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On the Spectrum of Orthomorphisms and Barbashin Operators

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Abstract. The paper is concerned with the spectrum of an operator $A = C+K$, where C is an orthomorphism and K is a compact operator. The proofs are in a certain sense constructive. The results are applied to Barbashin equations $\frac{dx}{dt} = Ax$, where $A = C+K$ with a multiplication operator C and an integral operator K. In some particular cases even necessary and sufficient conditions for stability are given.

Keywords: *Barbashin equations, orthomorphisms in Banach lattices, essential spectrum, spectral estimates, perturbations*

AMS subject classification: 45 K 05, 47 B 38, 47 B 65

0. Introduction

The aim of this paper is two-fold: In Section 1, we calculate the essential spectrum of an orthomorphism in a Banach lattice and give some perturbation results for the spectrum. Although some of the results concerning the essential spectrum can already be found in the literature, we shall give constructive proofs (and some new perturbation results). In Section 2, we consider the stationary Barbashin equation

$$\frac{\partial x(t,s)}{\partial t} = c(s)x(t,s) + \int_a^b k(s,r)x(t,r)\,dr. \tag{1}$$

The connection with orthomorphisms is the following: Let $X = C([a,b])$, or let X be an ideal space over $[a,b]$. The latter means that X is a Banach space of (classes of) measurable functions, such that for any $x \in X$ and any measurable function y satisfying $|y(s)| \le |x(s)|$ a.e. we have $y \in X$ and $\|y\| \le \|x\|$ (see [17, 24]). Ideal spaces are also

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called (complete normed) *Köthe spaces* [20] or *Banach function spaces* (but be aware that the definitions sometimes differ slightly in the literature).

Under some natural conditions (see, e.g., [17]), equation (1) may be written as a stationary linear differential equation in X,

$$\frac{dx}{dt} = Ax \tag{2}$$

with

$$Ax(s) = Cx(s) + Kx(s).$$

Here, C is a multiplication operator, defined by

$$Cx(s) = c(s)\,x(s)$$

and K is the integral operator with kernel k. The stationary Barbashin equation (2) is exponentially stable if and only if the spectrum $\sigma(A)$ belongs to the interior of the left half-plane [3]. Thus we are interested in determining the spectral properties of A.

In the described situation, A is the sum of an orthomorphism C in the Banach lattice X and a (usually) compact operator K. Applying the results of Section 1, we may reduce the calculation of the spectrum of A to the calculation of the point spectrum and even get estimates for the latter. However, if K is a Volterra operator or degenerated, we can even calculate the spectrum precisely. The latter can be applied to get arbitrary sharp estimates in the general case.

1. The spectrum of orthomorphisms

In this section, X will always denote a Banach lattice. For basic definitions concerning Banach lattices, we refer to [11, 21, 22] (see also [12]).

Recall that a subspace $B \subseteq X$ is called a *band*, if it is an order ideal (i.e. $|y| \leq |x|$ and $x \in B$ imply $y \in B$), and if for any subset $M \subseteq B$ for which $x = \sup M$ exists in X we have $x \in B$. An order bounded operator $C : X \to X$ is called *orthomorphism* if $CB \subseteq B$ for any band B. It is an important result [21: Theorem 140.4 and Corollary 144.3] that the set of all orthomorphisms of a Banach lattice with the operator norm is itself a Banach lattice with $|C|x = |Cx|$ for $x \geq 0$, and that it is precisely the *centre* $Z(X)$ of X, i.e. the ideal generated by the identity operator I. In other words: A linear operator $C : X \to X$ is an orthomorphism if and only if there is some $\lambda \geq 0$ such that $|Cx| \leq \lambda x$ for all $x \geq 0$. It turns out that the infimum of all those λ is the operator norm $\|C\|$. With composition as multiplication, $Z(X)$ is a commutative *f-algebra* [20: Theorems 140.9 and 140.10].

Since we are interested in spectral properties, it is important that the above results turn over for complex Banach lattices: If $X_{\mathbb{R}}$ is a real Banach lattice, we define the complexification $X = X_{\mathbb{R}} + i\,X_{\mathbb{R}}$ as in [12: Chapter II,§11], [21: §91], or [22: Sections 13 and 15] (in the latter reference the application of representation theory is avoided; see also [21: Exercise 91.12] and [6]). Any complex linear operator $A : X \to X$ is uniquely determined by its values on $X_{\mathbb{R}}$, and on $X_{\mathbb{R}}$ it has the form $A = (\operatorname{Re} A) + i\,(\operatorname{Im} A)$, where $\operatorname{Re} A, \operatorname{Im} A : X_{\mathbb{R}} \to X_{\mathbb{R}}$ are linear. We put $\overline{A} = (\operatorname{Re} A) - i\,(\operatorname{Im} A)$.

For the following theorem we require only that $X_{\mathbb{R}}$ is a uniformly complete Archimedean Riesz space (any Banach lattice has this property, see [21: Theorem 100.4/(ii)] or [22: Theorem 15.3/(ii)]). Part 1 of the following proof is similar to the proof in the real case [21: Theorem 140.4].

Theorem 1.1. *The complex linear operator $C : X \to X$ is an orthomorphism if and only if $\operatorname{Re} C$ and $\operatorname{Im} C$ both are orthomorphisms. In this case, $|C|$ is defined, and*

$$|C|\,|x| = |\,C|x|\,| = |Cx| \qquad (x \in X). \tag{3}$$

For all orthomorphisms $C, D : X \to X$ we have the formulas $CD = DC$,

$$|CD| = |C|\,|D| \tag{4}$$

$$|C|^2 = (\operatorname{Re} C)^2 + (\operatorname{Im} C)^2, \tag{5}$$

and

$$\sup\{|\operatorname{Re} C|, |\operatorname{Im} C|\} \le |C| \le |\operatorname{Re} C| + |\operatorname{Im} C| \le 2\sup\{|\operatorname{Re} C|, |\operatorname{Im} C|\}. \tag{6}$$

Proof. The first statement follows by [21: Theorem 91.6] and the fact that the space of all order bounded operators of X is the complexification of the space of all order bounded operators of $X_{\mathbb{R}}$ [21: §92].

1. Now we show that $|C|$ is defined with $|C|x = |Cx|$ for $x \ge 0$. We assume first that X is Dedekind complete. Then $|C|$ is defined, and we have by [21: Theorem 92.6] (no representation theory is required in our situation, see [21: Exercise 92.7]) that

$$|C|x = \sup_{|z| \le x} |Cz| \qquad (x \ge 0). \tag{7}$$

In particular, $|Cx| \le |C|x$. For the converse inequality we apply the complex form of Freudenthal's spectral theorem [21: Theorem 91.5]: For each $\varepsilon > 0$ and each $z \in X$ with $|z| \le x$ there is some finite sum $y = \sum \lambda_k x_k$ with $|z - y| \le \varepsilon x$ with the following properties: $|\lambda_k| \le 1$, and $x_k \ge 0$ are pairwise disjoint components of x, i.e. $\inf(x_k, x - x_k) = \inf(x_k, x_j) = 0$ for $k \ne j$. We have $|Cy| \le |Cx|$. Indeed, since C is disjointness preserving we have by [21: Theorem 91.4/(i)] that

$$|Cy| = \left|\sum \lambda_k C x_k\right| = \sum |\lambda_k|\,|Cx_k| = \sup |\lambda_k|\,|Cx_k|$$

and

$$|Cx_k| \le |Cx_k| + |C(x - x_k)| = |Cx|.$$

By (7) we have $|Cz - Cy| \le |C|\,|z - y| \le \varepsilon\,|C|x$, and so in view of the triangle inequality [11: Theorem 12.1] that $|Cz| \le |Cy| + \varepsilon|C|x \le |Cx| + \varepsilon|C|x$. Since $X_{\mathbb{R}}$ is Archimedean, this implies $|Cz| \le |Cx|$. Hence (7) gives the desired estimate $|C|x \le |Cx|$.

Now we drop the assumption that X is Dedekind complete. We write $C = C_1 + i\,C_2$ with $C_k : X_{\mathbb{R}} \to X_{\mathbb{R}}$. By [: Lemma 140.2], C_k may be extended to an orthomorphism D_k of the Dedekind completion of $X_{\mathbb{R}}$. Put $D = D_1 + i\,D_2$. By what we just proved,

$$|D|x = |Dx| = |Cx| \qquad (x \in X,\ x \ge 0).$$

Hence, $|D| : X \to X$. It thus is easily checked that this restriction of $|D| = \sup_t (D_1 \cos t + D_2 \sin t)$ to X is the least upper bound for the set $\{C_1 \cos t + C_2 \sin t : t\}$, whence $|C| = |D|$ exists, and $|C|x = |D|x = |Cx|$ for $x \geq 0$.

2. Now we prove the second equality in (3). First assume that $C \geq 0$. Then C is a Riesz homomorphism, by [11: Theorem 18.13] and [21: Theorem 140.5] even a normal Riesz homorphism, i.e. C preserves arbitrary suprema. Consequently, we have for $x = x_1 + i\,x_2$ with $x_k \in X_{\mathbb{R}}$ that

$$\begin{aligned} |Cx| &= \sup_t (Cx_1 \cos t + Cx_2 \sin t) \\ &= \sup_t C(x_1 \cos t + x_2 \sin t) \\ &= C \sup_t (x_1 \cos t + x_2 \sin t) \\ &= C|x|. \end{aligned} \tag{8}$$

Now we consider an arbitrary orthomorphism C, but suppose that X is Dedekind complete. Then the space of all orthomorphisms of $X_{\mathbb{R}}$ is Dedekind complete [21: Corollary 142.9] and we may apply Freudenthal's spectral theorem [21: Theorem 91.5] for the complexification: For each $\varepsilon > 0$ we find some finite sum $E = \sum \lambda_k C_k$ with $|C - E| \leq \varepsilon |C|$ where $\lambda_k \in \mathbb{C}$ and $C_k \geq 0$ are pairwise disjoint orthomorphisms. By (7) we have for $k \neq j$

$$\inf \big\{ |C_k x|, |C_j x| \big\} \leq \inf \big\{ |C_k|\,|x|, |C_j|\,|x| \big\} = \big(\inf\{|C_k|, |C_j|\} \big)|x| = 0 \qquad (x \in X)$$

whence $C_k x$ are pairwise disjoint. Thus, [21: Theorem 91.4/(i)] implies

$$|Ex| = \Big| \sum \lambda_k C_k x \Big| = \sum |\lambda_k|\, |C_k x|.$$

By (8), the right-hand side does not change if we replace x by $|x|$, and so $|Ex| = |E|x|\,|$. By the inverse triangle inequality and (7) we have

$$\big|\, |Cx| - |Ex| \,\big| \leq |(C - E)x| \leq |C - E|\, |x| \leq \varepsilon\, |C|\, |x|.$$

Adding this formula for x and $|x|$, the triangle inequality gives

$$\big|\, |Cx| - |\, C|x|\,| \,\big| \leq 2\varepsilon |C|\, |x| \qquad (x \in X).$$

Since $X_{\mathbb{R}}$ is Archimedean, this gives the desired formula.

To drop the assumption that X is Dedekind complete we consider as in Step 1 the extension of C to the orthomorphism D in the (complex) Dedekind completion of X. For $x \in X$ we have then $|Cx| = |Dx| = |\,D|x|\,| = |\,C|x|\,|$.

3. The formula $CD = DC$ follows by considering the real and imaginary parts separately, since real orthomorphisms commute. Repeated application of (3) gives

$$|C|\,|D|x = |C|\,|Dx| = |CDx| = |CD|x \qquad (x \geq 0)$$

and so (4) holds. By [21: Theorem 142.1/(v)] we have $(\operatorname{Re} C)^2, (\operatorname{Im} C)^2 \geq 0$, whence by (4)

$$(\operatorname{Re} C)^2 + (\operatorname{Im} C)^2 = |(\operatorname{Re} C)^2 + (\operatorname{Im} C)^2| = |C\overline{C}| = |C|\,|\overline{C}| = |C|^2.$$

Formula (6) can be proved straightforwardly as in [21: Theorem 91.2] ∎

Formula (4) might be compared with the recent result in [6] which states that $|cd| = |c|\,|d|$ is valid in any complexification of a uniformly complete d-algebra.

Corollary 1.1. *The complex linear operator $C : X \to X$ is an orthomorphism if and only if it belongs to the centre $Z(X)$, i.e. if and only if there is some $\lambda \geq 0$ with $|Cx| \leq \lambda x$ for all $x \geq 0$. In this case, the minimum of all those λ is $\|C\|$.*

Proof. Write $C_1 = \operatorname{Re} C$ and $C_2 = \operatorname{Im} C$. By [21: Theorem 91.2] we have $|C_k x| \leq |Cx| \leq |C_1 x| + |C_2 x|$ for $k = 1, 2$ and $x \geq 0$. Theorem 1.1 thus reduces the first statement to the real case. Moreover, for orthomorphisms C the theorem implies that $|Cx| \leq \lambda x$ for $x \geq 0$ is equivalent to $|C| \leq \lambda I$. Since $Z(X_{\mathbb{R}})$ is Archimedean, there exists a minimal λ with this property, and since $|C|$ is real, we already know that this minimum is $\|\,|C|\,\| = \|C\|$ ∎

Lemma 1.1. *For any orthomorphism $C : X \to X$ the following statements are equivalent:*

1. *C is invertible and $C^{-1} : X \to X$ is an orthomorphism.*

2. *$|C| \geq \varepsilon I$ for some $\varepsilon > 0$.*

3. *$\sup\{|\operatorname{Re} C|, |\operatorname{Im} C|\} \geq \varepsilon I$ for some $\varepsilon > 0$.*

4. *$(\operatorname{Re} C)^2 + (\operatorname{Im} C)^2 \geq \varepsilon I$ for some $\varepsilon > 0$.*

Proof. Put $C_1 = \operatorname{Re} C$, $C_2 = \operatorname{Im} C$, and $E = C_1^2 + C_2^2$.

a) The equivalence of Statements 2 and 3 follows from (6). Since the Archimedean f-algebra $Z(X_{\mathbb{R}})$ is semiprime [21: Example 142.6/(i)], we have by [21: Theorem 142.3/(ii)] that

$$|C|^2 \geq \varepsilon^2 I = (\varepsilon I)^2 \iff |C| \geq \varepsilon I. \tag{9}$$

Hence (5) shows that Statements 2 and 4 are equivalent.

b) Assume that Statement 4 holds. Since $E \in Z(X_{\mathbb{R}})$, there is some $\lambda \geq \varepsilon$ such that $E \leq \lambda I$. Hence,

$$|I - \tfrac{1}{\lambda} E| = I - \tfrac{1}{\lambda} E \leq I - \tfrac{\varepsilon}{\lambda} I$$

and so $\|I - \frac{1}{\lambda} E\| \leq 1 - \frac{\varepsilon}{\lambda} < 1$. Using the Neumann series, we get that $\frac{1}{\lambda} E$ is invertible in the Banach algebra $Z(X_{\mathbb{R}})$ (alternatively, we could also have applied [21: Theorem 146.3] to see this). Since $Z(X_{\mathbb{R}})$ is commutative, we may conclude that $\frac{1}{\lambda}(\frac{1}{\lambda} E)^{-1}(C_1 - i\,C_2)$ is inverse to $C = C_1 + i\,C_2$.

c) Conversely, if C is invertible with $C^{-1} = D = D_1 + i\,D_2$ such that $D_k \in Z(X_{\mathbb{R}})$, then we put $F = D_1^2 + D_2^2$. Observe that $E, F \geq 0$. Since there is some $\lambda > 0$ with $F \leq \lambda I$, we get

$$I = CD\overline{CD} = C\overline{C}D\overline{D} = EF \leq E(\lambda I)$$

and Statement 4 holds for $\varepsilon = \lambda^{-1}$ ∎

Given an orthomorphism $C : X \to X$ and a non-trivial band $B \subseteq X$, we denote the norm of the restriction $C : B \to B$ by $\|C\|_B$, i.e.

$$\|C\|_B = \sup_{x \in B, \|x\| \leq 1} \|Cx\|.$$

Theorem 1.2. *Let the* (*real or complex*) *orthomorphism* $C : X \to X$ *satisfy*

$$\inf \Big\{\|C\|_B : \ B \textit{ is a non-trivial band}\Big\} > 0. \tag{10}$$

Then C *is invertible, and* $C^{-1} : X \to X$ *is an orthomorphism.*

Proof. Assume the conclusion is false. Then Lemma 1.1 implies that for no $\varepsilon > 0$ the operator $D = |C| - \varepsilon I$ is positive, i.e. $D^- \neq 0$. The null space of D^+ is a band B_ε [21: Theorem 140.5/(i)] which in view of $D^+D^- = 0$ (see [21: Theorem 142.1/(iv)]) and $D^- \neq 0$ is non-trivial. For each $0 \leq x \in B_\varepsilon$ we have $Dx = (D^+x - D^-x) = -D^-x \leq 0$, i.e. $|Cx| \leq \varepsilon x$. Corollary 1.1 thus implies $\|C\|_{B_\varepsilon} \leq \varepsilon$, and (10) must fail ∎

The theorem has a converse; even more holds. Recall that an element $x \neq 0$ is called an *atom* of a Riesz space, if it follows from $0 \leq u, v \leq |x|$ and $\inf\{u, v\} = 0$ that either $u = 0$ or $v = 0$.

Theorem 1.3. *Let the* (*real or complex*) *orthomorphism* $C : X \to X$ *have closed range and a finite-dimensional null space. Assume that either* $X_\mathbb{R}$ *is non-atomic or that* C *is one-to-one. Then* (10) *holds.*

Proof. The 'real' null space $N(C) = \{x \in X_\mathbb{R} : Cx = 0\} = N(\operatorname{Re} C) \cap N(\operatorname{Im} C)$ is a band in $X_\mathbb{R}$ [21: Theorem 140.5/(i)]. In particular, if $N(C)$ has finite but positive dimension, it is a finite-dimensional Riesz space which thus contains atoms [11: Theorem 26.3]. Since $N(C)$ is a band, each atom in $N(C)$ is also an atom in $X_\mathbb{R}$. Our assumptions thus imply in both cases that C is one-to-one. Since the range Y of C is a Banach space, the open mapping theorem implies that $C^{-1} : Y \to X$ is a bounded linear operator; denote the operator norm by $N = \|C^{-1}\| > 0$. Then we have for all $x \in X$ that $\|x\| = \|C^{-1}Cx\| \leq N\|Cx\|$, which shows that $\|C\|_B \geq N^{-1}$ for any non-trivial band B ∎

Corollary 1.2. *If the* (*real or complex*) *orthomorphism* $C : X \to X$ *is one-to-one and has closed range, then* C *is onto.*

Corollary 1.3. *If the* (*real or complex*) *orthomorphism* $C : X \to X$ *is invertible, then* $C^{-1} : X \to X$ *is an orthomorphism.*

For any orthomorphism $C : X \to X$ we call

$$\operatorname{ess} C = \Big\{\lambda \in \mathbb{C} : \inf\{\|C - \lambda I\|_B : \ B \text{ a non-trivial band}\} = 0\Big\}$$

the *essential range* of C.

Corollary 1.4. *For any orthomorphism* $C : X \to X$*, we have for the spectrum* $\sigma(C) = \operatorname{ess} C$.

Proof. Observe that $C_\lambda = C - \lambda I$ is an orthomorphism. If $\lambda \notin \sigma(C)$, then C_λ is invertible, and Theorem 1.3 implies that $0 \notin \operatorname{ess} C_\lambda$, i.e. $\lambda \notin \operatorname{ess} C$. Conversely, if $\lambda \notin \operatorname{ess} C$, then Theorem 1.2 implies that C_λ is invertible, i.e. $\lambda \notin \sigma(C)$ ∎

We call a point $\lambda \in \mathbb{C}$ *Fredholm point* of a linear operator $A : X \to X$, if

1. $A - \lambda I$ has closed range.
2. The codimension m of the range of $A - \lambda I$ is finite.
3. The null space of $A - \lambda I$ has finite dimension n.

If additionally $m = n$, we call λ Fredholm point of *index* 0. We consider only bounded operators A, and so the first condition for Fredholm points is actually superfluous, because it follows from the second.

We denote by $\sigma_{ew}(A)$ the essential spectrum of A in the sense of Wolf [19], i.e. the complement of the set of all Fredholm points, and by $\sigma_{em}(A)$ the complement of all Fredholm points of index 0 (which is the essential spectrum of A in the sense of Schechter [13]). Both sets are invariant under compact perturbations [13: Corollary 2] (see also [1: Theorem 2.3.7]).

Corollary 1.5. *Let $X_{\mathbb{R}}$ be non-atomic, $C : X \to X$ be an orthomorphism, $K : X \to X$ be compact, and $A = C + K$. Then*

$$\sigma_{ew}(A) = \sigma_{em}(A) = \sigma(C) = \operatorname{ess} C.$$

Proof. Since $\sigma_{ew}(A) = \sigma_{ew}(A - K)$ and $\sigma_{em}(A) = \sigma_{em}(A - K)$ it suffices to prove the statement for $A = C$. Evidently, $\sigma_{ew}(C) \subseteq \sigma_{em}(C) \subseteq \sigma(C)$. Applied for the orthomorphisms $C_\lambda = C - \lambda I$, Theorem 1.2 implies $\sigma(C) \subseteq \operatorname{ess} C$, and Theorem 1.3 gives similarly $\operatorname{ess} C \subseteq \sigma_{ew}(C)$ ∎

Lemma 1.1 implies in view of Corollary 1.3 other representations for $\operatorname{ess} C$:

Corollary 1.6. *For any orthomorphim $C : X \to X$, the spectrum $\sigma(C) = \operatorname{ess} C$ is the (identical) null set of the functions*

$$\begin{aligned}
\delta_2(\lambda; C) &= \max\left\{\varepsilon \ge 0 :\ |C - \lambda I| \ge \varepsilon I\right\} \\
&= \max\left\{\varepsilon \ge 0 :\ \big(\operatorname{Re}(C - \lambda I)\big)^2 + \big(\operatorname{Im}(C - \lambda I)\big)^2 \ge \varepsilon^2 I\right\}
\end{aligned}$$

and

$$\delta_\infty(\lambda; C) = \max\left\{\varepsilon \ge 0 :\ \sup\left\{|\operatorname{Re}(C - \lambda I)|, |\operatorname{Im}(C - \lambda I)|\right\} \ge \varepsilon I\right\}.$$

We have $\delta_\infty \le \delta_2 \le 2\delta_\infty$.

Observe that the maxima in the definition of δ_2 and δ_∞ indeed exist for all λ, since $Z(X_{\mathbb{R}})$ is Archimedean. The identity for δ_2 in the corollary follows from (5) and (9), and the estimate follows from (6).

Now we prove a perturbation result for δ_2 and δ_∞:

Lemma 1.2. *Let $C, D : X \to X$ be orthomorphisms. Then*

$$\delta_2(\lambda; C - D) \ge \delta_2(\lambda; C) - \|D\| \tag{11}$$

and

$$\delta_\infty(\lambda; C - D) \ge \delta_\infty(\lambda; C) - \max\left\{\|\operatorname{Re} D\|, \|\operatorname{Im} D\|\right\}. \tag{12}$$

Proof. Without loss of generality, let $\lambda = 0$ (consider $\widetilde{C} = C - \lambda I$ instead of C). Recall that $|D| \leq \|D\| I$. By the triangle inequality

$$|C - D| \geq |C| - |D| \geq \big(\delta_2(0; C) - \|D\|\big) I$$

which implies (11). To prove (12), we write $C = C_1 + i\, C_2$ and $D = D_1 + i\, D_2$ with real operators C_1, C_2 and D_1, D_2 and put $\alpha = \max\{\|D_1\|, \|D_2\|\}$. In view of $|D_k| \leq \|D_k\| I$ we have

$$|C_k - D_k| \geq |C_k| - |D_k| \geq |C_k| - \alpha I.$$

By [11: Theorem 11.5/(v)] this implies

$$\sup\big\{|C_1 - D_1|, |C_2 - D_2|\big\} \geq \sup\big\{|C_1| - \alpha I, |C_2| - \alpha I\big\} \geq \big(\delta_\infty(0; C) - \alpha\big) I$$

which gives (12) ∎

For sets in the complex plane, we denote the Euclidean distance by dist_2, and the distance with respect to the maximum-norm by dist_∞.

Corollary 1.7. *For any orthomorphism C we have*

$$\delta_2(\lambda; C) \geq \mathrm{dist}_2(\lambda, \sigma(C)) \tag{13}$$

and

$$\delta_\infty(\lambda; C) \geq \mathrm{dist}_\infty(\lambda, \sigma(C)). \tag{14}$$

Proof. If $|\mu| < \delta_2(\lambda; C)$, then

$$\delta_2(\lambda + \mu; C) = \delta_2(\lambda; C - \mu I) \geq \delta_2(\lambda; C) - |\mu| > 0.$$

Hence $\lambda + \mu \notin \sigma(C)$. This shows (13). The proof of (14) is analogous and omitted ∎

We can prove now that in Corollary 1.5 not only the essential spectrum but also the spectrum of $A = C + K$ is mainly determined by C, provided that K is 'small'. Let $B(r)$ denote the set of all complex numbers z with $|z| \leq r$.

Theorem 1.4. *If X is non-atomic, $C : X \to X$ is an orthomorphism, $K : X \to X$ is compact, and $A = C + K$, then*

$$\sigma(A) \subseteq \sigma(C) + B(\|K\|).$$

Proof. If the statement is false, there is some $\lambda \in \sigma(A)$ with $\|K\| < \mathrm{dist}_2(\lambda, \sigma(C)) \leq \delta_2(\lambda; C)$. By Corollary 1.5, $\lambda \notin \sigma(C) = \sigma_{em}(A)$. Hence, $\lambda \in \sigma(A)$ must be an eigenvalue of A to some eigenvector $x \neq 0$. But $Cx + Kx = \lambda x$ implies by (3)

$$|Kx| = |(C - \lambda I)x| = |C - \lambda I|\,|x| \geq \delta_2(\lambda; C)\,|x|$$

and so $\|Kx\| \geq \delta_2(\lambda; C)\|x\|$ gives the contradiction $\|K\| \geq \delta_2(\lambda; C)$ ∎

2. The spectrum of Barbashin operators

Let us now return to the Barbashin equation (2). For the rest of this paper, let X be an ideal space over $[a,b]$, or $X = C([a,b])$. For simplicity, we will assume that the ideal space X has full support which means that X contains a non-vanishing function (see [17, 24]). Let c be a (real or complex) function such that the multiplication operator defined by $Cx(s) = c(s)x(s)$ acts in X. Then c is (essentially) bounded (see, e.g., [17: Theorem 5.1.4]). Hence, C is an orthomorphism in X, and

$$|C|x(s) = |c(s)|\, x(s).$$

The formulas of Theorem 1.1 are evident in this case. In view of the fact that X contains a non-vanishing function, Corollary 1.1 reduces to the following statement: If C is an orthomorphism in X, then c is (essentially) bounded, and $\|C\| = \text{ess sup } |c(s)|$. Theorem 1.2 just means that the condition ess inf $|c(s)| > 0$ implies that C is invertible with C^{-1} being an orthomorphism (of course, $C^{-1}x(s) = (c(s))^{-1}x(s)$). Theorem 1.3 becomes the not so obvious fact that the condition ess inf $|c(s)| > 0$ must be satisfied if C has a closed range and a finite-dimensional null space. In particular, if c is continuous with finitely many zero's, then CX is not closed in X. If one wants to see this elementary, one will probably like to apply the open mapping theorem; but this is precisely what we did in the proof of Theorem 1.3.

For the Barbashin equation, we have also a compact and linear operator $K : X \to X$ be given (for our results we do not need that K is an integral operator). Let

$$\operatorname{ess} c(a,b) = \Big\{ u : \operatorname*{ess\,inf}_{s\in[a,b]} |c(s) - u| = 0 \Big\}$$

be the *essential range* of c (in the case $X = C([a,b])$ the essential range is of course the range $c([a,b])$). Then Corollary 1.5 immediately implies:

Theorem 2.1. *Under the above assumptions,*

$$\sigma_{ew}(A) = \sigma_{em}(A) = \sigma(C) = \operatorname{ess} c(a,b).$$

If one is only interested in Theorem 2.1, one can of course simplify the proof: As we have just seen, the only deeper statement needed for the proof is a corresponding variant of Theorem 1.3. However, the main steps of such an 'elementary' proof are the same that we used. A result analogous to Theorem 2.1 was proved (independently) for matrix-valued functions in L_2 in [5].

Theorem 2.1 implies that ess sup $c(s) < 0$ is a necessary condition for the exponential stability of equation (2). Conversely, if ess sup $c(s) < 0$, the equation is exponentially stable if and only if A has no eigenvalues in the right half plane or the imaginary axis. Thus in the following we will concentrate on the point spectrum of A.

If K is an integral operator with a degenerate kernel, we may reduce the problem to a finite-dimensional system:

Example 2.1. Let

$$Kx(s) = \int_a^b k(s,r)x(r)\,dr \qquad \text{with } k(s,r) = \sum_{i=1}^n a_i(s)\,b_i(r) \tag{15}$$

where the linearly independent functions a_i belong to X, and b_i belong to the associate space X' [17, 20, 24] (for $X = C([a,b])$ put $X' = L_1[a,b]$)). To determine $\sigma(A)$ it suffices to calculate the eigenvalues $\lambda \notin \operatorname{ess} c(a,b)$. For such λ there exists $x \neq 0$ with

$$c(s)x(s) + \sum_{i=1}^n a_i(s) \int_a^b b_i(r)x(r)\,dr = \lambda\,x(s).$$

Putting

$$\beta_i = \int_a^b b_i(r)x(r)\,dr \qquad (i = 1, \ldots, n) \tag{16}$$

we find by $\operatorname{ess\,inf}_{s\in[a,b]} |\lambda - c(s)| > 0$ that

$$x(s) = \sum_{j=1}^n \frac{a_j(s)}{\lambda - c(s)}\,\beta_j. \tag{17}$$

Equations (16) now become

$$\Gamma_\lambda \beta = \beta \tag{18}$$

where $\beta = (\beta_i)$ and $\Gamma_\lambda = (\gamma_{ij})$ with

$$\gamma_{ij} = \int_a^b \frac{b_i(r)a_j(r)}{\lambda - c(r)}\,dr.$$

(17) implies $\beta \neq 0$ by $x \neq 0$. Thus a necessary condition for $\lambda \notin \operatorname{ess} c(a,b)$ to be an eigenvalue is

$$\det(\Gamma_\lambda - I) = 0. \tag{19}$$

But (19) is also sufficient, since in this case there exists a non-trivial solution $\beta = (\beta_i)$ of (18), and a straightforward calculation shows that (17) is an eigenfunction of A for λ. In other words: $\sigma(A)$ is the union of $\operatorname{ess} c(a,b)$ and all other values λ satisfying (19).

In order to estimate the spectrum of A in the general case, we may apply the results of Section 1. Note that the functions δ_2 and δ_∞ for the multiplication operator C just become $\delta_2(\lambda; C) = \operatorname{ess\,inf} |c(s) - \lambda|$ and $\delta_\infty(\lambda; C) = \operatorname{ess\,inf} |c(s) - \lambda|_\infty$ (with the notation $|z|_\infty := \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}$). The statement of Corollary 1.7 is evident in this case, but Theorem 1.4 provides a non-trivial and very useful estimate for the spectrum:

Theorem 2.2. *In the situation of Theorem* 2.1 *we have* $\sigma(A) \subseteq \operatorname{ess} c(a,b) + B(\|K\|)$.

For some important cases we also have sharper estimates:

Example 2.2. If additionally X is a Hilbert space and C and K are self-adjoint, then $\sigma(A)$ is contained in the interval

$$\Big[\text{ess inf}\, c(s) + \min \sigma(K),\ \text{ess sup}\, c(s) + \max \sigma(K)\Big].$$

This follows by $(Ax, x) = (Cx, x) + (Kx, x)$ and $\sigma(C) = \text{ess}\, c(a, b)$.

Theorem 2.3. *If K in Theorem 2.2 is a compact Volterra integral operator, i.e.*

$$Kx(s) = \int_a^s k(s, r)x(r)\, dr,$$

then $\sigma(A) = \text{ess}\, c(a, b)$.

Proof. Assume, there exists some $\lambda \in \sigma(A) \setminus \text{ess}\, c(a, b)$. Then λ is an eigenvalue of A, i.e. there is some $x_\lambda \neq 0$ satisfying a.e.

$$0 = (A - \lambda I)x_\lambda(s) = (c(s) - \lambda)x_\lambda(s) + Kx_\lambda(s). \tag{20}$$

Since $\text{ess inf}_{s\in[a,b]} |c(s) - \lambda| > 0$ we have that the linear operator $(C - \lambda I)^{-1}x(s) = (c(s) - \lambda)^{-1}x(s)$ is bounded. Hence, $(C - \lambda I)^{-1}K$ again is a compact Volterra operator. Thus it has spectral radius 0 (see [18]; an alternative proof for the special cases that X is regular or that $X = C([a, b])$ can be found in [23]). In particular, -1 can not be an eigenvalue of $(C - \lambda I)^{-1}K$, i.e. x_λ can not satisfy (20) ∎

By the above results one might suspect that it is always possible to replace $\|K\|$ by the spectral radius $r(K)$ of K in Theorem 2.2, i.e. $\sigma(A) \subseteq \text{ess}\, c(a, b) + B(r(K))$. However, this is not true in general. It is not even true that the maximal real part of $\sigma(A)$ is bounded by $\text{ess sup}\, c(s) + r(K)$ (as in Example 2.2). We give a class of counterexamples of integral operators with degenerate kernels:

Example 2.3. Consider the interval $[-1, 1]$ and

$$Kx(s) = \int_{-1}^{1} a(s)b(r)x(r)\, dr.$$

Example 2.1 shows for $c \equiv 0$ that the spectrum of the integral operator consists of the points 0 and

$$\int_{-1}^{1} a(r)b(r)\, dr.$$

We now consider the special function $c(s) = -1$ for $s < 0$ and $c(s) = 0$ for $s \geq 0$, i.e. $\text{ess}\, c(a, b) = \{-1, 0\}$. Example 2.1 shows that $\lambda \neq -1, 0$ belongs to $\sigma(A)$ if and only if

$$\frac{\alpha}{\lambda + 1} + \frac{\beta}{\lambda} = 1 \tag{21}$$

where we have put

$$\alpha = \int_{-1}^{0} a(r)b(r)\,dr \qquad \text{and} \qquad \beta = \int_{0}^{1} a(r)b(r)\,dr.$$

Given any numbers $\lambda \geq \delta > 0$, choose functions a and b such that

$$\alpha = -\delta(\lambda + 1) \qquad \text{and} \qquad \beta = (\delta + 1)\lambda.$$

Then (21) holds, i.e. λ is an eigenvalue of the corresponding operator A. However, the spectrum of the integral operator is just $\{0, \alpha + \beta\} = \{0, \lambda - \delta\}$. In particular, if $r(K)$ denotes the spectral radius of the integral operator K, then $\lambda = \text{ess sup } c(s) + r(K) + \delta$ belongs to the spectrum of A. This means that the difference of the maximal real value in $\sigma(A)$ and ess sup $c(s) + r(K)$ is positive and may even be arbitrarily large.

3. Remarks

1. All proofs given in this paper are constructive in the following sense: The results remain true, if one replaces the axiom of choice by the axiom of dependent choices (see, e.g., [7]) which only allows countably many (recursive or non-recursive) choices.

There is only one exception to this rule: We do not know whether it can be proved under this axiom that the essential spectrum (in the sense of Wolf or Schechter) is invariant under compact perturbations. However, there do exist constructive proofs, if one requires in the definition of "Fredholm point" additionally that the null space of $A - \lambda I$ be topologically complemented in X. If one assumes the axiom of choice, this modification does not change the definition because each finite-dimensional subspace is topologically complemented by Hahn-Banach's extension theorem. However, one may construct an operator A in the space $X = l_\infty/c_0$ with a one-dimensional null space; the axiom of dependent choice is not sufficent to prove that this null space is topologically complemented (because it can not be disproved that X^* is trivial). See [10] for details.

2. The centre $Z(X)$ of a (real or complex) Banach lattice X is an AM-space with the strong norm unit I. If we assume the axiom of choice, $Z(X)$ thus is algebraically and isometrically isomorphic to a representation space $Y = C(S)$ of continuous functions over an appropriate compact Hausdorff space S [21: Theorem 121.1] (to see that also the multiplicative structure is preserved, apply the two directions of [21: Theorem 141.1] in the f-algebras Y and $Z(X)$, respectively).

If we have such a representation space Y – it also suffices that $Y = L_\infty(S)$ with some measure space S – Lemma 1.1 is evident. Moreover, we may interpret $\operatorname{ess} C$ as the (essential) range of the function which corresponds to the orthomorphism C in the representation space Y (this follows from Corollary 1.4 together with Corollary 1.3). This explains our name 'essential range' for this set.

One might also try to use this as a *definition* of $\operatorname{ess} C$. In this case, Corollary 1.5 already follows from known results: In [14: Theorem 1.11] the equality $\sigma(C) = \sigma_{ew}(C)$

for orthomorphisms C in non-atomic Banach lattices has been proved (but the proof requires further applications of the axiom of choice); also Corollary 1.3 has been proved in [14: Theorem 1.8] by considering the representation space Y for $Z(X)$.

However, our proofs are 'constructive', and for our definition of $\operatorname{ess} C$ no representation space Y for $Z(X)$ has to be known. Without the axiom of choice it is not clear at all, whether such a representation space must exist.

3. The reader may find more results on representations of Banach lattices by spaces of continuous or measurable functions in the monographs [2, 9, 12, 16, 21] and the references therein. As an example, let us give a short proof of Theorem 1.1 for Banach lattices X, using representation theory (and thus the axiom of choice):

Fix $x \in X$, and let Y be the ideal in $X_{\mathbb{R}}$ generated by $|x|$. Then the Archimedean Riesz space Y has the strong unit $|x|$ and is uniformly complete. Hence the Yosida representation theorem [11: Theorem 45.4] implies that Y is Riesz isomorphic to some space $C(S)$ with a compact Hausdorff space S. The orthomorphisms of $X_{\mathbb{R}}$ belong to $Z(X_{\mathbb{R}})$ and thus map the ideal Y into itself. Hence they may be interpreted precisely as the orthomorphisms of $C(S)$, and thus precisely as the multiplication operators (see [21: Example 142.2]). Considering complexifications, we may thus interpret x as a function in the complex Banach lattice $C(S)$, and orthomorphisms C precisely as multiplication operators $Cx(s) = c(s)\,x(s)$. Now (3) is evident. The remaining formulas follow immediately by interpreting $Z(X)$ as some space $C(S)$.

4. The crucial Theorem 2.1 (respectively Corollary 1.5) fails without the condition that X is non-atomic. It may even fail for complete subspaces of ideal spaces over $[a,b]$:

Let Y be some ideal space over $[a,b] = [0,2]$ containing the constant functions, and X be the subspace of Y containing all those functions, which are constant on $[0,1]$. Then for any $c \in L_\infty([0,2])$, $c(s) = 0$ on $[0,1]$ and $c(s) \neq 0$ otherwise, we have that 0 is a Fredholm point of $C : X \to X$ with index 0 $(m = n = 1)$, whence 0 does not belong to the essential spectrum of C (or of $A = C + K$ with compact $K : X \to X$).

5. Theorem 2.1 illustrates the formula for the radius $R(A)$ of $\sigma_{ew}(A)$,

$$R(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|^\gamma}$$

[1: Theorem 2.6.11/(h)], where

$$\|A\|^\gamma = \inf\Big\{L : \gamma(AM) \le L\,\gamma(M) \text{ for all } M \subseteq X\Big\}$$

is the measure of non-compactness of A with respect to the Hausdorff measure γ of non-compactness (i.e. $\gamma(M)$ is the infimum of all $\varepsilon > 0$, such that there exists a finite ε-net for M in X). In fact, we have $\|A^n\|^\gamma = \|c\|^n_{L_\infty([a,b])}$, whence $R(A) = \|c\|_{L_\infty([a,b])}$ as in Theorem 2.1. To see this, observe that $A^n = C^n + K_n$ with some compact operator K_n, and thus that $\|A^n\|^\gamma = \|C^n\|^\gamma$ by [1: Lemma 2.6.7/(e,j)]. Now the statement follows by the fact that $\|C^n\|^\gamma = \|C^n\| = \|c\|^n_{L_\infty([a,b])}$ (for the first equality, apply, e.g., [15: Theorem 2.6]). For further studies of the measure of non-compactness in Banach lattices see also [4].

6. Example 2.1 may be used to estimate numerically the borders of $\sigma(A)$: Any compact set R of resolvent points of A still belongs to the resolvent set under small (depending on R) perturbations of the operator [8: Theorem 3.1]. In particular, for estimates it suffices to approximate $A = C + K$ by more simple operators (in operator norm).

If K may be approximated by finite rank operators, and if X is a regular ideal space (i.e. any $x \in X$ satisfies $\|P_{D_n} x\| \to 0$ for $D_n \downarrow \emptyset$), these approximating operators have the form (15). Indeed, let a finite rank operator K_0 be given. Then K_0 may be written as

$$K_0 x = \sum_{i=1}^{n} a_i \, l_i(x)$$

with $a_i \in X$ and continuous linear functionals l_i. By the regularity of X any continuous linear functional has the form

$$l_i(x) = \int_a^b b_i(r) x(r) \, dr$$

with $b_i \in X'$ [20]. Thus for the spectrum of the approximating operators one may use Example 2.1.

But one may even simplify the calculation, by observing that (19) becomes a polynomial equation, if c is a simple function. Now use the fact that any $c \in L_\infty([a,b])$ may be approximated by simple functions in the L_∞-norm: Given $\varepsilon > 0$, divide $\operatorname{ess} c(a,b)$ in a finite number of Borel sets I_n with $\operatorname{diam} I_n < \varepsilon$. Let $E_n = c^{-1}(I_n)$ and $c_n \in I_n$. Then

$$\Big\| c - \sum c_n \chi_{E_n} \Big\|_{L_\infty} < \varepsilon.$$

7. Theorem 2.1 may be interpreted as a result for the integral equation of the third kind,

$$c(s)x(s) + \int_a^b k(s,r)x(r) \, dr = y(s) \tag{22}$$

with $\operatorname{ess\,inf} |c(s)| = 0$: If the integral operator occuring in (22) is compact in an ideal space X or in $X = C([a,b])$, then the operator A defined by the left-hand side of (22) is not a Fredholm operator, i.e. either the null space of A has infinite dimension or its range has infinite codimension. In particular, it may not happen that (22) has for each $y \in X$ a unique solution.

8. All results remain unchanged, if the interval $[a,b]$ is replaced by some non-atomic σ-finite measure space (respectively by a compact Hausdorff space without discrete elements).

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