Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 18 (1999), No. 3, 611–624

The Generalized Riemann-Hilbert Boundary Value Problem for Non-Homogeneous Polyanalytic Differential Equation of Order n in the Sobolev Space $W_{n,p}(D)$

Ali Seif Mshimba

Abstract. Given is a nonlinear non-homogeneous polyanalytic differential equation of order n in a simply-connected domain D in the complex plane. Initially we prove (under certain conditions) the existence of its general solution in $W_{n,p}(D)$ by first transforming it into a system of integro-differential equations. Next we prove the solvability of a generalized Riemann-Hilbert problem for the differential equation. This is effected by first reducing the boundary value problem posed to a corresponding one for a polyanalytic function. The latter is then transformed into n classical Riemann-Hilbert problems for holomorphic functions, whose solutions are known in the literature.

Keywords: *Polyanalytic functions, generalized Cauchy-Pompeiu integral operators of higher order, Riemann-Hilbert problem*

AMS subject classification: 30 G 30, 35 J 40, 47 G 10

1. Introduction

We consider the following non-homogeneous polyanalytic differential equation of order n in a given simply-connected bounded domain D in the complex plane $\mathbb{C}$:

$$\frac{\partial^n w}{\partial \overline{z}^n} = F\left(z, w, \left\{\frac{\partial^{m+k} w}{\partial z^m \partial \overline{z}^k}\right\}\right) \tag{1}$$
$$n \geq m, k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

The right-hand side is a continuous function of its variables $z \in D$, w and the partial derivatives of w of order not exceeding n and excluding $\frac{\partial^n w}{\partial \overline{z}^n}$, which are denoted here by $\left\{\frac{\partial^{m+k} w}{\partial z^m \partial \overline{z}^k}\right\}$. Following [5, 6] the general solution of equation (1) may be expressed in

Ali Seif Mshimba: University of Dar es Salaam, Department of Mathematics, P.O. Box 35062, Dar es Salaam, Tanzania; e-mail: amshimba@cs.udsm.ac.tz

ISSN 0232-2064 / $ 2.50

the form

$$\begin{aligned} w(z) &= \Phi(z) + T_{0,n,D}F\Big(\zeta, w(\zeta), \Big\{\frac{\partial^{m+k}w}{\partial\zeta^m\partial\overline{\zeta}^k}\Big\}\Big)(z) \\ &= \Phi(z) + \iint_D K_{0,n}(z-\zeta)F\Big(\zeta, w(\zeta), \Big\{\frac{\partial^{m+k}w}{\partial\zeta^m\partial\overline{\zeta}^k}\Big\}\Big)d\xi d\eta \end{aligned} \tag{2}$$

where Φ is a polyanalytic function of order n in D, and $T_{0,n,D}$ is a generalized Cauchy-Pompeiu type singular integral operator:

$$K_{m,n}(z) = \begin{cases} \dfrac{(-m)!(-1)^m}{(n-1)!\pi}\, z^{m-1}\overline{z}^{n-1} & \text{if } m \le 0 \\ \dfrac{(-n)!(-1)^n}{(m-1)!\pi}\, z^{m-1}\overline{z}^{n-1} & \text{if } n \le 0 \\ \dfrac{z^{m-1}\overline{z}^{n-1}}{(m-1)!(n-1)!\pi}\Big(\log|z|^2 - \displaystyle\sum_{r=1}^{m-1}\frac{1}{r} - \sum_{s=1}^{n-1}\frac{1}{s}\Big) & \text{if } m,n \in \mathbb{N}. \end{cases} \tag{3}$$

When $m = 1$ or $n = 1$, the corresponding summation in the formula is dropped. The kernel $K_{m,n}$ of the integral operator $T_{m,n,D}$ has no singularity on D, except possibly at the origin. Moreover, it follows from the properties of the operators $T_{m,k,D}$ $(m+k \le n)$ that $T_{0,n,D}f \in W_{n,p}(D)$, if $f \in L_p(D)$ $(1 < p < \infty)$ (cf. [2, 5, 6]).

Suppose $w \in W_{n,p}(D)$ is a solution of equation (1). Thus w may be expressed in the form (2), and hence we obtain

$$\begin{aligned} \frac{\partial w}{\partial z} &= \Phi_z + T_{-1,n,D}F\,, & \frac{\partial w}{\partial \overline{z}} &= \Phi_{\overline{z}} + T_{0,n-1,D}F \\ \frac{\partial^k w}{\partial z^k} &= \frac{\partial^k \Phi}{\partial z^k} + T_{-k,n,D}F\,, & \frac{\partial^k w}{\partial \overline{z}^k} &= \frac{\partial^k \Phi}{\partial \overline{z}^k} + T_{0,n-k,D}F \quad (0 \le k \le n) \end{aligned}$$

and, in general,

$$\frac{\partial^{m+k}w}{\partial z^m\partial\overline{z}^k} = \frac{\partial^{m+k}\Phi}{\partial z^m\partial\overline{z}^k} + T_{-m,n-k,D}F \qquad (n \ge m,k;\ m+k \le n).$$

Consequently, we arrive at the following result (cf. [12, 15, 21])).

Theorem 1. *The function $w \in W_{n,p}(D)$ $(2 < p < \infty)$ defined by equation (2) is a general solution of the non-homogeneous polyanalytic equation (1) if and only if for a given in the domain D polyanalytic function $\Phi \in W_{n,p}(D)$ of order n, $(w, \{h_{m,k}\})$ is a solution of the system*

$$\left.\begin{aligned} & w(z) = \Phi(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\ & h_{m,k}(z) = \frac{\partial^{m+k}\Phi}{\partial z^m\partial\overline{z}^k} + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\ & n \ge m, k \in \mathbb{N}_0, m+k \le n, (0,0) \ne (m,k) \ne (0,n), n \in \mathbb{N}. \end{aligned}\right\} \tag{4}$$

We note in passing that the integral operators $T_{m,k,D}$ $(m+k=0<m^2+k^2)$ are of singular Calderon-Zygmund type, and may be viewed as analogues of Vekua-type integral operators Π_D and $\overline{\Pi}_D$ defined by

$$\Pi_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{(\zeta-z)^2}\,d\xi d\eta$$
$$\overline{\Pi}_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{(\overline{\zeta}-\overline{z})^2}\,d\xi d\eta$$

(cf. [8, 11, 17, 18, 22]). They are singular and must be understood in the sense of Cauchy's principal value. Moreover, they satisfy the Calderon-Zygmund inequality (cf. [5, 6, 8, 17, 18, 22])

$$\|T_{m,n,D}f\|_{p,D} \le A_p \|f\|_{p,D} \tag{5}$$

where

$$A_p = \|T_{m,n,D}\|_p, \quad A_p \ge A_2 = 1 \quad (1<p<\infty).$$

On the other hand, if $m+k>0$, then $T_{m,k,D}$ are regular or weakly singular integral operators, and they may be viewed as generalizations of the Cauchy-Pompeiu integral operators T_D, $\overline{T}_D$, T_D^* and the potential operator P_D given by

$$T_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{\zeta-z}\,d\xi d\eta, \qquad \overline{T}_D f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{\overline{\zeta}-\overline{z}}\,d\xi d\eta$$
$$T_D^* f(z) = -\frac{1}{\pi}\iint_D \frac{f(\zeta)}{|\zeta-z|}\,d\xi d\eta, \qquad P_D f(z) = \frac{2}{\pi}\iint_D f(\zeta)\log|\zeta-z|\,d\xi d\eta.$$

Moreover, since $\|K_{m,n}\|_{1,D} \le C(m,n,D)$ =const, it follows from the convolution theorem of W. H. Young (see [17], for instance) that $T_{m,k,D}$ maps the Banach space $L_p(D)$ $(1\le p\le\infty)$ into itself, and the estimate

$$\|T_{m,k,D}f\|_{p,D} \le C(m,k,D)\,\|f\|_{p,D} \qquad (1\le p\le\infty, m+k>0) \tag{6}$$

holds.

2. Existence of the general solution

We make the following assumptions on the right-hand side of equation (1):

(A1) $F(z,w,\{h_{m,k}\})$ is a continuous function of its variables $z\in D$, w and the partial derivatives of w of order not exceeding n and excluding $\frac{\partial^n w}{\partial \overline{z}^n}$, which are denoted here by $\{h_{m,k}\}$.

(A2) There exists a tupel $(w^*,\{h^*_{m,k}\})$ $(w^*, h^*_{m,k}\in L_p(D), 2<p<\infty)$ such that $F(z,w^*,\{h^*_{m,k}\})\in L_p(D)$.

(A3) $F(z,w,\{h_{m,k}\})$ satisfies a Lipschitz condition of the form

$$\begin{aligned}&\Big|F(z,w(z),\{h_{m,k}(z)\}) - F(z,\widetilde{w}(z),\{\widetilde{h}_{m,k}(z)\}\Big|\\ &\qquad \le L_1 \max\Big\{\max_{m+k<n}|h_{m,k}(z)-\widetilde{h}_{m,k}(z)|,\ |w(z)-\widetilde{w}(z)|\Big\}\\ &\qquad\quad + L_2\max_{m+k=n}\big|h_{m,k}(z)-\widetilde{h}_{m,k}(z)\big|\end{aligned}$$

almost everywhere on D. While $0<L_2<1$, L_1 may take any positive value.

Note. It follows from assumptions (A2) and (A3) that $F(z,w,\{h_{m,k}\}) \in L_p(D)$ $(2<p<\infty)$ if w and all the elements of $\{h_{m,k}\}$ belong to $L_p(D)$.

We introduce the following Banach space $\mathcal{L}_p(D)$ $(2<p<\infty)$:

$$\mathcal{L}_p(D) = \left\{(w,\{h_{m,k}\}) \,\middle|\, w, h_{m,k} \in L_p(D)\right\}$$

$$n \geq m,k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

$$\|(w,\{h_{m,k}\})\| = \max\left\{\gamma\|w\|_{p,D},\, \gamma \max_{m+k<n} \|h_{m,k}\|_{p,D},\, \max_{m+k=n} \|h_{m,k}\|_{p,D}\right\} \quad (\gamma>0).$$

Next we define a mapping $\mathbb{P}$ in $\mathcal{L}_p(D)$ through the right-hand side of (4). For any tuple $(w,\{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$\begin{aligned}
&(W,\{H_{m,k}\}) = \mathbb{P}(w,\{h_{m,k}\}) \\
&W(z) = \Phi(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\
&H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \overline{z}^k}\Phi + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z) \\
&n \geq m,k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N} \\
&\Phi \in W_{n,p}(D) \;\; (2<p<\infty) \text{ a polyanalytic function in } D.
\end{aligned} \tag{7}$$

It follows immediately from the preceding discussion that $\mathbb{P}$ maps $\mathcal{L}_p(D)$ $(2<p<\infty)$ into itself.

We next show that, under certain conditions, $\mathbb{P}$ is contractive in $\mathcal{L}_p(D)$, so that we can apply the Banach fixed point theorem. To this end we consider the images $(W,\{H_{m,k}\})$, $(\widetilde{W},\{\widetilde{H}_{m,k}\})$ of $(w,\{h_{m,k}\})$, $(\widetilde{w},\{\widetilde{h}_{m,k}\}) \in \mathcal{L}_p(D)$, respectively, under the mapping $\mathbb{P}$. We then have

$$\begin{aligned}
\gamma\|W-\widetilde{W}\|_p &\leq \gamma\|T_{0,n,D}\|_p \big\|F(z,w,\{h_{m,k}\}) - F(z,\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|_{p,D} \\
&\leq \gamma\|T_{0,n,D}\|_p \Bigg(L_1 \max\left\{\max_{0<m+k<n} \|h_{m,k}-\widetilde{h}_{m,k}\|_{p,D}, \|w-\widetilde{w}\|_{p,D}\right\} \\
&\qquad + L_2 \max_{m+k=n} \|h_{m,k}-\widetilde{h}_{m,k}\|_{p,D}\Bigg) \\
&\leq \|T_{0,n,D}\|_p (L_1+\gamma L_2) \big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|.
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\gamma\|H_{m,k}-\widetilde{H}_{m,k}\|_{p,D} &\leq \|T_{-m,n-k,D}\|_p (L_1+\gamma L_2) \big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\| \\
\|H_{\alpha,\beta}-\widetilde{H}_{\alpha,\beta}\|_p &\leq \|T_{-\alpha,n-\beta,D}\|_p \Big(\frac{1}{\gamma}L_1 + L_2\Big) \big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|
\end{aligned}$$

for $0<m+k<n$ and $\alpha+\beta=n$ with $(\alpha,\beta) \neq (0,n)$. On account of the relations

$$\|T_{-m,n-k,D}\| = \begin{cases} C(m,k,D) & \text{for } 0<m+k<n \\ \|\Pi_D\|_p & \text{for } m+k=n \end{cases} \qquad (1<p<\infty), \tag{8}$$

where Π_D is the strongly singular Vekua-type integral operator, we arrive at the estimate

$$\begin{aligned}&\big\|(W,\{H_{m,k}\}) - (\widetilde{W},\{\widetilde{H}_{m,k}\})\big\| \\ &\quad \le \Big(\frac{1}{\gamma}L_1 + L_2\Big)\max\Big\{\gamma\|T_{0,n,D}\|_p, \gamma \max_{m+k<n}\|T_{-m,n-k,D}\|_p, \|\Pi_D\|_p\Big\} \\ &\qquad \times \big\|(w,\{h_{m,k}\}) - (\widetilde{w},\{\widetilde{h}_{m,k}\})\big\|\end{aligned}$$

and $\mathbb{P}$ is contractive in $\mathcal{L}_P(D)$ $(2 < p < \infty)$ if

$$\Big(\frac{1}{\gamma}L_1 + L_2\Big)\max\Big\{\gamma\|T_{0,n,D}\|_p,\ \gamma \max_{m+k<n}\|T_{-m,n-k,D}\|_p,\ \|\Pi_D\|_p\Big\} < 1. \tag{9}$$

This condition may be satisfied if the constants L_1, L_2 and γ can be chosen properly and the domain D made sufficiently small. It is known (cf. [5, 6, 8, 12, 17, 18], for instance) that

$$\|\Pi_D\|_p \ge 1 \quad (1 < p < \infty) \qquad \text{and} \qquad \|\Pi_D\|_2 = 1.$$

Thus for a chosen $\mathcal{L}_p(D)$ $(2 < p < \infty)$ we need L_2, $0 < L_2 < 1$, such that $L_2\|\Pi_D\|_p < 1$. Next we choose the constant $\gamma > 0$ large enough so that, for the given $L_1 > 0$, $(\frac{1}{\gamma}L_1 + L_2)\|\Pi_D\|_p < 1$ also holds. Finally, since $\|T_{0,n,D}\|_p$ and $\|T_{-m,n-k,D}\|_p$, $0 < m + k < n$, vary directly with the area of the domain D, we may satisfy estimate (9) eventually be reducing the size of D.

If estimate (9) is realized, then $\mathbb{P}$ has a unique fixed element $(\dot{w}, \{\dot{h}_{m,k}\}) \in \mathcal{L}_p(D)$ $(2 < p < \infty)$ and $\dot{w}$ is the general solution of equation (1) corresponding to the given in D polyanalytic function $\Phi \in W_{n,p}(D)$ of order n. Moveover, $\dot{w} \in W_{n,p}(D)$ $(2 < p < \infty)$:

$$\dot{w}(z) = \Phi(z) + T_{0,n,D}F\big(\zeta, \dot{w}(\zeta), \{\dot{h}_{m,k}(\zeta)\}\big)(z)$$

$$\dot{h}_{m,k}(z) = \frac{\partial^{m+k}\Phi}{\partial z^m \partial \overline{z}^k} + T_{-m,n-k,D}F\big(\zeta, \dot{w}(\zeta), \{\dot{h}_{m,k}(\zeta)\}\big)(z)$$

$$n \ge m, k \in \mathbb{N}_0, m + k \le n, (0,0) \ne (m,k) \ne (0,n), n \in \mathbb{N}.$$

Theorem 2. *Under assumptions* (A1) - (A3) *and* (9) *the non-homogeneous polyanalytic differential equation* (1) *admits a uniquely defined solution* $w \in W_{n,p}(D)$ $(2 < p < \infty)$ *given by equation* (2) *for every prescribed in the domain* D *polyanalytic function* $\Phi \in W_{n,p}(D)$. *This solution defines a mapping from* $\Phi \longrightarrow w = R(\Phi)$.

3. The generalized Riemann-Hilbert problem for polyanalytic functions

We consider the following boundary value problem for a polyanalytic function Φ of order n:

$$\begin{aligned} \frac{\partial^n \Phi}{\partial \overline{z}^n} &= 0 && \text{on } D = \{z : \, |z| < 1\} \\ \operatorname{Re}\left[(a_k + ib_k)\frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}}\right](t) &= c_k(t) && \text{on } \partial D \quad (k = 1, \ldots, n) \end{aligned} \tag{10}$$

where $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$ are prescribed real-valued functions on ∂D. Moreover, $(a_k + ib_k)(t) \neq 0$ for all $t \in \partial D$.

A polyanalytic function Φ of order n may be expressed as

$$\Phi = \Phi(z, \overline{z}) = \sum_{\rho=0}^{n-1} \overline{z}^{\rho} \varphi_\rho(z) \qquad (\varphi_\rho \ \text{ holomorphic}).$$

Thus

$$\begin{aligned} \frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}} &= i^{k-1}\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\Big)^{n-k}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big)^{k-1} \sum_{\rho=0}^{n-1} \overline{z}^{\rho}\varphi_\rho(z) \\ &= i^{k-1}\sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1}(-1)^\beta \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\quad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, \overline{z}^{\rho-\alpha-\beta}\, \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}}\varphi_\rho(z). \end{aligned}$$

Since on $\partial D\, \overline{t} = \frac{1}{t}$ we shall replace $\overline{z}$ by $\frac{1}{z}$ in the expression above and then reduce the given boundary conditions for the polyanalytic function Φ to n eqivalent Riemann-Hilbert boundary value problems for some holomorphic functions G_k $(k = 1, \ldots, n)$ which are defined in terms of the holomorphic functions φ_ρ $(\rho = 0, \ldots, n-1)$. Thus

$$\begin{aligned} \frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}} &= i^{k-1}\sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1}(-1)^\beta \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\quad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, z^{\alpha+\beta-\rho}\, \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}}\varphi_\rho(z). \end{aligned}$$

Hence

$$\begin{aligned} &\operatorname{Re}\left[(a_k + ib_k)\frac{\partial^{n-1}\Phi}{\partial x^{n-k}\partial y^{k-1}}\right](t) \\ &\quad = \operatorname{Re}\bigg[(a_k + b_k)(t)\, t^{1-n} i^{k-1}\sum_{\alpha=0}^{n-k}\sum_{\beta=0}^{k-1}(-1)^\beta \binom{n-k}{\alpha}\binom{k-1}{\beta} \\ &\qquad \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, t^{n+\alpha+\beta-\rho-1}\, \frac{d^{n-\alpha-\beta-1}}{dt^{n-\alpha-\beta-1}}\,\varphi_\rho(t)\bigg], \end{aligned}$$

i.e.

$$\mathrm{Re}\left[(a_k + ib_k)(t)\, i^{k-1} t^{1-n} G_k(t)\right] = c_k(t) \qquad (k = 1, \ldots, n) \tag{11}$$

where

$$\begin{aligned} G_k(z) = & \sum_{\alpha=0}^{n-k} \sum_{\beta=0}^{k-1} (-1)^\beta \binom{n-k}{\alpha} \binom{k-1}{\beta} \\ & \times \sum_{\rho=\alpha+\beta}^{n-1} \frac{\rho!}{(\rho-\alpha-\beta)!}\, z^{n+\alpha+\beta-\rho-1} \frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}} \varphi_\rho(z). \end{aligned} \tag{12}$$

The solution of the Riemann-Hilbert problem (11) is known (see [9, 11, 14, 16]). If $\kappa_k := \mathrm{index}\,[(a_k - ib_k), \partial D] \geq 0$, then the general solution G_k is given with the aid of the Schwarz integral as

$$z^{1-n} G_k(z) = \frac{X_k(z)}{2\pi i} \left[\int_{\partial D} \frac{i^{1-k} c_k(t)}{(a_k + ib_k)(t)\, X_k^+(t)} \frac{t+z}{(t-z)t}\, dt + P_{\kappa_k}(z) \right] \tag{13}$$

where P_{κ_k} is a polynomial of degree not exceeding κ_k and X_κ is the canonical solution of the corresponding homogeneous problem

$$\begin{aligned} X_k(z) &= z^{\kappa_k} \exp \Gamma_k(z) \\ \Gamma_k(z) &= \frac{1}{4\pi i} \int_{\partial D} \log\left((-1)^{k-2} t^{-2\kappa_k} \frac{(a_k - ib_k)(t)}{(a_k + ib_k)(t)} \right) \frac{t+z}{(t-z)t}\, dt\,. \end{aligned}$$

If any of the κ_k is negative, then the corresponding Riemann-Hilbert problem has a unique solution, bounded at infinity for instance, if and only if the conditions

$$\int_{\partial D} \frac{c_k(t)}{(a_k + ib_k)(t)\, X_k^+(t)}\, t^j\, dt = 0 \qquad (j = 0, \ldots, -2\kappa_k - 2) \tag{14}$$

are fulfilled, and in that case the solution is given by (13) as well, with the obvious modification that we set $P_{\kappa_k}(z) \equiv 0$ (cf. [9, 11, 16]).

We investigate the possibility for the satisfaction of the solvability conditions (14). For this purpose we consider the modified Riemann-Hilbert problem (cf. [4, 25])

$$\mathrm{Re}\left[i^{k-1} t^{1-n} (a_k + ib_k)(t)\, G_k(t)\right] = c_k(t) - \sum_{s=\kappa_k+1}^{-\kappa_k-1} \lambda_s t^s \qquad \text{on } \partial D \tag{11$'$}$$

where $\lambda_{-s} = \overline{\lambda_s}$ are constants yet to be determined appropriately. The modified problem is uniquely solvable for $\kappa_k < 0$, and the solution G_k to the original Riemann-Hilbert problem (11) has the representation

$$z^{1-n} G_k(z) = \frac{X_k(z)}{2\pi i} \int_{\partial D} \frac{i^{1-k}}{(a_k + ib_k)(t) X_k^+(t)} \left(c_k(t - \sum_{s=\kappa_k+1}^{-\kappa_k-1} \lambda_s t^s \right) \frac{t+z}{(t-z)t}\, dt \tag{13$'$}$$

(cf. [4, 25]).

In order that a Riemann-Hilbert problem with non-negative index to be uniquely solvable $2\kappa_k + 1$ point conditions need to be imposed on the solution G_k. These conditions can be expressed in terms of the solution Φ of the given polyanalytic equation (10). Suppose r among the n Riemann-Hilbert problems (11) have non-negative indices, whose sum is N. Then, we demand that

$$\begin{gathered}\operatorname{Im}\left[i^{k-1}(a_k + ib_k)(\tau_j)\,\tau_j^{1-n} G_k(\tau_j)\right] = dj \quad (j = 1, 2, \ldots, N + r) \\ \tau_j \in \partial D, \tau_m \neq \tau_n \text{ for } m \neq n, d_j \in \mathbb{R}.\end{gathered} \tag{15}$$

It can be shown that $z^{1-n} G_k(z) \in W_{1,p}(0)$ $(2 < p < \infty, k = 1, \ldots, n)$ and estimates of the form

$$\begin{aligned}\|z^{1-n} G_k\|_{p,D} &\le C_k(a_k, b_k, p, D)\,\|c_k\|_{p,\partial D} \\ \|z^{1-n} G_k\|_{1,p,D} &\le K_k(a_k, b_k, p, D)\,\|c_k\|_{1-\frac{1}{p},p,\partial D}\end{aligned} \qquad (k = 1, \ldots, n; 2 < p < \infty) \tag{16}$$

hold (cf. [12, 14, 15]).

Suppose we have determined all n holomorphic functions G_k $(k = 1, \ldots, n)$ uniquely. We proceed to compute the required polyanalytic function Φ by expressing the holomorphic functions φ_p $(p = 0, \ldots, n-1)$ in terms of G_k. We shall make use of the following three facts:

1. Derivatives of Φ with respect to x, y can easily be expressed by the holomorphic functions G_k. It follows from (12) that

$$\begin{aligned}\frac{\partial^{n-j}\Phi}{\partial x^{n-q-\nu-j}\partial y^{q+\nu}} &= i^{q+\nu}\Big(\frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}\Big)^{n-q-\nu-j}\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\Big)^{q+\nu}\sum_{\rho=0}^{n-1}\overline{z}^{\rho}\varphi_\rho(z) \\ &= i^{q+\nu}\sum_{\alpha=0}^{n-q-\nu-j}\sum_{\beta=0}^{q+\nu}(-1)^{\beta}\binom{n-q-\nu-j}{\alpha}\binom{q+\nu}{\beta} \\ &\quad\times\sum_{\rho=\alpha+\beta}^{n-1}\frac{\rho!}{(\rho-\alpha-\beta)!}\,\overline{z}^{\rho-\alpha-\beta}\,\frac{d^{n-\alpha-\beta-1}}{dz^{n-\alpha-\beta-1}}\,\varphi_\rho(z) \\ &= i^{q+\nu} z^{1-n} G_{q+\nu+j}(z)\end{aligned}$$

on ∂D (i.e. $\overline{z} = \frac{1}{z}$).

2. Derivatives of Φ with respect to $z, \overline{z}$ can be expressed by the derivatives with respect to x, y, and hence in terms of G_k. Indeed, on ∂D we have

$$\begin{aligned}\frac{\partial^{n-j}\Phi}{\partial z^{n-k}\partial \overline{z}^{k-j}} &= 2^{j-n}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)^{n-k}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)^{k-j}\Phi \\ &= 2^{j-n}\sum_{q=0}^{n-k}\sum_{\nu=0}^{k-j}(-1)^q i^{q+\nu}\binom{n-k}{q}\binom{k-j}{\nu}\frac{\partial^{n-j}\Phi}{\partial x^{n-q-\nu-j}\partial y^{q+\nu}} \\ &= 2^{j-n}\sum_{q=0}^{n-k}\sum_{\nu=0}^{k-j}(-1)^q i^{q+\nu}\binom{n-k}{q}\binom{k-j}{\nu} i^{q+\nu} z^{1-n} G_{q+\nu+j}(z) \\ &= 2^{j-n} z^{1-n}\sum_{q=0}^{n-k}\binom{n-k}{q}\sum_{\nu=0}^{k-j}(-1)^{\nu}\binom{k-j}{\nu} G_{q+\nu+j}(z)\end{aligned} \tag{17}$$

for $n \geq k \geq j$.

3. The holomorphic functions φ_ρ can be expressed by the derivatives of the poly-analytic function Φ with respect to z and $\overline{z}$.

Thus

$$\frac{\partial^{n-1}\Phi}{\partial \overline{z}^{n-1}} = \frac{\partial^{n-1}}{\partial \overline{z}^{n-1}} \sum_{\rho=0}^{n-1} \overline{z}^\rho \varphi_\rho(z) = (n-1)! \varphi_{n-1}(z).$$

On the other hand, it follows from (17), with $k = n$ and $j = 1$, that

$$\frac{\partial^{n-1}\Phi}{\partial \overline{z}^{n-1}} = (2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n-1}{\nu} G_{\nu+1}(z) \qquad \text{on } \partial D.$$

Hence we may conclude that

$$\varphi_{n-1}(z) = \frac{1}{(n-1)!} (2z)^{1-n} \sum_{\nu=0}^{n-1} (-1)^\nu \binom{n-1}{\nu} G_{\nu+1}(z) \qquad \text{on } \partial D.$$

Next we have, on the one hand,

$$\begin{aligned} \frac{\partial^{n-2}\Phi}{\partial \overline{z}^{n-2}} &= \frac{\partial^{n-2}}{\partial \overline{z}^{n-2}} \Big(\overline{z}^{n-2} \varphi_{n-2}(z) + \overline{z}^{n-1} \varphi_{n-1}(z) \Big) \\ &= (n-2)!\, \varphi_{n-2}(z) + (n-1)! \overline{z}\, \varphi_{n-1}(z). \end{aligned}$$

On the other hand, we deduce from (17), with $k = n$ and $j = 2$, that

$$\frac{\partial^{n-2}\Phi}{\partial \overline{z}^{n-2}} = 2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^\nu \binom{n-2}{\nu} G_{\nu+2}(z) \qquad \text{on } \partial D.$$

So we can obtain for φ_{n-2} the representation

$$\varphi_{n-2}(z) = \frac{1}{(n-2)!} \left[2^{2-n} z^{1-n} \sum_{\nu=0}^{n-2} (-1)^\nu \binom{n-2}{\nu} G_{\nu+2}(z) - (n-1)!\, \overline{z}\, \varphi_{n-1}(z) \right]$$

on ∂D.

Similarly we compute $\varphi_{n-3}, \ldots, \varphi_1, \varphi_0$. Suppose we have computed $\varphi_{n-1}, \varphi_{n-2}, \ldots,$ φ_{n-j+1}. Then we compute φ_{n-j} as

$$\frac{\partial^{n-j}\Phi}{\partial \overline{z}^{n-j}} = \frac{\partial^{n-j}}{\partial \overline{z}^{n-j}} \sum_{\rho=n-j}^{n-1} \overline{z}^\rho \varphi_\rho(z) = (n-j)! \varphi_{n-j}(z) + \sum_{\rho=n-j+1}^{n-1} \varphi_\rho(z) \frac{\partial^{n-j} \overline{z}^\rho}{\partial \overline{z}^{n-j}}.$$

On the other hand, for $k = n$ formula (17) yields

$$\frac{\partial^{n-j}\Phi}{\partial \overline{z}^{n-j}} = 2^{j-n} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^\nu \binom{n-j}{\nu} G_{\nu+j}(z).$$

We thus arrive at the general representation

$$\varphi_{n-j}(z) = \frac{1}{(n-j)!}\left[2^{j-n} z^{1-n} \sum_{\nu=0}^{n-j} (-1)^{\nu} \binom{n-j}{\nu} G_{\nu+j}(z) - \sum_{\rho=n-j+1}^{n-1} \varphi_{\rho}(z) \frac{\partial^{n-j} \overline{z}^{\rho}}{\partial \overline{z}^{n-j}}\right]$$

for φ_{n-j} $(j = 1, \ldots, n)$. Hence all n holomorphic functions φ_j $(j = 0, \ldots, n-1)$ are uniquely determinable, and with them the polyanalytic function Φ as well. Furthermore, since $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$, we conclude that $z^{1-n} G_k(z) \in W_{1,p}(D)$ (cf. [1, 10, 12, 14, 15, 19, 20]). It thus follows from (12) that

$$t^{1-n} G_1(t) = \sum_{\alpha=0}^{n-1} \binom{n-1}{\alpha} \sum_{j=\alpha}^{n-1} \frac{j!}{(j-\alpha)!} t^{\alpha-j} \frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D)$$

and, in particular,

$$\frac{d^{n-\alpha-1}}{dt^{n-\alpha-1}} \varphi_j(t) \in W_{1-\frac{1}{p},p}(\partial D) \qquad (j, \alpha = 0, \ldots, n-1).$$

Hence

$$\frac{d^{n-1}}{dt^{n-1}} \varphi_j \in W_{1-\frac{1}{p},p}(\partial D) \qquad (j = 0, 1, \ldots, n-1).$$

It now follows from the properties of traces of functions that

$$\varphi_j \in W_{n-\frac{1}{p},p}(\partial D) \quad \text{and} \quad \varphi_j,\ \Phi \in W_{n,p}(D) \qquad (j = 1, \ldots, n)$$

and the estimates

$$\left.\begin{aligned} \|\Phi\|_{p,D} &\le C_1(p, D) \max_k \|c_k\|_{p,\partial D} \\ \|\Phi\|_{j,p,D} &\le C_2(p, D) \max_k \|c_k\|_{1-\frac{1}{p},p,\partial D} \end{aligned}\right\} \qquad (j, k = 1, \ldots, n; 2 < p < \infty) \tag{18}$$

hold (cf. [1, 10, 12, 14, 15, 19, 20]).

4. The generalized Riemann-Hilbert problem for equation (1)

We now take up the following boundary value problem for the function w:

$$\frac{\partial^n w}{\partial \overline{z}^n} = F\left(z, w, \left\{\frac{\partial^{m+k} w}{\partial z^m \partial \overline{z}^k}\right\}\right) \quad \text{on } D$$

$$\operatorname{Re}\left[(a_k + i b_k) \frac{\partial^{n-1} w}{\partial x^{n-k} \partial y^{k-1}}\right](t) = c_k(t) \quad \text{on } \partial D \qquad (k = 1, \ldots, n) \tag{19}$$

$$n \ge m, k \in \mathbb{N}_0, m + k \le n, (0,0) \ne (m,k) \ne (0,n), n \in \mathbb{N}$$

where $a_k, b_k, c_k \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty)$ are prescribed real-valued functions on ∂D with $(a_k + ib_k)(t) \neq 0$ for all $t \in \partial D$.

It was shown earlier that for every polyanalytic function $\Phi \in W_{n,p}(D)$ $(2 < p < \infty)$ there exists a unique solution $w \in W_{n,p}(D)$ to the partial differential equation (1). This solution is represented by (2). We shall now exploit the arbitrariness of the polyanalytic function Φ to construct the solution of the boundary value problem (1), (19). For this purpose we shall write Φ as

$$\Phi = \Phi_c + \Phi_{(w,h)}$$

where Φ_c, $\Phi_{(w,h)}$ are solutions of the boundary value problems

$$\begin{aligned} &\mathrm{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}\Phi_c}{\partial x^{n-j}\partial y^{j-1}}\right](t) = c_j(t) \quad \text{on} \quad \partial D \\ &\mathrm{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}}{\partial x^{n-j}\partial y^{j-1}}\,\Phi_{(w,h)}\right](t) \\ &\qquad = -\mathrm{Re}\left[(a_j + ib_j)\,\frac{\partial^{n-1}}{\partial x^{n-j}\partial y^{j-1}} T_{0,n,D}F(\cdot, w, \{h_{m,k}\})\right](t) \\ &\qquad := g_{w,h),j}(t) \quad \text{on} \quad \partial D \end{aligned} \tag{20}$$

for $j = 1, \ldots, n$. Since $F(z, w, \{h_{m,k}\}) \in L_p(D)$ $(2 < p < \infty)$, then $T_{0,n,D}F \in W_{n,p}(D)$ (cf. [5, 6]). Moreover,

$$\begin{aligned} &\frac{\partial^{n-1}}{\partial x^{n-k}\partial y^{k-1}}\,T_{0,n,D}F(z) \\ &\quad = i^{k-1}\sum_{\alpha=0}^{n-k}\binom{n-k}{\alpha}\sum_{\beta=0}^{k-1}(-1)^{\beta}\binom{k-1}{\beta}\frac{\partial^{n-1}}{\partial z^{n-\alpha-\beta-1}\partial \overline{z}^{\alpha+\beta}}\,T_{0,n,D}F(z) \\ &\quad = i^{k-1}\sum_{\alpha=0}^{n-k}\binom{n-k}{\alpha}\sum_{\beta=0}^{k-1}(-1)^{\beta}\binom{k-1}{\beta}T_{\alpha+\beta+1-n,n-\alpha-\beta,D}F(z) \\ &\quad \in W_{1,p}(D), \end{aligned}$$

i.e. $g_{(w,h),j} \in W_{1-\frac{1}{p},p}(\partial D)$ $(2 < p < \infty; j = 1, \ldots, n)$ (cf. [1, 10, 11, 12, 19, 20]).

Polyanalytic functions which satisfy boundary conditions of the form (20) have been constructed earlier, and we deduce from there that Φ_c, $\Phi_{(w,h)} \in W_{n,p}(D)$ $(2 < p < \infty)$ and, in particular, the estimates

$$\begin{aligned} \|\Phi_{(w,h)}\|_{p,D} &\leq C(p, D)\,\|g_{(w,h)j}\|_{p,\partial D} \leq C_1(p, D)\,\|T_{0,n,D}F\|_{1,p,D} \\ \|\Phi_{(w,h)}\|_{k,p,D} &\leq C_2(p, D)\,\|T_{0,n,D}F\|_{n+1-k,p,D} \end{aligned} \tag{21}$$

hold for $k, j = 1, \ldots, n$ and $2 < p < \infty$.

We now define a mapping $\mathbb{Q}$ in the Banach space $\mathcal{L}_p(D)$ $(2 < p < \infty)$. For any tuple $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$ we set

$$(W, \{H_{m,k}\}) = \mathbb{Q}(w, \{h_{m,k}\})$$

where

$$W(z) = \Phi_c(z) + \Phi_{(w,h)}(z) + T_{0,n,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z)$$

$$H_{m,k}(z) = \frac{\partial^{m+k}}{\partial z^m \partial \overline{z}^k}\big(\Phi_c(z) + \Phi_{(w,h)}(z)\big) + T_{-m,n-k,D}F\big(\zeta, w(\zeta), \{h_{m,k}(\zeta)\}\big)(z)$$

$$n \geq m, k \in \mathbb{N}_0, m+k \leq n, (0,0) \neq (m,k) \neq (0,n), n \in \mathbb{N}.$$

The operator $\mathbb{Q}$ is uniquely defined, and it maps the Banach space $\mathcal{L}_p(D)$ $(2 < p < \infty)$ into itself. Moreover, the following result holds.

Theorem 3. *If $(w, \{h_{m,k}\})$ is a fixed point of the operator $\mathbb{Q}$, then w is the solution of the given differential equation* (1) *which also satisfies the boundary conditions* (19)*.*

We next derive the conditions to be imposed in order that $\mathbb{Q}$ has a fixed point. Suppose $(W, \{H_{m,k}\}), (\widetilde{W}, \{\widetilde{H}_{m,k}\})$ are the respective images of $(w, \{h_{m,k}\})$, $(\widetilde{w}, \{\widetilde{h}_{m,k}\}) \in \mathcal{L}_p(D)$ $(2 < p < \infty)$. If we set

$$\varphi = \Phi_{(w,h)} - \Phi_{(\widetilde{w},\widetilde{h})} \qquad \text{and} \qquad f = F\big(z, w, \{h_{m,k}\}\big) - F\big(z, \widetilde{w}, \{\widetilde{h}_{m,k}\}\big),$$

then

$$W - \widetilde{W} = \varphi + T_{0,n,D}f, \qquad H_{m,k} - \widetilde{H}_{m,k} = \frac{\partial^{m+k}\varphi}{\partial z^m \partial \overline{z}^k} + T_{-m,n-k,D}F$$

and

$$\begin{aligned}
\gamma \, \|W - \widetilde{W}\|_{p,D} &\leq \gamma\Big(C_1(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big)\|f\|_{p,D} \\
&\leq \Bigg(L_1 \max\Big\{\max_{m+k<n}\|h_{m,k} - \widetilde{h}_{m,k}\|_{p,D}, \|w - \widetilde{w}\|_{p,D}\Big\} \\
&\qquad + L_2 \max_{m+k=n}\|h_{m,k} - \widetilde{h}_{m,k}\|_{p,D}\Bigg)\gamma\Big(C_1(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big) \\
&\leq \Big(C_1(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\Big)(L_1 + \gamma L_2)\big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|.
\end{aligned}$$

Similary we arrive at

$$\begin{aligned}
\gamma \, \|H_{m,k} - \widetilde{H}_{m,k}\|_{p,D} \leq \Big(&C_2(p,D)\,\|T_{0,n,D}\|_{n-m-k,p} + \|T_{-m,n-k,D}\|_p\Big) \\
&\times (L_1 + \gamma L_2)\,\big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|
\end{aligned}$$

and

$$\begin{aligned}
\|H_{\alpha,\beta} - \widetilde{H}_{\alpha,\beta}\|_{p,D} \leq \Big(&C_3(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{-\alpha,n-\beta,D}\|_p\Big) \\
&\times \Big(\frac{1}{\gamma}L_1 + L_2\Big)\big\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\big\|
\end{aligned}$$

for $0 < m+k < n, \alpha + \beta = n$ and $(\alpha, \beta) \neq (0, n)$. Consequently, on account of relations (8), we arrive at the estimate

$$\|(W, \{H_{m,k}\}) - (\widetilde{W}, \{\widetilde{H}_{m,k}\})\|_p \leq \kappa\,\|(w, \{h_{m,k}\}) - (\widetilde{w}, \{\widetilde{h}_{m,k}\})\| \tag{22}$$

where $\left(\frac{1}{\gamma}L_1 + L_2\right)^{-1}\kappa$ is the maximum of the three quantities

$$\begin{aligned}
&\gamma\left(C_1(p,D)\,\|T_{0,n,D}\|_{1,p} + \|T_{0,n,D}\|_p\right)\\
&\gamma \max_{m+k<n}\left\{C_2(p,D)\,\|T_{0,n,D}\|_{n-m-k+1,p} + \|T_{-m,n-k,D}\|_p\right\}\\
&C_3(p,D)\,\|T_{0,n,D}\|_{1,p} + \|\Pi_D\|_p
\end{aligned}$$

If $\kappa < 1$, then the mapping $\mathbb{Q}$ is contractive in $\mathcal{L}_p(D)$ $(2 < p < \infty)$ and it has therefore exactly one fixed element $(w, \{h_{m,k}\}) \in \mathcal{L}_p(D)$, by the Banach fixed point theorem.

The contractiveness of $\mathbb{Q}$ imposes certain restrictions on the constants L_1, L_2, γ and the size of the domain D. Going through an argument similar to the one presented earlier for the case of the existence of a general solution, we can secure the contractiveness of $\mathbb{Q}$, and hence the existence of a solution $w \in W_{n,p}(D)$ $(2 < p < \infty)$ of the boundary value problem posed. It is easy to establish that the solution is unique.

Theorem 4. *Under the assumptions* (A1) - (A3), (15) *and* $\kappa < 1$ *the generalized Riemann-Hilbert boundary value problem* (1), (19) *admits a unique solution* $w \in W_{n,p}(D)$ $(2 < p < \infty)$.

Acknowledgement. This work was carried out during the author's 1998 visit to the Free University of Berlin in the Federal Republic of Germany under the DAAD sponsorship. The author wishes to thank DAAD and his host, Prof. Dr. Heinrich Begehr of the I. Mathematics Institute for the support and encouragement.

References

[1] Adams, R. A.: *Sobolev Spaces.* New York: Academic Press 1975.

[2] Akal, M. S.: *Boundary Value Problems for Complex Elliptic Partial Differential Equations of Higher Order.* Dissertation. Berlin: Free University 1996. Aachen: Shaker-Verlag 1996.

[3] Balk, M. B.: *Polyanalytic Functions.* Berlin: Akademie Verlag 1991.

[4] Begehr, H.: *Complex Analytic Methods for Partial Differential Equations.* Singapore: World Sci. Publ. Co. 1994.

[5] Begehr, H. and G. N. Hile: *A hierarchy of integral operators.* Rocky Mountains J. Math. 27 (1997), 669 – 706.

[6] Begehr, H. and G. N. Hile: *Higher Order Cauchy-Pompeiu operators.* In: Proceedings of Operator Theory for Complex and Hypercomplex Analysis, Mexico City 1994 (eds.: E. Remires de Arellano et al.). Providence: Amer. Math. Soc. 1998, pp. 41 – 49.

[7] Begehr, H. and G. C. Wen: *Nonlinear Elliptic Boundary Value Problems and Their Applications.* Harlow: Longman Ltd. 1996.

[8] Calderon, A. P. and A. Zygmund: *On the existence of singular integrals.* Acta Math. 88 (1957), 85 – 139.

[9] Gakhov, F. D.: *Boundary Value Problems.* Oxford: Pergamon Press 1966.

[10] Kufner, A., John, O. and S. Fucik: *Function Spaces.* Leyden: Noordhoff Int. Publ. 1977.

[11] Monakhov, V. N.: *Boundary Value Problems.* New Jersey: Amer. Math. Soc. 1983.

[12] Mshimba, A. S.: *Construction of the solution to the Dirichlet boundary value problem in $W_{1,p}(G)$ for systems of elliptic partial differential equations in the plane.* Math. Nachr. 99 (1980), 145 – 163.

[13] Mshimba, A. S.: *On the L_p norms of some singular integral operators.* Afrika Mathematika 5 (1983), 34 – 46.

[14] Mshimba, A. S.: *The Hilbert boundary value problem for holomorphic functions in Sobolev spaces.* Appl. Anal. 30 (1988), 87 – 99.

[15] Mshimba, A. S.: *The Cauchy-Lebesgue integral and boundary value problems.* Compl. Var. 16 (1991), 307 – 313.

[16] Muskhelishvili, N. I.: *Singuläre Integralgleichungen.* Berlin: Akademie-Verlag 1965.

[17] Neri, U.: *Singular Integrals.* Berlin et al.: Springer-Verlag 1971.

[18] Stein, E. M.: *Singular Integrals and Differentiability of Functions.* Princeton: Univ. Press 1970.

[19] Triebel, H.: *Interpolation Theory, Functions Spaces, Differential Operators.* Amsterdam: North Holland Publ. Co. 1978.

[20] Triebel, H.: *Theory of Function Spaces I.* Basel: Birkhäuser Verlag 1983.

[21] Tutschke, W.: *Die neuen Methoden der Komplexen Analysis und ihre Anwendungen auf nichtlineare Differentialgleichungssysteme* (Sitzungsber. Akad. Wiss. DDR: Vol. 17N). Berlin: Dt. Verlag Wiss. 1976.

[22] Vekua, I. N.: *Generalized Analytic Functions.* Oxford: Pergamon Press 1962.

[23] Vekua, I. N.: *New Methods of Solving Elliptic Equations.* Amsterdam: North Holland Publ. Co. 1967.

[24] Wen, G. C. and H. Begehr: *Boundary Value Problems for Elliptic Equations and Systems.* Harlow: Longman Ltd. 1990.

[25] Xu, Z. Y.: *Nonlinear Poincaré problem for a system of first order elliptic equations in the plane.* Compl. Var. 7 (1987), 363 – 381.

Received 17.07.1998; in revised form 16.02.1999