Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 19 (2000), No. 1, 131–145

On the Stabilizability of a Slowly Rotating Timoshenko Beam

W. Krabs and G. M. Sklyar

Abstract. In this paper we continue our investigation of a slowly rotating Timoshenko beam in a horizontal plane whose movement is controlled by the angular acceleration of the disk of a driving motor into which the beam is clamped. We show how to choose a feedback control allowing to stabilize our system (the beam plus the disk) in a preassigned position of rest.

Keywords: *Stabilizability, Timoshenko beam, feedback control*

AMS subject classification: 93 D 15

1. Introduction and statement of the stabilizability problem

In [1] we considered the following linear model of a slowly rotating Timoshenko beam in a horizontal plane derived in [2]:

$$\left.\begin{aligned} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) &= -\ddot{\theta}(t)(r+x) \\ \ddot{\xi}(x,t) - \xi''(x,t) + \xi(x,t) + w'(x,t) &= \ddot{\theta}(t) \end{aligned}\right\} \quad (x \in (0,1), t > 0) \tag{1.1}$$

where $w(x,t)$ means the deflection of the center line of the beam at the location $x \in [0,1]$ and time $t \geq 0$, $\xi(x,t)$ means the rotation angle of the cross section area at x and t, $\dot{w} = w_t$, $\dot{\xi} = \xi_t$, $w' = w_x$, $\xi' = \xi_x$, θ is the rotation angle of the motor disk, $\dot{\theta} = \frac{d\theta}{dt}$, and r is the radius of the disk. The boundary conditions are of the form

$$\left.\begin{aligned} w(0,t) = \xi(0,t) &= 0 \\ w'(1,t) + \xi(1,t) &= 0 \\ \xi'(1,t) &= 0 \end{aligned}\right\} \quad (t \geq 0). \tag{1.2}$$

It is assumed that the motion of the beam is controlled by the acceleration $\ddot{\theta}(t)$ of the rotation of the motor disk. The problem we deal with in [1] is to transfer the beam from a position of rest into a position of rest under a given angle within a given time T by

W. Krabs: Techn. Univ., FB Math., Schloßgartenstr. 7, D-64289 Darmstadt
G. M. Sklyar: State Univ., Dept. Math. Anal., Svoboda sqr. 4, 310077 Kharkov, Ukraine; also: Inst. Math. Univ., ul. Wielkopolska 15, 70-451 Szczecin, Poland

This research was supported by NATO Linkage Grant No. CRG-LG 972980

ISSN 0232-2064 / $ 2.50

means of a choice of control $\ddot{\theta}(\cdot) \in L_2[0,T]$. It has been shown that such a control exists, if the time T is large enough. This result relies essentially on the moment method, in particular, we made use of [3: Theorem 1.2.17].

In the present paper we consider another problem for this model of a Timoshenko beam.

Problem of strong stabilizability. Find a linear continuous functional

$$p(w, \dot{w}, \xi, \dot{\xi}, \theta, \dot{\theta})$$

such that every solution of problem (1.1) - (1.2) with feedback control

$$\ddot{\theta}(t) = p\big(w(\cdot,t), \dot{w}(\cdot,t), \xi(\cdot,t), \dot{\xi}(\cdot,t), \theta(t), \dot{\theta}(t)\big)$$

tends to 0 when $t \to +\infty$ in the following sense:

$$\int_0^1 w'(x,t)^2 dx \to 0, \qquad \int_0^1 \xi'(x,t)^2 dx \to 0 \tag{1.3}$$

$$\int_0^1 \dot{w}(x,t)^2 dx \to 0, \qquad \int_0^1 \dot{\xi}(x,t)^2 dx \to 0 \tag{1.4}$$

$$\theta(t) \to 0, \qquad \dot{\theta}(t) \to 0 \tag{1.5}$$

as $t \to +\infty$. Note that conditions (1.3) - (1.4) mean the extinguishing of the total energy of the beam and conditions (1.5) mean the stabilization of the disk in the position $\theta = 0$ which, obviously, can be considered as an arbitrary preassigned one. In addition note that conditions (1.3), due to Friedrichs inequality, imply

$$\int_0^1 w(x,t)^2 dt \to 0, \quad \text{and} \quad \int_0^1 \xi(x,t)^2 dt \to 0 \qquad (t \to +\infty).$$

The main result of the paper is based on the theorem on strong stabilizability of contractive systems [4: p. 1324] which, in turn, is a consequence of the Sekefal'vy-Nagy and Foyas theorem on strong convergence to zero of powers of a contractive operator [5: p. 102].

2. The singular values of the disk radius

Following [1] we rewrite (1.1) - (1.2) in the operator form

$$\begin{pmatrix} \ddot{w}(\cdot,t) \\ \ddot{\xi}(\cdot,t) \end{pmatrix} + A \begin{pmatrix} w(\cdot,t) \\ \xi(\cdot,t) \end{pmatrix} = b\,\ddot{\theta}(t) \qquad (t > 0) \tag{2.1}$$

where $H = L^2((0,1), \mathbb{C}^2)$, the linear operator $A : D(A) \to H$ is defined by

$$A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'' - z' \\ y' - z'' + z \end{pmatrix}$$

on

$$D(A) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H^2((0,1),\mathbb{C}^2) \,\middle|\, \begin{matrix} y(0) = z(0) = 0 \\ y'(1) + z(1) = z'(1) = 0 \end{matrix} \right\}$$

and $b = \binom{-r-x}{1} \in H$.

In [1] we proved that this operator A is positive and self-adjoint and the smallest eigenvalue of A is larger than 1.

The main part of this Section is devoted to the question on the orthogonality of eigenvectors of the operator A to the vector b. Let $\binom{y}{z}$ be a given eigenvector of A corresponding to eigenvalue λ ($\lambda > 1$). Bearing in mind that $\binom{y}{z} \in D(A)$, we obtain

$$\begin{aligned}
\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, b \right\rangle_H &= -\int_0^1 (r+x)y(x)\,dx + \int_0^1 z(x)\,dx \\
&= \frac{1}{\lambda}\int_0^1 (r+x)\big(y''(x) + z'(x)\big)\,dx + \int_0^1 z(x)\,dx \\
&= \frac{r}{\lambda}\big(y'(1) + z(1) - y'(0) - z(0)\big) \\
&\quad + \frac{1}{\lambda}\left(y'(1) + z(1) - \big(y(1) - y(0)\big) - \int_0^1 z(x)\,dx\right) + \int_0^1 z(x)\,dx \\
&= -\frac{r}{\lambda}y'(0) - \frac{1}{\lambda}y(1) + \frac{\lambda - 1}{\lambda}\int_0^1 z(x)\,dx \\
&= -\frac{r}{\lambda}y'(0) - \frac{1}{\lambda}y(1) + \frac{1}{\lambda}\big(-z'(1) + z'(0) + y(1) - y(0)\big) \\
&= \frac{1}{\lambda}\big(-ry'(0) + z'(0)\big).
\end{aligned}$$

Further, we take advantage of the direct form of eigenvectors obtained in [1: Section 2]:

$$\left.\begin{aligned}
y(x) &= C_1 e^{\mu_1 x} + C_2 e^{-\mu_1 x} + C_3 e^{\mu_3 x} + C_4 e^{-\mu_3 x} \\
z(x) &= -C_1 \frac{\sqrt{\lambda}}{\mu_1} e^{\mu_1 x} + C_2 \frac{\sqrt{\lambda}}{\mu_1} e^{-\mu_1 x} + C_3 \frac{\sqrt{\lambda}}{\mu_3} e^{\mu_3 x} - C_4 \frac{\sqrt{\lambda}}{\mu_3} e^{-\mu_3 x}
\end{aligned}\right\}$$

where

$$\mu_1 = \sqrt{-\lambda + \sqrt{\lambda}}, \quad \mu_3 = \sqrt{-\lambda - \sqrt{\lambda}}, \quad C_1, \quad C_2, \quad C_3, \quad C_4$$

are some complex constants defined from boundary conditions.

With the notation of this work we have from (2.2)

$$\begin{aligned}
&-ry'(0) + z'(0) \\
&\quad = -ir\big(C_1\sigma_1 - C_2\sigma_1 + C_3\sigma_3 - C_4\sigma_3\big) + \sqrt{\lambda}\big(-C_1 - C_2 + C_3 + C_4\big)
\end{aligned} \tag{2.3}$$

where $\sigma_1 = \sqrt{\lambda - \sqrt{\lambda}}$ and $\sigma_3 = \sqrt{\lambda + \sqrt{\lambda}}$. Reality of the functions y and z means that $C_2 = \overline{C_1}$ and $C_4 = \overline{C_3}$. Let

$$C_1 = \alpha + i\beta, \quad C_2 = \alpha - i\beta, \quad C_3 = \gamma + i\delta, \quad C_4 = \gamma - i\delta.$$

Then (2.3) reads

$$-ry'(0)+z'(0)=2r(\beta\sigma_1+\delta\sigma_3)+2\sqrt{\lambda}(\gamma-\alpha).$$

The boundary conditions $y(0)=0$ and $z(0)=0$ give

$$\left.\begin{aligned} \alpha+\gamma&=0\\ \sigma_3\beta+\sigma_1\delta&=0.\end{aligned}\right\} \tag{2.4}$$

Therefore,

$$-ry'(0)+z'(0)=2r\delta\left(\frac{\sigma_1^2}{\sigma_3}+\sigma_3\right)+4\sqrt{\lambda}\gamma=4\left(r\delta\frac{\lambda}{\sigma_3}+\sqrt{\lambda}\gamma\right). \tag{2.5}$$

It follows from here that the equality $\left\langle\binom{y}{z},b\right\rangle_H=0$ is possible (for some value of the radius of the disk) if and only if $\delta\gamma<0$. To analize this possibility we make use of the other pair of boundary conditions which with our notation reads

$$\left.\begin{aligned} \sigma_3(\alpha\sin\sigma_1+\beta\cos\sigma_1)+\sigma_1(\gamma\sin\sigma_3+\delta\cos\sigma_3)&=0\\ \alpha\cos\sigma_1-\beta\sin\sigma_1-\gamma\cos\sigma_3+\delta\sin\sigma_3&=0.\end{aligned}\right\}$$

With regard to (2.4) this gives

$$\left.\begin{aligned} P\gamma+\sigma_1R\delta&=0\\ \sigma_3R\gamma+Q\delta&=0\end{aligned}\right\} \tag{2.6}$$

where

$$\left.\begin{aligned} P&=\sigma_1\sin\sigma_3-\sigma_3\sin\sigma_1\\ Q&=\sigma_1\sin\sigma_1-\sigma_3\sin\sigma_3\\ R&=\cos\sigma_3+\cos\sigma_1.\end{aligned}\right\}$$

It follows from (2.6) that

$$PQ=\sigma_1\sigma_3R^2\geq 0.$$

Let us show that actually

$$PQ=\sigma_1\sigma_3R^2>0. \tag{2.7}$$

In fact, let

$$R=\cos\sigma_3+\cos\sigma_1=2\cos\frac{\sigma_3+\sigma_1}{2}\cos\frac{\sigma_3-\sigma_1}{2}=0.$$

On the other hand, from [1: Formula (2.9)] we have

$$(\sigma_3-\sigma_1)^2\cos^2\frac{\sigma_3-\sigma_1}{2}=(\sigma_3+\sigma_1)^2\cos^2\frac{\sigma_3+\sigma_1}{2}$$

and, therefore, $\cos\frac{\sigma_3-\sigma_1}{2}=0$. But the latter equality is impossible because it is easy to see that $0<1<\sigma_3-\sigma_1\leq\sqrt{2}<\frac{\pi}{2}$ as $\lambda\geq 1$. This proves (2.7).

Remark 2.1. Note that (2.7) implies immediately that all the eigenvalues of the operator A are simple. In fact, since $R \neq 0$, then from (2.4), (2.6) follows that the dimension of the eigenspace corresponding to the eigenvalue λ equals 1.

Taking (2.7) into account we get

$$\frac{P+Q}{R} = -(\sigma_3 - \sigma_1)\tan\frac{\sigma_1+\sigma_3}{2}.$$

Thus, $\delta\gamma < 0$ if and only if

$$2\pi k < \sigma_1 + \sigma_3 < \pi + 2\pi k \qquad (k \in \mathbb{Z}).$$

Lemma 2.1. *For a given eigenvector* $\binom{y}{z}$, $A\binom{y}{z} = \lambda\binom{y}{z}$, *there exists a value* r *of the disk radius such that* $\left\langle \binom{y}{z}, b\right\rangle_H = 0$ *if and only if*

$$2\pi k < \sqrt{\lambda + \sqrt{\lambda}} + \sqrt{\lambda - \sqrt{\lambda}} < \pi + 2\pi k \qquad (k \in \mathbb{Z}). \tag{2.8}$$

As follows from (2.5) – (2.6) *the corresponding value is given by*

$$r = \frac{\sigma_1 \sin\sigma_1 - \sigma_3 \sin\sigma_3}{\sqrt{\lambda}(\cos\sigma_3 + \cos\sigma_1)}. \tag{2.9}$$

We call all such values of the disk radius the *singular* ones. It is shown in [1] that for every $k \in \mathbb{N}$ there exists exactly one eigenvalue $\lambda = \lambda_{2k+1}$ of the operator A satisfying (2.8). Therefore, (2.9) defines a sequence $\{r_k\}_{k\geq 1}$ of the singular values. Let us determine the asymptotic behaviour of this sequence. To this end we use the next two properties of the eigenvalues λ_k [1]:

i) $\sqrt{\lambda_{2k+1}} \sim \frac{\pi}{2} + \pi k$ if $k \to \infty$ and, as a consequence,

$$\sigma_3^{(k)} = \sqrt{\lambda_{2k+1} + \sqrt{\lambda_{2k+1}}} \sim \frac{\pi}{2} + \pi k + \frac{1}{2}$$
$$\sigma_1^{(k)} = \sqrt{\lambda_{2k+1} - \sqrt{\lambda_{2k+1}}} \sim \frac{\pi}{2} + \pi k - \frac{1}{2}$$

if $k \to \infty$.

ii) $(\sigma_3^{(k)} + \sigma_1^{(k)})^2\big(1 + \cos(\sigma_3^{(k)} + \sigma_1^{(k)})\big) = (\sigma_3^{(k)} - \sigma_1^{(k)})^2\big(1 + \cos(\sigma_3^{(k)} - \sigma_1^{(k)})\big)$ and, as a consequence, taking into account of (2.8),

$$(\sigma_3^{(k)} + \sigma_1^{(k)})\cos\frac{\sigma_3^{(k)} + \sigma_1^{(k)}}{2} = (\sigma_3^{(k)} - \sigma_1^{(k)})\cos\frac{\sigma_3^{(k)} - \sigma_1^{(k)}}{2}.$$

Applying i) - ii) we obtain from (2.9)

$$\begin{aligned}
r_k &= \frac{(\sigma_1^{(k)} - \sigma_3^{(k)}) \sin \sigma_3^{(k)} + \sigma_1^{(k)} (\sin \sigma_1^{(k)} - \sin \sigma_3^{(k)})}{\sqrt{\lambda_{2k+1}} (\cos \sigma_3^{(k)} + \cos \sigma_1^{(k)})} \\
&= \frac{(\sigma_1^{(k)} - \sigma_3^{(k)}) \sin \sigma_3^{(k)}}{((\sigma_3^{(k)})^2 - (\sigma_1^{(k)})^2) \cos \sigma_3^{(k)} - \sigma_1^{(k)} 2 \cos \sigma_3^{(k)} + \sigma_1^{(k)} 2} \\
&\quad + \sqrt{1 - \frac{1}{\sqrt{\lambda_{2k+1}}}} \tan \frac{\sigma_3^{(k)} - \sigma_1^{(k)}}{2} \\
&\to \frac{\sin \frac{1}{2}}{\cos^2 \frac{1}{2}} - \tan \frac{1}{2} \\
&= \left(1 - \cos \frac{1}{2}\right) \tan \frac{1}{2}
\end{aligned}$$

as $k \to \infty$.

Lemma 2.2. *The set of the singular values of the disk radius is given by a sequence* $\{r_k\}_{k=1}^{\infty}$ *which is convergent to* $(1 - \cos \frac{1}{2}) \tan \frac{1}{2}$ *as* $k \to \infty$.

Obviously, in the case when the disk radius has a singular value there exists a fundamental frequency of the beam which is invariable under the influence of the control, i.e. system (2.1) is not controllable. So later on we shall assume the disk radius to be of a non-singular value.

3. The operator equation of motion

For our further purpose it will be necessary to describe the motion of the system (the beam plus the disk) by means of a single operator equation of first order in a Hilbert space.

In [1] we have shown that the operator A in (2.1) is strictly positive and self-adjoint. Further, we have shown that A has a complete orthogonal sequence of eigenfunctions $\binom{y_j}{z_j} \in D(A)$ $(j \in \mathbb{N})$ and a corresponding sequence of eigenvalues $\lambda_j \in \mathbb{R}$ of multiplicity one (see Remark 2.1) such that $0 < \lambda_j \uparrow \infty$ as $j \to \infty$. We even know that $\lambda_1 > 1$. All this implies that

$$D(A) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H \,\middle|\, \sum_{j=1}^{\infty} \lambda_j^2 \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H^2 < \infty \right\}$$

and

$$A \begin{pmatrix} y \\ z \end{pmatrix} = \sum_{j=1}^{\infty} \lambda_j \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H \begin{pmatrix} y_j \\ z_j \end{pmatrix}$$

on $D(A)$. Further, there exists a "square root" $A^{\frac{1}{2}}$ of A which is also a self-adjoint linear operator with domain

$$D(A^{\frac{1}{2}}) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H \,\middle|\, \sum_{j=1}^{\infty} \lambda_j \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H^2 < \infty \right\}$$

and given by

$$A^{\frac{1}{2}}\begin{pmatrix} y \\ z \end{pmatrix} = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H \begin{pmatrix} y_j \\ z_j \end{pmatrix}$$

on $D(A^{\frac{1}{2}})$. It is easy to see that

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, A\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_H = \left\langle A^{\frac{1}{2}}\begin{pmatrix} y \\ z \end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_H$$

for all $\widetilde{\binom{y}{z}} \in D(A)$ and all $\binom{y}{z} \in D(A^{\frac{1}{2}})$. If we introduce vector functions

$$\begin{aligned} Y(x,t) &= \big(w(x,t), \xi(x,t), \dot{w}(x,t), \dot{\xi}(x,t)\big)^T \\ \widetilde{b}(x) &= \big(0, 0, b(x)^T\big)^T \end{aligned} \qquad \big(x \in [0,1], t \geq 0\big)$$

and define a matrix operator by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \qquad D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}}),$$

then (2.1) can be rewritten in the form

$$\dot{Y}(\cdot,t) = \mathcal{A}Y(\cdot,t) + \widetilde{b}(\cdot)u(t) \tag{3.1}$$

for $t > 0$, where $u(t) = \ddot{\theta}(t)$.

Let $\mathcal{H} = D(A^{\frac{1}{2}}) \times H$. Then $\mathcal{H}$ is a Hilbert space with scalar product

$$\langle v_1, v_2 \rangle_{\mathcal{H}} = \left\langle A^{\frac{1}{2}}\begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_H + \left\langle \begin{pmatrix} \widetilde{y}_1 \\ \widetilde{z}_1 \end{pmatrix}, \begin{pmatrix} \widetilde{y}_2 \\ \widetilde{z}_2 \end{pmatrix} \right\rangle_H$$

for all $v_1 = (y_1, z_1, \widetilde{y}_1, \widetilde{z}_1)^T$ and $v_2 = (y_2, z_2, \widetilde{y}_2, \widetilde{z}_2)^T$ in $\mathcal{H}$. Further, it follows for every $v = (y, z, \widetilde{y}, \widetilde{z})^T \in D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}})$ that

$$\begin{aligned} \langle \mathcal{A}v, v \rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix}, \begin{pmatrix} y \\ z \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \\ -A\begin{pmatrix} y \\ z \end{pmatrix} \end{pmatrix}, \begin{pmatrix} y \\ z \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle A^{\frac{1}{2}}\begin{pmatrix} y \\ z \end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_H - \left\langle A\begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} \right\rangle_H \\ &= 0 \end{aligned}$$

which implies that $\mathcal{A} : D(A) \times D(A^{\frac{1}{2}}) \to \mathcal{H}$ is monotone. Let $v_1 = (y_1, z_1, \widetilde{y}_1, \widetilde{z}_1)^T$ and $v_2 = (y_2, z_2, \widetilde{y}_2, \widetilde{z}_2)^T$ in $D(\mathcal{A})$ be given. Then it follows that

$$\begin{aligned}\langle \mathcal{A}v_1, v_2\rangle_{\mathcal{H}} &= \left\langle A^{\frac{1}{2}}\begin{pmatrix}\widetilde{y}_1\\ \widetilde{z}_1\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix}y_2\\ z_2\end{pmatrix}\right\rangle_H - \left\langle A\begin{pmatrix}y_1\\ z_1\end{pmatrix}, \begin{pmatrix}\widetilde{y}_2\\ \widetilde{z}_2\end{pmatrix}\right\rangle_H\\ &= -\left\langle A^{\frac{1}{2}}\begin{pmatrix}\widetilde{y}_2\\ \widetilde{z}_2\end{pmatrix}, A^{\frac{1}{2}}\begin{pmatrix}y_1\\ z_1\end{pmatrix}\right\rangle_H + \left\langle A\begin{pmatrix}y_2\\ z_2\end{pmatrix}, \begin{pmatrix}\widetilde{y}_1\\ \widetilde{z}_1\end{pmatrix}\right\rangle_H\\ &= -\langle v_1, \mathcal{A}v_2\rangle_{\mathcal{H}}\end{aligned}$$

which implies that $\mathcal{A}$ is skew-adjoint, i.e. $\mathcal{A} = -\mathcal{A}^*$ where $\mathcal{A}^* : D(\mathcal{A}) \to \mathcal{H}$ denotes the adjoint operator. Since $\overline{D(\mathcal{A})} = \mathcal{H}$, it follows that $\mathcal{A}$ is maximal monotone. This in turn implies that $(\mathcal{A}, D(A) \times D(A^{\frac{1}{2}}))$ generates a C°-semigroup $\{\mathcal{T}(t) | \, 0 \le t < \infty\}$ on $\mathcal{H}$ which is a contraction, i.e. $\|\mathcal{T}(t)\| \le 1$ for all $t \ge 0$. For every $v_0 \in D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}})$ the unique solution $Y : [0, \infty) \to D(A) \times D(A^{\frac{1}{2}})$ of the problem

$$\left.\begin{aligned}&\dot{Y}(\cdot, t) = \mathcal{A}Y(\cdot, t) \quad (t > 0)\\ &Y(\cdot, 0) = v_0\end{aligned}\right\}$$

is given by $Y(\cdot, t) = \mathcal{T}(t)v_0$ for $t \ge 0$. Now let us define a linear operator $\widetilde{\mathcal{A}} : D(\mathcal{A}) \times \mathbb{C}^2 \to \mathcal{H} \times \mathbb{C}^2$ by

$$\widetilde{\mathcal{A}}\begin{pmatrix}Y\\ \theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\mathcal{A}Y\\ \theta_2\\ 0\end{pmatrix} \qquad \text{for all } \; Y \in D(\mathcal{A}) \text{ and } (\theta_1, \theta_2)^T \in \mathbb{C}^2.$$

Then $\widetilde{\mathcal{A}}$ turns out to be an infinitesimal generator of a C°-semigroup $\{\widetilde{\mathcal{T}}(t) | \, 0 \le t < \infty\}$ on $\mathcal{H} \times \mathbb{C}^2$ which is given by

$$\widetilde{\mathcal{T}}(t)\begin{pmatrix}Y\\ \theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\mathcal{T}(t)Y\\ \theta_1 + t\theta_2\\ \theta_2\end{pmatrix} \qquad \text{for } Y \in \mathcal{H} \text{ and } (\theta_1, \theta_2)^T \in \mathbb{C}^2.$$

If we introduce a vector function $Z : [0, \infty] \to D(\mathcal{A}) \times \mathbb{C}^2$ by

$$Z(t) = \begin{pmatrix}Y(\cdot, t)\\ \theta_1(t)\\ \theta_2(t)\end{pmatrix} \qquad (t \ge 0)$$

and define $\widetilde{\widetilde{b}} = (\widetilde{b}^T, 0, 1)^T$, then (3.1) with $u(t) = \ddot{\theta}(t)$ can be rewritten in the form

$$\dot{Z}(t) = \widetilde{\mathcal{A}}Z(t) + \widetilde{\widetilde{b}}u(t) \qquad (t > 0). \tag{3.2}$$

Let $p \in (\mathcal{H} \times \mathbb{C}^2)^* = \mathcal{H} \times \mathbb{C}^2$ be an arbitrary continuous linear functional. Then the operator $\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p$ is a finite-dimensional perturbation of $\widetilde{\mathcal{A}}$ and consequently (see [6]) generates a strongly continuous semigroup which we denote by $\{\widetilde{\mathcal{T}}_p(t), t \geq 0\}$. Thus the problem of strong stabilizability as formulated in Section 1 turns out to be the problem of existence (and construction) of a functional $p \in (\mathcal{H} \times \mathbb{C}^2)^*$ such that $\widetilde{\mathcal{T}}_p(t)Z \to 0$ as $t \to +\infty$ for every $Z \in \mathcal{H} \times \mathbb{C}^2$.

It is easy to show that the operator $\mathcal{A} : D(A) \times D(A^{\frac{1}{2}}) \to \mathcal{H} = D(A^{\frac{1}{2}}) \times H$ has an orthogonal complete sequence of eigenelements

$$Y_j = \begin{pmatrix} y_j \\ z_j \\ \mu_j y_j \\ \mu_j z_j \end{pmatrix} \qquad \text{and} \qquad Y_{-j} = \begin{pmatrix} y_j \\ z_j \\ -\mu_j y_j \\ -\mu_j z_j \end{pmatrix} \qquad (j \in \mathbb{N})$$

with corresponding eigenvalues $\mu_{\pm j} = \pm i\sqrt{\lambda_j}$ $(j \in \mathbb{N})$. This implies that the operator $\widetilde{\mathcal{A}} : D(\mathcal{A}) \times \mathbb{C}^2 \to \mathcal{H} \times \mathbb{C}^2$ also has an orthogonal sequence of eigenelements $Z_k = (Y_k, 0, 0)^T$ $(k \in \mathcal{Z}\backslash\{0\})$ with corresponding eigenvalues $\widetilde{\mu}_k = \mu_k$ $(k \in \mathcal{Z}\backslash\{0\})$ and in addition an eigenelement $Z_0 = (0, 1, 0)^T$ with eigenvalue $\widetilde{\mu}_0 = 0$. For all $k \in \mathcal{Z}\backslash\{0\}$ it follows that

$$\langle \widetilde{\widetilde{b}}, Z_k \rangle_{\mathcal{H}\times\mathbb{C}^2} = \langle \widetilde{b}, Y_k \rangle_{\mathcal{H}} = \mu_k \left\langle b, \begin{pmatrix} y_{|k|} \\ z_{|k|} \end{pmatrix} \right\rangle_H . \tag{3.3}$$

This leads to the following

Theorem 3.1. *If the radius of the disk is of singular value, then strong stabilizability is impossible.*

Proof. By assumption there is some $k \in \mathbb{N}$ such that $\langle b, \binom{y_k}{z_k} \rangle_H = 0$ which implies because of (3.3) that

$$\langle \widetilde{\widetilde{b}}, Z_{\pm k} \rangle_{\mathcal{H}\times\mathbb{C}^2} = 0. \tag{3.4}$$

From here we infer

$$(\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p)^* Z_k = \widetilde{\mathcal{A}}^* Z_k + (\widetilde{\widetilde{b}}p)^* Z_k = \widetilde{\mathcal{A}}^* Z_k + \underbrace{\langle \widetilde{\widetilde{b}}, Z_k \rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} p^* = \begin{pmatrix} -\mathcal{A}Y_k \\ 0 \\ 0 \end{pmatrix} = -\mu_k Z_k.$$

This implies $\widetilde{\mathcal{T}}_p^*(t) Z_k = e^{-\mu_k t} Z_k$ and therefore

$$\widetilde{\mathcal{T}}_p^*(t)(Z_k + Z_{-k}) = \begin{pmatrix} 2\cos\sqrt{\lambda_k}t \begin{pmatrix} y_k \\ z_k \end{pmatrix} \\ 2\sqrt{\lambda_k}\sin\sqrt{\lambda_k}t \begin{pmatrix} y_k \\ z_k \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}.$$

This, finally, leads to

$$
\begin{aligned}
&\big\langle \widetilde{\mathcal{T}}_p(t)(Z_k + Z_{-k}), Z_k + Z_{-k}\big\rangle_{\mathcal{H}\times\mathbb{C}^2} \\
&\quad = \big\langle Z_k + Z_{-k}, \widetilde{\mathcal{T}}_p^*(t)(Z_k + Z_{-k})\big\rangle_{\mathcal{H}\times\mathbb{C}^2} \\
&\quad = \left\langle \begin{pmatrix} 2\begin{pmatrix} y_k \\ z_k \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} 2\cos\sqrt{\lambda_k}t \begin{pmatrix} y_k \\ z_k \end{pmatrix} \\ 2\sqrt{\lambda_k}\sin\sqrt{\lambda_k}t \begin{pmatrix} y_k \\ z_k \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} \right\rangle_{\mathcal{H}\times\mathbb{C}^2} \\
&\quad = 4\cos\sqrt{\lambda_k}t
\end{aligned}
$$

which implies $\widetilde{\mathcal{T}}_p(t)(Z_k + Z_{-k}) \not\to 0$ as $t \to \infty$ and shows that strong stabilizability is impossible ∎

4. Construction of the stabilizing control

In order to prove stabilizability of (3.2) we make use of the following theorem on the strong stabilizability of contractive systems [4: Theorem 5]. Consider a system of the form

$$\frac{dx}{dt} = Ax + Bu \qquad (x \in H, u \in U)$$

where H and U are Hilbert spaces, the operator A generates a strongly continuous contractive semigroup $\{T(t) : t \geq 0\}$ and $b \in [U, H]$. Let there exist $t_0 > 0$ such that the set $\sigma(T(t_0)) \cap \{z \in \mathbb{C} : |z| = 1\}$ is at most countable. Then in order that the system is strongly stabilizable it is necessary and sufficient that there does not exist an eigenvector x_0 of the operator A corresponding to an eigenvalue λ ($\operatorname{Re}\lambda = 0$) such that $x_0 \in \operatorname{Ker} B^*$. At the same time notice that this theorem cannot be applied directly, because the semigroup $\{\widetilde{\mathcal{T}}(t) : t \geq 0\}$ is not contractive. Therefore, first of all we find a perturbation $\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p$ of the operator $\widetilde{\mathcal{A}}$ such that the generated semigroup $\{\widetilde{\mathcal{T}}_p(t) : t \geq 0\}$ is contractive in a suitable norm of $\mathcal{H} \times \mathbb{C}^2$.

Let $\mu > 0$ be given. Then we define $p_\mu \in (\mathcal{H} \times \mathbb{C}^2)^* = (\mathcal{H} \times \mathbb{C}^2)$ by

$$p_\mu(Z) = -\mu\langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} \tag{4.1}$$

where $\widetilde{Z}_0 = (0, 0, 1)^T$. Then

$$
\left.\begin{aligned}
&(\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu)Z_0 = 0 \\
&(\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu)Z_k = \widetilde{\mathcal{A}}Z_k - \mu\langle Z_k, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}\widetilde{\widetilde{b}} = \mu_k Z_k \quad (k \in \mathbb{Z}\backslash\{0\})
\end{aligned}\right\}
$$

follows. Further, one can show that $-\mu$ is a simple eigenvalue of $\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu$ whose corresponding eigenelements are multiples of

$$Z_\mu = \begin{pmatrix} -\mu(A+\mu^2 I)^{-1}b \\ \mu^2(A+\mu^2 I)^{-1}b \\ -\frac{1}{\mu} \\ 1 \end{pmatrix}.$$

With the aid of this eigenelement we define a scalar product in $\mathcal{H}\times\mathbb{C}^2$ by

$$\begin{aligned} \langle Z^1, Z^2\rangle_\mu = {} & \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}\langle Z^1,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}\langle Z^2,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ & + \Big\langle Z^1 - \langle Z^1,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} Z_\mu,\ Z^2 - \langle Z^2,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} Z_\mu\Big\rangle_{\mathcal{H}\times\mathbb{C}^2} \end{aligned} \tag{4.2}$$

for $Z^1, Z^2 \in \mathcal{H}\times\mathbb{C}^2$ which leads to the norm

$$\|Z\|_\mu = \left(\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}\left|\langle Z,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}\right|^2 + \left\|Z - \langle Z,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} Z_\mu\right\|^2_{\mathcal{H}\times\mathbb{C}^2}\right)^{\frac{1}{2}}$$

for $Z\in\mathcal{H}\times\mathbb{C}^2$. It is easy to see that $\|Z\|_{\mathcal{H}\times\mathbb{C}^2}\to 0$ as $\|Z\|_\mu \to 0$ and vice versa. Thus these two norms are equivalent to each other and, therefore, stabilization in one of them implies stabilization in the other.

Lemma 4.1. *The eigenvectors Z_k $(k\in\mathbb{Z})$ and Z_μ of the operator $\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu$ form a complete orthogonal system in $\mathcal{H}\times\mathbb{C}^2$ with respect to the scalar product* (4.2)*.*

Proof. Let $k, j\in\mathbb{Z}$ be chosen such that $k\neq j$. Then

$$\begin{aligned} \langle Z_k, Z_j\rangle_\mu = {} & \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}\underbrace{\langle Z_k,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0}\underbrace{\langle Z_j,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} \\ & + \Big\langle Z_k - \underbrace{\langle Z_k,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} Z_\mu,\ Z_j - \underbrace{\langle Z_j,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} Z_\mu\Big\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ = {} & \langle Z_k, Z_j\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ = {} & \langle Y_k, Y_j\rangle_{\mathcal{H}} \\ = {} & 0. \end{aligned}$$

Let $k\in\mathbb{Z}$ be chosen arbitrarily. Then

$$\begin{aligned} \langle Z_k, Z_\mu\rangle_\mu = {} & \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}\underbrace{\langle Z_k,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0}\underbrace{\langle Z_\mu,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} \\ & + \Big\langle Z_k - \underbrace{\langle Z_k,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} Z_\mu,\ \underbrace{Z_\mu - \underbrace{\langle Z_\mu,\widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=1} Z_\mu}_{=0}\Big\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ = {} & 0 \end{aligned}$$

which shows the orthogonality of the system $\{Z_k : k \in \mathbb{Z}\} \bigcup \{Z_\mu\}$ with respect to the scalar product (4.2).

Now let $Z \in \mathcal{H} \times \mathbb{C}^2$ be chosen arbitrarily. Then one calculates

$$\left.\begin{aligned} \langle Z, Z_\mu\rangle_\mu &= \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} \langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ \langle Z, Z_k\rangle_\mu &= \big\langle Z - \langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} Z_\mu,\, Z_k\big\rangle_{\mathcal{H}\times\mathbb{C}^2} \quad (k \in \mathbb{Z}). \end{aligned}\right\}$$

Now let

$$\left.\begin{aligned} \langle Z, Z_\mu\rangle_\mu &= 0 \\ \langle Z, Z_k\rangle_\mu &= 0 \quad (k \in \mathbb{Z}). \end{aligned}\right\} \tag{4.3}$$

Then $\langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} = 0$ and $\langle Z, Z_k\rangle_\mu = \langle Z, Z_k\rangle_{\mathcal{H}\times\mathbb{C}^2}$ $(k \in \mathbb{Z})$ follow, hence $\langle Z, Z_k\rangle_{\mathcal{H}\times\mathbb{C}^2} = 0$ for all $k \in \mathbb{Z}$. Let $Z = (Y, \sigma_1, \sigma_2)^T$. Then this implies $\langle Y, Y_k\rangle_{\mathcal{H}} = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$, hence $Y = 0$.

From $\langle Z, Z_0\rangle_{\mathcal{H}\times\mathbb{C}^2} = \sigma_1$ we obtain $\sigma_1 = 0$. Finally, $\langle Z, \widetilde{Z}_0\rangle = 0$ implies $\sigma_2 = 0$. Therefore, assumption (4.3) implies $Z = 0$ which shows that the system $\{Z_k : k \in \mathbb{Z}\} \bigcup \{Z_\mu\}$ is complete and concludes the proof ∎

Next we prove

Theorem 4.1.

(i) *The semigroup $\{\widetilde{\mathcal{T}}_{p_\mu}(t) \,|\, t \geq 0\}$ which is generated by $\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu$ is contractive with respect to $\|\cdot\|_\mu$, i.e. $\|\widetilde{\mathcal{T}}_{p_\mu}(t)\|_\mu \leq 1$ for all $t \geq 0$.*

(ii) *For every $n \in \mathbb{N}$ the spectrum $\sigma(\widetilde{\mathcal{T}}_{p_\mu}(n))$ of the operator $\widetilde{\mathcal{T}}_{p_\mu}(n)$ is an at most countable set.*

(iii) *Under the assumption of non-singularity of the disk radius there does not exist an eigenvector $\widetilde{Z} \neq 0$ of $\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu$ such that $\langle \widetilde{Z}, \widetilde{\widetilde{b}}\rangle_\mu = 0$.*

Proof. (i) Let $Z \in D(\widetilde{\mathcal{A}})$ be given. Then by Lemma 4.1 $Z = \alpha Z_\mu + \sum_{k=-\infty}^{\infty} \alpha_k Z_k$ (in the sense of $\|\cdot\|_\mu$-convergence) which implies

$$(\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu)Z = -\mu\alpha Z_\mu + \sum_{k=-\infty}^{\infty} \alpha_k \mu_k Z_k.$$

Since $\{Z_\mu, Z_k \,|\, k \in \mathbb{Z}\}$ is an orthogonal system with respect to $\langle\cdot,\cdot\rangle_\mu$ and $\mathrm{Re}(\mu_k) = 0$ for all $k \in \mathbb{Z}$, it follows that

$$\begin{aligned} \mathrm{Re}\big\langle(\widetilde{\mathcal{A}} + \widetilde{\widetilde{b}}p_\mu)Z, Z\big\rangle_\mu &= -\mu|\alpha|^2\|Z_\mu\|^2_\mu + \sum_{k=-\infty}^{\infty} |\alpha_k|^2 \underbrace{\mathrm{Re}(\mu_k)}_{=0} \|Z_k\|^2_\mu \\ &= -\mu|\alpha|^2\|Z_\mu\|^2_\mu \qquad\qquad\qquad\qquad (Z \in D(\widetilde{\mathcal{A}})). \\ &\leq 0 \end{aligned}$$

This implies $\operatorname{Re}\big\langle(\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu)^*Z, Z\big\rangle_\mu \le 0$ for all $Z \in D(\widetilde{\mathcal{A}}^*) = D(\widetilde{\mathcal{A}})$. Hence $(\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu)^*$ is a monotone linear operator. This implies that $(\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu)^*$ is accretive, i.e.

$$\big\|\big((\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu)^* - \lambda I\big)Z\big\|_\mu \ge \lambda\|Z\|_\mu \qquad \text{for all } \ \lambda > 0 \text{ and } Z \in D(\widetilde{\mathcal{A}}^*).$$

In turn this implies that the range $\mathrm{R}\big((\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu)-\lambda I\big) = \mathcal{H}\times\mathbb{C}^2$ for all $\lambda > 0$, since $\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu$ is closed. Hence $\widetilde{\mathcal{A}}+\widetilde{\widetilde{b}}p_\mu$ is maximal monotone and therefore generates a contractive semigroup $\{\widetilde{\mathcal{T}}_{p_\mu}(t)\,|\,t \ge 0\}$.

(ii) Let us define $\widetilde{Z}_\mu = \|Z_\mu\|_\mu^{-1}Z_\mu$ and $\widetilde{Z}_k = \|Z_k\|_\mu^{-1}Z_k$ for $k \in \mathbb{Z}$. Then, for every

$$Z = \langle Z, \widetilde{Z}_\mu\rangle_\mu \widetilde{Z}_\mu + \sum_{k=-\infty}^{\infty} \langle Z, \widetilde{Z}_k\rangle_\mu \widetilde{Z}_k \in \mathcal{H}\times\mathbb{C}^2$$

we obtain

$$\widetilde{\mathcal{T}}_{p_\mu}(n)Z = e^{-\mu n}\langle Z, \widetilde{Z}_\mu\rangle_\mu \widetilde{Z}_\mu + \sum_{k=-\infty}^{\infty} e^{\mu_k n}\langle Z, \widetilde{Z}_k\rangle_\mu \widetilde{Z}_k$$

for every $n \in \mathbb{N}$. From this we infer that the eigenvalues of $\widetilde{\mathcal{T}}_{p_\mu}(n)$ are given by $e^{-\mu n}$ and $e^{\mu_k n}$ $(k \in \mathbb{Z})$, with corresponding normalized eigenelements $\widetilde{Z}_\mu$ and $\widetilde{Z}_k$ $(k \in \mathbb{Z})$. By the spectral mapping theorem [7] it follows that

$$\sigma(\widetilde{\mathcal{T}}_{p_\mu}(n)) = \overline{\big\{e^{-\mu n}, e^{\mu_k n} \quad (k \in \mathbb{Z})\big\}}$$

where $\overline{M}$ denotes the closure of the set M. Since $\mu_{\pm k} = \pm i\sqrt{\lambda_k}$ for $k \in \mathbb{N}_0$ and (see [1])

$$\lambda_{2\ell} \sim \lambda_{2\ell-1} \sim \frac{1}{4}((2\ell-1)\pi)^2 \qquad \text{as } \ \ell \to \infty$$

it follows that $\sigma(\widetilde{\mathcal{T}}_{p_\mu}(n))$ can only have a finite number of accumulation points and is therefore at most countable.

(iii) From (4.2) it follows that

$$\left.\begin{aligned} \langle Z_\mu, \widetilde{\widetilde{b}}\rangle_\mu &= \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} \ne 0 \\ \langle Z_0, \widetilde{\widetilde{b}}\rangle_\mu &= -\langle Z_0, Z_\mu\rangle_\mu = \frac{1}{\mu} \ne 0. \end{aligned}\right\}$$

Therefore it remains to prove that $\langle Z_k, \widetilde{\widetilde{b}}\rangle_\mu \ne 0$ for $k \in \mathbb{Z}\backslash\{0\}$. From (4.2) it follows that

$$\begin{aligned} \langle Z_k, \widetilde{\widetilde{b}}\rangle_\mu &= \|Z_\mu\|_{\mathcal{H}\times\mathbb{C}^2} \underbrace{\langle Z_k, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} \underbrace{\langle \widetilde{\widetilde{b}}, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=1} \\ &\quad + \Big\langle Z_k - \underbrace{\langle Z_k, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=0} Z_\mu, \widetilde{\widetilde{b}} - \underbrace{\langle \widetilde{\widetilde{b}}, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2}}_{=1} Z_\mu \Big\rangle_{\mathcal{H}\times\mathbb{C}^2} \\ &= \langle Z_k, \widetilde{\widetilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2} - \langle Z_k, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2}. \end{aligned}$$

From (3.3) $\langle Z_k, \tilde{\tilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2} = \mu_k\big\langle\binom{y_{|k|}}{z_{|k|}}, b\big\rangle_H$ follows. Further we obtain

$$\begin{aligned}
\langle Z_k, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} &= \frac{1}{\mu_k}\langle\widetilde{\mathcal{A}}Z_k, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} = \frac{1}{\mu_k}\left\langle\begin{pmatrix}\mathcal{A}Y_k\\0\\0\end{pmatrix}, \begin{pmatrix}Y_\mu\\ \theta_{1\mu}\\ \theta_{2\mu}\end{pmatrix}\right\rangle_{\mathcal{H}\times\mathbb{C}^2}\\
&= \frac{1}{\mu_k}\langle\mathcal{A}Y_k, Y_\mu\rangle_{\mathcal{H}} = -\frac{1}{\mu_k}\langle Y_k, \mathcal{A}Y_\mu\rangle_{\mathcal{H}} = -\frac{1}{\mu_k}\langle Z_k, \widetilde{\mathcal{A}}Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2}\\
&= \frac{1}{\mu_k}\langle Z_k, \underbrace{p_\mu(Z_\mu)}_{-\mu}\tilde{\tilde{b}} + \mu Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2}.
\end{aligned}$$

This implies

$$(\mu_k - \mu)\langle Z_k, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} = -\mu\langle Z_k, \tilde{\tilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2},$$

hence

$$\langle Z_k, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} = \frac{\mu}{\mu_k - \mu}\langle Z_k, \tilde{\tilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2}$$

since $\mu_k \neq \mu$ for all $k \in \mathbb{Z}\backslash\{0\}$. Summarizing we obtain

$$\langle Z_\mu, \tilde{\tilde{b}}\rangle_\mu = \left(\mu_k - \frac{\mu\,\mu_k}{\mu - \mu_k}\right)\left\langle\begin{pmatrix}y_{|k|}\\ z_{|k|}\end{pmatrix}, b\right\rangle_H = -\frac{\mu_k^2}{\mu - \mu_k}\left\langle\begin{pmatrix}y_{|k|}\\ z_{|k|}\end{pmatrix}, b\right\rangle_H.$$

Because of $\frac{\mu_k^2}{\mu-\mu_k} \neq 0$ and $\big\langle\binom{y_{|k|}}{z_{|k|}}, b\big\rangle_H \neq 0$ for all $k \in \mathbb{Z}\backslash\{0\}$ (due to the non-singularity of the radius of the disk) it follows that $\langle Z_k, \tilde{\tilde{b}}\rangle_\mu \neq 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. This concludes the proof of Theorem 4.1 ∎

Thus all assumptions of [4: Theorem 5] are satisfied, if the radius r of the disk is non-singular. This leads to the strong stabilizability of the system

$$\dot{Z} = (\widetilde{\mathcal{A}} + \tilde{\tilde{b}}p_\mu)Z + \tilde{\tilde{b}}u.$$

According to the proof of [4: Theorem 5] this can be achieved by the control $u = -\langle Z, \tilde{\tilde{b}}\rangle_\mu$. Hence, system (3.2) is strongly stabilizable with the aid of the control

$$u = -\mu\langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} - \langle Z, \tilde{\tilde{b}}\rangle_\mu. \tag{4.4}$$

On using $Z(t) = \big(Y_1(t), Y_2(t), \theta_1(t), \theta_2(t)\big)^T$ and $\tilde{\tilde{b}} = (0, b, 0, 1)^T$ we obtain $\langle Z, \widetilde{Z}_0\rangle_{\mathcal{H}\times\mathbb{C}^2} = \theta_2$.

Further, it follows from (4.2) that

$$\begin{aligned}
\langle Z, \tilde{\tilde{b}}\rangle_\mu &= \theta_2\,\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} + \big\langle Z - \theta_2 Z_\mu, \tilde{\tilde{b}} - Z_\mu\big\rangle_{\mathcal{H}\times\mathbb{C}^2}\\
&= \langle Z, \tilde{\tilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2} - \langle Z, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} - \theta_2\langle Z_\mu, \tilde{\tilde{b}}\rangle_{\mathcal{H}\times\mathbb{C}^2} + 2\theta_2\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}.
\end{aligned}$$

Let $Z_\mu = (Y^*, -\mu Y^*, -\frac{1}{\mu}, 1)^T$ where $Y^* = -\mu(A + \mu^2 I)^{-1} b$. Then we obtain

$$\begin{aligned}\langle Z, \widetilde{\widetilde{b}} \rangle_\mu &= \langle -\mu Y^*, b\rangle_H - \langle Z, Z_\mu\rangle_{\mathcal{H}\times\mathbb{C}^2} + 2\,\theta_2 \|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} \\ &= -\langle Y_1, Y^*\rangle_{D(A^{\frac{1}{2}})} + \langle Y_2, b + \mu Y^*\rangle_H - \tfrac{1}{\mu}\theta_1 \\ &\quad + \big(-1 + \mu \langle Y^*, b\rangle_H + 2\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2}\big)\theta_2\end{aligned}$$

where

$$\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} = \langle Y^*, Y^*\rangle_{D(A^{\frac{1}{2}})} + \mu^2 \langle Y^*, Y^*\rangle_H + \tfrac{1}{\mu^2} + 1. \tag{4.5}$$

If we put $Y_1 = Y$ and $\theta_1 = \theta$, then $Y_2 = \dot{Y}$ and $\theta_2 = \dot{\theta}$, and summarizing we obtain

$$\begin{aligned}u = &-\langle Y, Y^*\rangle_{D(A^{\frac{1}{2}})} + \langle \dot{Y}, b + \mu Y^*\rangle_H - \tfrac{1}{\mu}\theta \\ &+ \big(-1 + \mu\langle Y^*, b\rangle_H + 2\|Z_\mu\|^2_{\mathcal{H}\times\mathbb{C}^2} - \mu\big)\dot{\theta}.\end{aligned} \tag{4.6}$$

Theorem 4.2. *If the radius of the disk is non-singular, then system* (3.2) *is strongly stabilizable with u given by* (4.6), (4.5) *which is a real function.*

References

[1] Krabs, W. and G. M. Sklyar: *On the controllability of a slowly rotating Timoshenko beam.* Z. Anal. Anw. 18 (1999), 437 – 448.

[2] Xiong, Xiao-Jin: *Modelling, Control and Computer Simulation of a Rotating Timoshenko Beam.* Ph.D. Thesis. Montréal: McGill University 1997.

[3] Krabs, W.: *On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes* (Lecture Notes in Control and Information Sciences: Vol. 173). Berlin et al.: Springer-Verlag 1992.

[4] Korobov, V. I. and G. M. Sklyar: *On the question of the strong stabilizability of contractive systems in Hilbert spaces* (in Russian). Diff. Uravn. 20 (1984), 1862 – 1869; Engl. transl.: Diff. Equ. 20 (1984), 1320 – 1326.

[5] Sekefal'vy-Nagy, B. B. and S. Foyas: *The Harmonic Analysis of Operators in Hilbert Space.* Russian transl. by Mir: Moscow 1970.

[6] Kato, T.: *Perturbation Theory for Linear Operator.* Berlin et al.: Springer-Verlag 1966.

[7] Dunford, N. and J. T. Schwartz: *Linear Operators.* Part II: *Spectral Theory.* Russ. transl. by Mir: Moscow 1966.

Received 18.03.1999; in revised form 02.11.1999