+1,\alpha} \equiv s^a_{1,\alpha}. \tag{2}$$

Local spin 1/2 operators $s^a_{m,\alpha}$ ($a = x, y, z$; $m = 1, 2, \dots, L$; $\alpha = 1, 2$) are described by the Pauli matrices with commutation relations

$$[s^a_{m,\alpha}, s^b_{n,\beta}] = 2i\delta_{mn}\delta_{\alpha\beta}\epsilon^{abc}s^c_{m,\alpha};$$
$$s^\pm_{m,\alpha} = \frac{1}{2}(s^x_{m,\alpha} \pm is^y_{m,\alpha}). \tag{3}$$

The constant C,

$$C = (\widetilde{U} - \widetilde{h}_1 - \widetilde{h}_2)L, \tag{4}$$

is chosen so that the energy of the ferromagnetic (all spins up) reference state $|0\rangle$ equals zero,

$$|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle;\ \ \langle 0|0\rangle = 1;\ \ s^+_{m,\alpha}|0\rangle = 0, \qquad H_U|0\rangle = 0. \tag{5}$$

Under the Jordan–Wigner transformation,

$$c_{m,1} = e^{i\pi N_{m-1,1}}s^+_{m,1};\qquad c_{m,2} = e^{i\pi N_1}e^{i\pi N_{m-1,2}}s^+_{m,2};$$
$$c^+_{m,1} = s^-_{m,1}e^{-i\pi N_{m-1,1}};\quad c^+_{m,2} = s^-_{m,2}e^{-i\pi N_1}e^{-i\pi N_{m-1,2}}, \tag{6}$$

the Hamiltonian H_U (1) becomes just the Hubbard model Hamiltonian,

$$H_U = -\sum_{m=1}^{L}\sum_{\alpha=1,2}\left(c^+_{m,\alpha}c_{m+1,\alpha} + c^+_{m+1,\alpha}c_{m,\alpha}\right)$$
$$-h\sum_{m=1}^{L}(n_{m,1}+n_{m,2}) + B\sum_{m=1}^{L}(n_{m,1}-n_{m,2}) + U\sum_{m=1}^{L}n_{m,1}n_{m,2} \tag{7}$$

with parameters

$$U = 4\widetilde{U}; \quad h = 2\widetilde{U} - \widetilde{h}_1 - \widetilde{h}_2; \quad B = \widetilde{h}_1 - \widetilde{h}_2. \tag{8}$$

The boundary conditions for the Hubbard operators are not, however, periodic; due to (2), one has

$$c_{L+1,\alpha} = (-1)^{N_\alpha + 1} c_{1,\alpha}. \tag{9}$$

We define also the operators of the "number of particles" of sort α at the site m, the operators of the number of pariticles at the first m sites of the lattice, and the operators of the total number of particles of sort α on the lattice, correspondingly,

$$n_{m,\alpha} = \frac{1}{2}(1 - s^z_{m,\alpha}) = c^+_{m,\alpha} c_{m,\alpha}; \quad N_{m,\alpha} = \sum_{l=1}^{m} n_{l,\alpha}; \quad N_\alpha \equiv N_{L,\alpha}. \tag{10}$$

The Hubbard model can be solved by means of the Bethe Ansatz [32]. Hence the ladder model with the Hamiltonian (1) is also exactly solvable.

In what follows we will consider this model in the strong coupling limit

$$U \to +\infty, \qquad h, B \quad \text{fixed}. \tag{11}$$

In the case of finite coupling the space $\mathcal{F}$ where the Hamiltonian (1) acts is of dimension 4^L. In the limit (11), the energy of a state becomes infinitely large if the state contains at least one pair of spins $\mathbf{s}_{m,1}$ and $\mathbf{s}_{m,2}$ (with the same m) which are both down. These states should be excluded, hence the physical space $\mathcal{H}$ is of dimension 3^L. It means that in the limit (11) the model is described by the effective Hamiltonian

$$\begin{aligned} H = &\mathcal{P} \Big[- \sum_{\alpha=1,2} \sum_{m=1}^{L} (s^+_{m,\alpha} s^-_{m+1,\alpha} + s^-_{m,\alpha} s^+_{m+1,\alpha}) \\ &- h \sum_{m=1}^{L} (n_{m,1} + n_{m,2}) + B \sum_{m=1}^{L} (n_{m,1} - n_{m,2}) \Big] \mathcal{P}, \end{aligned} \tag{12}$$

similarly to the Hubbard Hamiltonian with infinitely strong repulsion [27, 28, 33, 34]. The projector $\mathcal{P}$,

$$\mathcal{P} = \prod_{m=1}^{L} (1 - n_{m,1} n_{m,2}), \tag{13}$$

is the projector onto the physical states removing the states with infinitely large energies (operators $n_{m,\alpha}$ are expressed in terms of spins $s^z_{m,\alpha}$, see (10)).

Considerations similar to those given for the Hubbard model in [27, 28] show that the complete orthogonal set of eigenstates of the effective Hamiltonian (12),

$$H\,|\Psi_{NM}(k,\lambda)\rangle = E_{NM}(k)\,|\Psi_{NM}(k,\lambda)\rangle\,, \tag{14}$$

in the space $\mathcal{H}$ is given by

$$|\Psi_{NM}(k,\lambda)\rangle = \sum_{m_1\ldots m_N=1}^{L}\ \sum_{\alpha_1\ldots\alpha_N=1,2} \chi_{NM}^{\alpha_1\ldots\alpha_N}(m_1\ldots m_N|k,\lambda)\cdot$$
$$s^-_{m_1\alpha_1}\ldots s^-_{m_N\alpha_N}\,|0\rangle\,. \tag{15}$$

The wave function χ_{NM} is not equal zero only if exactly M values of α's equal 2 among $\alpha_1,\ldots,\alpha_N$). The corresponding eigenenergies are

$$E_{NM}(k) = -2\sum_{a=1}^{N}\cos k_a - hN + B(N-2M). \tag{16}$$

The requirement of the periodicity of the wave functions results in the system of nested Bethe equation for the quasimomenta $k\equiv k_1,\ldots,k_N$ and auxiliary quasimomenta $\lambda\equiv\lambda_1,\ldots,\lambda_M$ parametrizing the eigenstates,

$$\begin{aligned} e^{ik_aL} &= (-1)^{N+1}e^{i\Lambda}, \qquad & a=1,\ldots,N,\\ e^{i\lambda_bL} &= (-1)^{M+1}, & b=1,\ldots,M, \end{aligned} \tag{17}$$

where Λ denotes the total quasimomentum of the auxiliary lattice problem,

$$\Lambda = \sum_{b=1}^{M}\lambda_b. \tag{18}$$

Each quasimomentum k_a is a function of the total quasimomentum Λ of the auxiliary lattice problem only, so that one can decompose k_a as

$$k_a = \widetilde{k}_a + \frac{1}{L}\Lambda, \tag{19}$$

where $\widetilde{k}_a$ does not depend on $\lambda_1,\ldots,\lambda_M$.

The wave function is

$$\chi_{NM}^{\alpha_1\ldots\alpha_N}(m_1\ldots m_N|k,\lambda) = \frac{1}{N!}\left[\sum_{P\in S_N}(-1)^{[P]}\xi_{NM}^{\alpha_{P_1}\ldots\alpha_{P_N}}(\lambda)\theta(m_{P_1}<\ldots<m_{P_N})\right]\det_N\left\{e^{ik_a m_s}\right\}. \tag{20}$$

We denote $\theta(m_{P_1}<\ldots<m_{P_N})$ the multidimensional step function. The subscript N in $\det_N A_{ab}$ is for the dimension of matrix A. The functions $\xi_{NM}^{\alpha_1\ldots\alpha_N}(\lambda)$ are components of 2^N-dimensional vector $\xi_{NM}(\lambda)$ which has to be an eigenvector of the cyclic shift operator acting in a 2^N-dimensional vector space which can be identified with the space of states for the spin-$\frac{1}{2}$ auxiliary chain with N sites. This vector can be chosen as an eigenvector of an auxiliary $XX0$ Heisenberg chain with periodic boundary conditions. Following ([23]) we write the expression for it in the form used in paper ([19]):

$$\xi_{NM}(\lambda) = \sum_{n_1\ldots n_M=1}^{N}\varphi_{NM}(n_1\ldots n_m|\lambda)\;\sigma_{n_1}^{-}\ldots\sigma_{n_m}^{-}\,|\Uparrow_N\rangle\,, \tag{21}$$

where $|\Uparrow_N\rangle$ (all spins up) is the pseudovacuum for the auxiliary chain, $\langle\Uparrow|\Uparrow_N\rangle = 1$, and $\boldsymbol{\sigma}_n$ are the Pauli matrices for the auxiliary problem with commutation relations

$$[\sigma_m^a,\sigma_n^b] = 2i\delta_{mn}\epsilon^{abc}\sigma_m^c;\quad \sigma_m^{\pm} = \frac{1}{2}(\sigma_m^x \pm i\sigma_m^y). \tag{22}$$

The auxiliary wave function is

$$\varphi_{NM}(n_1\ldots n_M|\lambda) = \frac{1}{M!}\left[\prod_{1\le j<l\le M}\operatorname{sign}(n_l-n_j)\right]\det_M\left\{e^{i\lambda_a n_b}\right\}. \tag{23}$$

The norm of the eigenvector is

$$\langle\Psi_{NM}(k,\lambda)|\Psi_{NM}(k,\lambda)\rangle = L^N N^M. \tag{24}$$

2. Form factors

Form factors of local spin operators are defined as their matrix elements between the Bethe eigenstates. They can be computed using the commutation relations (3), (22) between the Pauli matrices, like it was done earlier in [19] for the one-component $XX0$ model . We give here only the results.

Consider the local spin operators $s^{\pm}_{l,\alpha}$. They are hermitian conjugate, and it is enough to consider the formfactor of one of them,

$$F^{+}_{l,\alpha}(q,\mu|k,\lambda) = \left\langle \Psi_{N\widetilde{M}}\,(q,\mu)\left| s^{+}_{l} \right| \Psi_{N+1,M}\,(k,\lambda)\right\rangle_{N+1,M}$$

$$= e^{-i\frac{\Lambda-\Theta}{2}N}\left(2\cos(\tfrac{\Lambda-\Theta}{2})\right)^{N} e^{il\left(\sum\limits_{a=1}^{N+1}k_a-\sum\limits_{a=1}^{N}q_a\right)} \det\nolimits_M B_\alpha \det\nolimits_{N+1} D. \quad (25)$$

It is not equal to zero only if the state vector in the left hand side contains one particle less than the state vector in the right hand side. In (25) (and also further) we assume that the quasimomenta k,λ correspond to the eigenstates with numbers $N+1, M$ of quasiparticles, and the quasimomenta q,μ are prescribed to the eigenstates with the numbers $N,\widetilde{M}$. The set of the quasimomenta $k_1,\ldots,k_{N+1}$ and $\lambda_1,\ldots,\lambda_M$ satisfies the corresponding Bethe equations (see (17)),

$$\begin{aligned}
&e^{ik_aL} = (-1)^N e^{i\Lambda}, && a=1,\ldots,N+1,\\
&e^{i\lambda_b(N+1)} = (-1)^{M+1}, && b=1,\ldots,M,\\
&k_a = \widetilde{k}_a + \frac{\Lambda}{L}, && \Lambda=\sum_{b=1}^{M}\lambda_b, \qquad (26)
\end{aligned}$$

where $\widetilde{k}_a$ does not depend on $\lambda_1,\ldots,\lambda_M$. The second set of quasimomenta satisfies the equations

$$\begin{aligned}
&e^{iq_aL} = (-1)^{N+1} e^{i\Theta}, && a=1,\ldots,N,\\
&e^{i\mu_bN} = (-1)^{\widetilde{M}+1}, && b=1,\ldots,\widetilde{M},\\
&q_a = \widetilde{q}_a + \frac{\Theta}{L}, && \Theta=\sum_{b=1}^{\widetilde{M}}\mu_b,
\end{aligned}$$

$$\widetilde{M} = \begin{cases} M & (\alpha=1)\\ M-1 & (\alpha=2),\end{cases} \qquad (27)$$

and $\widetilde{q}_a$ does not depend on $\mu_1,\ldots,\mu_M$.

The matrix elements of the $(N+1)\times(N+1)$ matrix D are

$$\begin{aligned}
&D_{a,b} = \frac{1}{1-e^{-i(k_a-q_b)}}; && D_{a,N+1}=1,\\
&a=1,\ldots,N+1; && b=1,\ldots,N, \qquad (28)
\end{aligned}$$

and the $M \times M$ matrices B_α are defined as

$$
\begin{aligned}
(B_1)_{ab} &= \sum_{n=1}^{N} e^{i(\lambda_a - \mu_b)n}, && a, b = 1, \ldots, M, \\
(B_2)_{ab} &= -\sum_{n=1}^{N} e^{i(\lambda_a - \mu_b)n}, && b = 1, \ldots, M-1, \\
(B_2)_{a,M} &= 1, && a = 1, \ldots, M. \qquad (29)
\end{aligned}
$$

The simple determinant representations for the form factors obtained here could be useful for the calculation of different quantities on a finite lattice. In the next Section they are used for the calculation of the normalized mean values of bilocal operators on a finite lattice.

3. Normalized mean values for bilocal operators

To calculate the correlation functions in the thermodynamic limit, one needs first the normalized mean values of the products of two local spin operators with respect to Bethe eigenstates. In this Section we consider two mean values of this kind.

Consider first the mean value

$$
\left\langle s^-_{l+1,\alpha}(t) s^+_{1,\alpha}(0) \right\rangle_{N+1,M} \equiv \frac{\left\langle \Psi_{N+1,M}(k,\lambda) \left| s^-_{l+1,\alpha}(t) s^+_{1,\alpha}(0) \right| \Psi_{N+1,M}(k,\lambda) \right\rangle}{\left\langle \Psi_{N+1,M}(k,\lambda) | \Psi_{N+1,M}(k,\lambda) \right\rangle} \quad (\alpha = 1, 2). \qquad (30)
$$

As usual, the time dependence of operators is described as

$$
\mathbf{s}_{l,\alpha}(t) = e^{iHt} \mathbf{s}_{l,\alpha}(0) e^{-iHt}. \qquad (31)
$$

The set of the quasimomenta $k_1, \ldots, k_{N+1}; \quad \lambda_1, \ldots, \lambda_M$ satisfies the Bethe equations (26).

To calculate the mean value (30), one inserts the complete set of states (15) between the two operators $s^\pm$ entering the numerator obtaining, in the usual way, the sum over all intermediate states of an expression bilinear in the form factors of two local spin operators. The corresponding states possess all the possible quasimomenta $q_1, \ldots, q_N; \quad \mu_1, \ldots, \mu_{\widetilde{M}}$ permitted by the corresponding Bethe equations (27). Then one makes use of the explicit determinant representation (25) for the formfactors.

The calculation (which is a little bit bulky) can be performed along the lines of Sect. 4 of paper [23]. We give only the result here.

The mean value (30) is represented as

$$\left\langle s^-_{l+1,\alpha}(t) s^+_{1,\alpha}(0)\right\rangle_{N+1,M}$$
$$= \frac{1}{N} e^{-ith_\alpha} \sum_{n,p=0}^{N-1} e^{-\frac{2\pi}{N} ipn} \; F^{-+}_{p,\alpha} \frac{\partial}{\partial z} \det{}_{N+1}(S_n + zR)|_{z=0}, \quad (32)$$

where we denote h_α the magnetic field acting on the particle of the sort α,

$$h_1 \equiv h - B; \qquad h_2 \equiv h + B. \quad (33)$$

The "auxiliary spin" factors $F^{-+}_{p,\alpha}$ are expressed in terms of the determinants of $M \times M$-dimensional matrices $U_{p,\alpha}$ and P

$$F^{-+}_{p,1} = \det{}_M U_{p,1}, \quad (34)$$
$$F^{-+}_{p,2} = \frac{\partial}{\partial z} \det{}_M (U_{p,2} + zP)\Big|_{z=0}, \quad (35)$$

with matrix elements

$$(U_{p,\alpha})_{a,b} = \frac{1}{N(N+1)} \sum_{n,m=1}^{N} \sum_{\mu} e^{i(m-n+p)\mu} e^{i(\lambda_a n - \lambda_b m)}, \quad (36)$$
$$(P)_{ab} = \frac{1}{N+1}. \quad (37)$$

The sum here is taken over all the values of qusimomenta μ permitted by the Bethe equations (27). It is to be emphasized that matrices $U_{p,1}$ and $U_{p,2}$ are in fact different because the set of the qusimomenta μ is different for $\alpha = 1$ and $\alpha = 2$.

The $(N+1)\times(N+1)$-matrix S_n entering the right hand side of equation (32) can be obtained as

$$S_n = S\Big|_{\Theta = 2\pi n/N} \quad (38)$$

where the matrix $S(\Theta)$ is represented in the form

$$S = I + \frac{1 + \cos(\Lambda - \Theta)}{2} V_1 + \frac{\sin(\Lambda - \Theta)}{2} V_2, \quad (39)$$

and the matrix elements of the matrices V_1, V_2 are

$$(V_1)_{ab} = \frac{1}{L} e_0(k_a) e_0(k_b) \left\{ \cot \frac{k_a - k_b}{2} [e_1(k_a) - e_1(k_b)] - g(l,t) \right\}$$
$$(V_2)_{ab} = \frac{1}{L} e_0(k_a) e_0(k_b) \cot \frac{k_a - k_b}{2} \left\{ \frac{1}{e_0^2(k_a)} - \frac{1}{e_0^2(k_b)} \right\}. \tag{40}$$

Functions e_0, g and e_1 are defined as

$$e_0(k) \equiv e_0(k,l,t) = e^{-(2it\cos k + ilk)/2},$$
$$g \equiv g(l,t) = \frac{1}{L} \sum_{\widetilde{q}} e^{2it\cos q + ilq},$$
$$e_1(k) \equiv e_1(k,l,t) = \sum_{\widetilde{q}} \cot \frac{q-k}{2} \left(e^{2it\cos q + ilq} - e^{2it\cos k + ilk} \right). \tag{41}$$

It should be emphasized that obtaining matrices S_n (38) from matrix S (39) one should put

$$q = \widetilde{q} + \frac{2\pi n}{NL}$$

also in the sums defining the functions g and e_1 (entering the matrix elements of matrices V_1 and V_2).

The analogous representation can be also obtained for the normalized mean value $\left\langle s^+_{l+1,\alpha}(t) s^-_{1,\alpha}(0) \right\rangle_{N,\widetilde{M}}$.

The determinant representation for the normalized mean value obtained in this Section allows one to get the determinant representation for the corresponding correlation function in the thermodynamic limit. It is done in the next Section.

4. Correlation functions in the thermodynamic limit

In this Section we consider two-point correlation functions of local spins. On the finite lattice, the temperature correlation functions are defined in a standard way as temperature normalized mean values,

$$\langle s^+_{m+1,\alpha}(t) s^-_{1,\alpha}(0) \rangle^{(T,L)}$$
$$= \frac{1}{Z} \sum_{\text{states}} e^{-\frac{E_{NM}(k)}{T}} \frac{\langle \Psi_{NM}(k,\lambda) | \, s^+_{m+1,\alpha}(t) s^-_{1,\alpha}(0) \, | \Psi_{NM}(k,\lambda) \rangle}{\langle \Psi_{NM}(k,\lambda) | \, \Psi_{NM}(k,\lambda) \rangle},$$

$$\langle s^-_{m+1,\alpha}(t) s^+_{1,\alpha}(0) \rangle^{(T,L)}$$
$$= \frac{1}{Z} \sum_{\text{states}} e^{-\frac{E_{NM}(k)}{T}} \frac{\langle \Psi_{NM}(k,\lambda) | \, s^-_{m+1,\alpha}(t) s^+_{1,\alpha}(0) \, | \Psi_{NM}(k,\lambda) \rangle}{\langle \Psi_{NM}(k,\lambda) | \, \Psi_{NM}(k,\lambda) \rangle}. \tag{42}$$

Here $\alpha = 1, 2$; $E_{N,M}(k)$ is eigenenergy (16), T is the temperature, and Z is the partition function,

$$Z = \sum_{\text{states}} e^{-\frac{E_{N,M}(k)}{T}}. \tag{43}$$

The sum is taken over all the Bethe eigenstates (15) of the Hamiltonian (12),

$$\sum_{\text{states}} \equiv \sum_{N=0}^{L} \sum_{M=0}^{N} \sum_{\substack{k_1 < \cdots < k_N \\ \lambda_1 < \cdots < \lambda_M}} . \tag{44}$$

The correlation functions are mostly interesting in the thermodynamic limit ($L \to \infty$; h and B are kept fixed):

$$\langle \cdots \rangle^{(T)} = \lim_{L \to \infty} \langle \cdots \rangle^{(T,L)}. \tag{45}$$

The correlation functions with $\alpha = 1$ and $\alpha = 2$ are related in an obvious way, by changing $B \to -B$. So below only the correlation functions for spins $s_{l,1}^{\pm}$,

$$\begin{aligned} g_T^{+}(m,t;h,B) &= \langle s_{m+1,1}^{+}(t) s_{1,1}^{-}(0) \rangle^{(T)}, \\ g_T^{-}(m,t;h,B) &= \langle s_{m+1,1}^{-}(t) s_{1,1}^{+}(0) \rangle^{(T)} \end{aligned} \tag{46}$$

are considered.

Putting the expressions for the normalized mean values on the finite lattice obtained in Section 3 into equations (42), one has the explicit representation also for the correlation functions. This representation is, of course, very bulky, being considerably simplified only in the thermodynamic limit. To derive the representations given below from the representations on a finite lattice we use the technique described in detail (for the two-component gas) in paper [23]. In the thermodynamic limit, the sums over the intermediate states in the expressions for the correlators on a finite lattice can be reduced to expansions of the Fredholm determinants of some linear integral operators.

Thus, the correlation functions considered are represented as Fredholm determinants of linear integral operators. The integral operators which enter the representations for the correlators act on arbitrary function $f(k)$ according to the rule:

$$(\widehat{\mathcal{A}} \cdot f)(k) = \int_{-\pi}^{\pi} dk' \, \mathcal{A}(k,k') f(k'), \tag{47}$$

where the function $\mathcal{A}(k,k')$ is the kernel of the integral operator $\widehat{\mathcal{A}}$.

In order to write down the representations for the correlators, let us introduce some notations. Define the functions

$$E(k) = E(k;m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq \frac{e^{-it\epsilon(q)+imq} - e^{-it\epsilon(k)+imk}}{\tan(\frac{q-k}{2})},$$

$$G = G(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq e^{-it\epsilon(q)+imq}. \tag{48}$$

The distance m is an arbitrary integer; the dispersion is $\epsilon(q) = -2\cos q$ (see (16)) where the momenta q and k. In (48), $q,\ k \in [-\pi,\pi]$. In particular, $G(m,t) = i^m J_m(2t)$ where J_m is the Bessel function ($G(m,0) = \delta_{m,0}$). The functions $E(k)$ and G are the thermodynamic limits of the functions $e_1(k)$ and g (41), respectively.

The following pair of functions play an important role:

$$\begin{aligned} \ell_+(\eta|k) &= \left\{\frac{1+\cos\eta}{2}\, E_+(k) + \frac{\sin\eta}{2}\,\frac{1}{E_-(k)}\right\}\sqrt{\vartheta(k)}, \\ \ell_-(k) &= E_-(k)\sqrt{\vartheta(k)}, \end{aligned} \tag{49}$$

where $\eta \in [-\pi,\pi]$ is a parameter and $\vartheta(k)$ is the Fermi weight,

$$\vartheta(k) = \frac{e^{-B/T}}{2\cosh\frac{B}{T} + e^{(\epsilon(k)-h)/T}}. \tag{50}$$

Here h and B are the external constant fields. The functions $E_\pm(k)$ are defined as

$$E_+(k) = E(k)\, E_-(k), \qquad E_-(k) = \exp\left(\frac{it\epsilon(k) - imk}{2}\right). \tag{51}$$

We introduce also the function

$$F(\gamma;\eta) := 1 + \sum_{p=1}^{\infty} \gamma^{-p}(e^{ip\eta} + e^{-ip\eta}), \tag{52}$$

where

$$\gamma = 1 + e^{2B/T}. \tag{53}$$

It is worth mentioning that $\gamma \in [1,\infty)$ for any real external field B. At the point $\gamma = 1$ ($B = -\infty$) one has $F(1;\eta) = 2\pi\Delta(\eta)$ where $\Delta(\eta)$ is the 2π-periodic delta-function.

Now we are ready to formulate our main results which are the determinant representations for the temperature correlation functions in the thermodynamic limit.

For the correlation functions (46) we obtain the following representations:

$$g_T^-(m,t;h,B) = e^{-it(h-B)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta\, F(\gamma;\eta)$$

$$\left[\det\left(\widehat{\mathcal{I}}+\gamma\,\widehat{\mathcal{Q}}(\eta)+\widehat{\mathcal{R}}^{(-)}\right)-\det\left(\widehat{\mathcal{I}}+\gamma\,\widehat{\mathcal{Q}}(\eta)\right)\right],$$

$$g_T^+(m,t;h,B) = e^{it(h-B)}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\eta\, F(\gamma;\eta)$$

$$\left[\det\left(\widehat{\mathcal{I}}+\gamma\,\widehat{\mathcal{Q}}(\eta)-\gamma\,\widehat{\mathcal{R}}^{(+)}(\eta)\right)+(G-1)\det\left(\widehat{\mathcal{I}}+\gamma\,\widehat{\mathcal{Q}}(\eta)\right)\right], \tag{54}$$

where $\widehat{\mathcal{I}}$ is the unit operator on the interval $[-\pi,\pi]$, and

$$\widehat{\mathcal{Q}}(\eta)\equiv\widehat{\mathcal{V}}(\eta)-\frac{1+\cos\eta}{2}\,G\widehat{\mathcal{R}}^{(-)}. \tag{55}$$

The integral operators $\widehat{\mathcal{V}}(\eta)$, $\widehat{\mathcal{R}}^{(-)}$, $\widehat{\mathcal{R}}^{(+)}(\eta)$ (the latter two are of rank one) possess kernels

$$\mathcal{V}(\eta|k,k') = \frac{\ell_+(\eta|k)\,\ell_-(k')-\ell_-(k)\,\ell_+(\eta|k')}{2\pi\tan(\frac{k-k'}{2})},$$
$$\mathcal{R}^{(-)}(k,k') = \frac{1}{2\pi}\,\ell_-(k)\,\ell_-(k'),$$
$$\mathcal{R}^{(+)}(\eta|k,k') = \frac{1}{\pi(1+\cos\eta)}\,\ell_+(\eta|k)\,\ell_+(\eta|k'). \tag{56}$$

The representations for the correlators $\langle s^-_{m+1,2}(t)s^+_{1,2}(0)\rangle^{(T)}$ and $\langle s^+_{m+1,2}(t)s^-_{1,2}(0)\rangle^{(T)}$ can be obtained from (54) by inversing the sign of the external field, $B\to -B$.

In the equal-time case, $t=0$, the integrals over η in (54) can be taken explicitly. For $t=0$ and $m\neq 0$ one has

$$g_T^-(m,t=0;h,B) = g_T^+(-m,t=0;h,B) =$$
$$=\det\left(\widehat{\mathcal{I}}+(\gamma+1)\widehat{v}+\widehat{r}\right)-\det\left(\widehat{\mathcal{I}}+(\gamma+1)\widehat{v}\right), \tag{57}$$

where the kernels of integral operators $\widehat{v}$ and $\widehat{r}$ are

$$v(k,k') = -\sqrt{\vartheta(k)}\frac{\sin(|m|\frac{k-k'}{2})}{2\pi\tan(\frac{k-k'}{2})}\sqrt{\vartheta(k')},$$

$$r(k,k') = \sqrt{\vartheta(k)}e^{-im\frac{k+k'}{2}}\sqrt{\vartheta(k')}. \tag{58}$$

Conclusion

In conclusion, we would like to stress that the integral operators involved into the representations (54) and (57) are of the form usual for integrable models, i.e., they are "integrable integral operators" [2, 9]. This fact is important for constructing the corresponding matrix Riemann-Hilbert problem and for deriving integrable partial differential equations for the correlators. It is done quite similar to the case of the one-component $XX0$ Heisenberg model, see, e.g., [24]. This, in turn, will make possible the evaluation of different (e.g., large time and distance) asymptotics of the correlators considered.

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